ON IRREDUCIBLE ALGEBRAIC SETS OVER LINEARLY ORDERED SEMILATTICES II

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Equations over finite linearly ordered semilattices are studied. It is assumed that the order of a semilattice is not less than the number of variables in an equation. For any equation \( t(X) = s(X) \), we find irreducible components of its solution set. We also compute the average number \( \text{Irr}(n) \) of irreducible components for all equations in \( n \) variables. It turns out that \( \text{Irr}(n) \) and the function \( \frac{4}{9} n! \) are asymptotically equivalent.

Keywords: irreducible components, algebraic sets, semilattices.

Introduction

This paper is the sequel of [1], and we recall below the general problems studied in both papers.

Following [2], one can define a notion of an equation over a linearly ordered semilattice \( L_l = \{a_1, a_2, \ldots, a_l\} \) (the formal definition of an equation is given below in the paper). A set \( Y \) is algebraic if it is the solution set for a system of equations over \( L_l \). Let us consider an equation \( t(X) = s(X) \) in \( n \) variables over \( L_l \), and let \( Y \) be the solution set for \( t(X) = s(X) \). One can find algebraic sets \( Y_1, Y_2, \ldots, Y_m \) such that \( Y = \bigcup_{i=1}^{m} Y_i \). One can decompose each \( Y_i \) into a union of other algebraic sets, etc. This process terminates after a finite number of steps and gives a decomposition of \( Y \) into a union of irreducible algebraic sets \( Y_i \) (the sets \( Y_i \) are called the irreducible components of \( Y \)). Roughly speaking, irreducible algebraic sets are “atoms” which form any algebraic set. The size and the number of such “atoms” are important characteristics of the semilattice \( L_l \), since there are connections between irreducible algebraic sets and universal theory of linearly ordered semilattices [2]. Moreover, the number of irreducible components was involved in the estimation of lower bounds of algorithm complexity (see [3] for more details).

In the previous paper [1], we studied equations \( t(X) = s(X) \) with \( n > l \), i.e. the number of variables occurring in \( t(X) = s(X) \) is more than the order of the semilattice \( L_l \). In [1], we also studied algebraic sets and irreducible components and computed the average number of irreducible components of the solution sets for equations in \( n \) variables.

In this paper, we assume \( n \leq l \) (i.e. the order of the semilattice \( L_l \) is not less than the number of variables in \( t(X) = s(X) \)) and study the similar problems. Precisely, for any equation \( t(X) = s(X) \) in \( n \) variables, we study the number and properties of its solution set irreducible components, and for all equations in \( n \) variables, we count the average number \( \text{Irr}(n) \) of irreducible components of the solution sets.

Note that the cases \( n > l \) and \( n \leq l \) need a completely different techniques, and we can not directly use the results of [1] in the current paper. Moreover, almost all the results of [1] do not hold for the current case.

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1. Main definitions

Let \( L_l = \{a_1, a_2, \ldots, a_l\} \) be the linearly ordered semilattice of \( l \) elements and \( a_1 < a_2 < \ldots < a_l \). The multiplication in \( L_l \) is defined by \( a_i \cdot a_j = a_{\min(i,j)} \). Obviously, the linear order on \( L_l \) can be expressed by the multiplication as follows

\[
a_{i} \leq a_{j} \iff a_i a_j = a_i.
\]

A term \( t(X) \) in variables from \( X = \{x_1, x_2, \ldots, x_n\} \) is a commutative word in letters \( x_i \).

Let \( \text{Var}(t) \) be the set of all variables occurring in a term \( t(X) \). Following [2], an equation is an equality of some terms \( t(X) = s(X) \). Below we consider inequalities \( t(X) \leq s(X) \) as equations, since \( t(X) \leq s(X) \) is the short form of \( t(X)s(X) = t(X) \). Notice that we consider equations as ordered pairs of terms, i.e. the expressions \( t(X) = s(X) \) and \( s(X) = t(X) \) are different equations. Let \( \text{Eq}(n) \) denote the set of all equations in variables from \( X = \{x_1, x_2, \ldots, x_n\} \). We assume that each equation \( t(X) = s(X) \) in \( \text{Eq}(n) \) contains the occurrences of all variables \( x_1, x_2, \ldots, x_n \). An equation \( t(X) = s(X) \) in \( \text{Eq}(n) \) is said to be a \((k_1, k_2)\)-equation if \( |\text{Var}(t) \setminus \text{Var}(s)| = k_1 \) and \( |\text{Var}(s) \setminus \text{Var}(t)| = k_2 \). For example, \( x_1 x_2 = x_1 x_3 x_4 \) is a \((1,2)\)-equation. Let \( \text{Eq}(k_1, k_2, n) \) be the set of all \((k_1, k_2)\)-equations in \( \text{Eq}(n) \). Obviously,

\[
\text{Eq}(n) = \bigcup_{(k_1, k_2) \in K_n} \text{Eq}(k_1, k_2, n),
\]

where \( K_n = \{(k_1, k_2) : k_1 + k_2 \leq n\} \setminus \{(0, n), (n, 0)\} \).

Each equation \( t(X) = s(X) \) in \( \text{Eq}(k_1, k_2, n) \) is uniquely defined by \( k_1 \) variables in the left part and by \( k_2 \) other variables in the right part (the other \( n - k_1 - k_2 \) variables should occur in both parts of the equation). Thus,

\[
\#\text{Eq}(k_1, k_2, n) = \binom{n}{k_1} \binom{n - k_1}{k_2}.
\]

By (1), one can compute that \( \#\text{Eq}(n) = 3^n - 2 \).

Remark 1. Recall that we consider only equations \( t(X) = s(X) \) with \( n \leq l \), i.e. the number of variables occurring in \( t(X) = s(X) \) is not more than the order of the semilattice \( L_l \).

A point \( P \in L^n_l \) is a solution of an equation \( t(X) = s(X) \) if \( t(P) \) and \( s(P) \) define the same element in the semilattice \( L_l \). By the properties of linearly ordered semilattices, a point \( P = (p_1, p_2, \ldots, p_n) \) is a solution of \( t(X) = s(X) \) iff there exist variables \( x_i \) in \( \text{Var}(t) \) and \( x_j \) in \( \text{Var}(s) \) such that \( p_i = p_j \) and \( p_i \leq p_k \) for all \( k \), \( 1 \leq k \leq n \). The set of all solutions of an equation \( t(X) = s(X) \) is denoted by \( V(t(X) = s(X)) \).

An arbitrary set of equations is called a system. The set \( V(S) \) of all solutions of a system \( S = \{t_i(X) = s_i(X) : i \in I\} \) is defined as \( \bigcap_{i \in I} V(t_i(X) = s_i(X)) \). A subset \( Y \) of the set \( L^n_l \) is called algebraic over \( L_l \) if there exists a system \( S \) in \( n \) variables with \( V(S) = Y \). An algebraic set \( Y \) is irreducible if \( Y \) is not a proper finite union of other algebraic sets.

Proposition 1 [1, Proposition 2.2]. Any algebraic set \( Y \) over \( L_l \) is a finite union of irreducible sets, that is,

\[
Y = Y_1 \cup Y_2 \cup \ldots \cup Y_m,
\]

where \( Y_i \nmid Y_j \) for all \( i \) and \( j \) such that \( i \neq j \), and this decomposition is unique up to a permutation of components.
The subsets $Y_i$ from the union (2) are called the irreducible components of $Y$.

Let $Y$ be an algebraic set over $L_l$ defined by a system $S(X)$. One can define an equivalence relation $\sim_Y$ over the set of all terms in variables $X$ as follows

$$t(X) \sim_Y s(X) \iff t(P) = s(P) \text{ for any point } P \in Y.$$ 

The set of all $\sim_Y$-equivalence classes is called the coordinate semilattice of $Y$ and denoted by $\Gamma(Y)$ (see [2] for more details). The following statement describes the coordinate semilattices of irreducible algebraic sets.

**Proposition 2** [1, Proposition 2.3]. A set $Y$ is irreducible over $L_l$ iff $\Gamma(Y)$ is embedded into $L_l$.

There are different algebraic sets over $L_l$ with isomorphic coordinate semilattices. Such sets are called isomorphic. For example, the following sets

$$Y_1 = V(\{x_1 \leq x_2 \leq x_3\}), \ Y_2 = V(\{x_3 \leq x_2 \leq x_1\})$$

have the isomorphic coordinate semilattices

$$\Gamma(Y_1) = \langle x_1, x_2, x_3 \mid x_1 \leq x_2 \leq x_3 \rangle \cong L_3,$$

$$\Gamma(Y_2) = \langle x_1, x_2, x_3 \mid x_3 \leq x_2 \leq x_1 \rangle \cong L_3.$$ 

Thus, $Y_1$ and $Y_2$ are isomorphic.

**2. Example**

Let $n = 3$, $l = 3$. We have exactly $Eq(3) = 3^3 - 2 = 25$ equations in three variables over $L_3$. The Table on the page 52 contains the information about such equations over $L_3$. The second column contains systems which define irreducible components of the solution set for an equation in the first column. A cell of the table contains $\uparrow$ if an information in this cell is similar to the cell above.

Notice that $V(x_1 = x_2 \leq x_3)$ does not define an irreducible component for $Y = V(x_1 x_2 = x_1 x_3)$, since $V(x_1 = x_2 \leq x_3)$ is included into the solution set of another irreducible component $V(x_1 \leq x_2 \leq x_3)$. Similarly, $V(x_3 = x_1 \leq x_2)$ is not an irreducible component for $Y$, since it is contained in the irreducible component $V(x_1 \leq x_3 \leq x_2)$.

It turns out that the number of irreducible components does not depend on the semilattice order $l$. One can directly compute the average number of irreducible components of algebraic sets defined by equations in three variables:

$$\bar{\text{Irr}}(3) = \frac{6 + 2(2 + 2 + 2 + 2 + 3 + 3 + 4 + 4 + 4)}{25} = \frac{72}{25} = 2.88.$$ (3)
3. Decompositions of algebraic sets

Let $Y$ denote the solution set for an equation $t(X) = s(X)$ over the semilattice $L_l = \{a_1, a_2, \ldots, a_l\}$. The table on the page 52 shows that any irreducible component sorts the variables $X$ into some order. The following definition formalizes this property of irreducible components.

Let $\sigma$ be a permutation of the set $\{1, 2, \ldots, n\}$; $\sigma$ sorts the set $X$ as follows: $\{x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}\}$, i.e. $\sigma(i)$ is the $i$-th variable in the sorted set $X$. A permutation $\sigma$ is called a permutation of the first (second) kind if $x_{\sigma(1)} \in \text{Var}(t) \cap \text{Var}(s)$ (respectively, $x_{\sigma(2)} \in \text{Var}(t) \setminus \text{Var}(s)$, $x_{\sigma(1)} \in \text{Var}(s) \setminus \text{Var}(t)$). Let $\chi(\sigma) \in \{1, 2\}$ denote the kind of a permutation $\sigma$.

**Example 1.** Let us consider an algebraic set $Y_0 = V(x_1x_2 = x_1x_3)$. By the table, $Y_0$ is the union of the following irreducible components:

\[
Y_1 = V(x_1 \leq x_2 \leq x_3), \quad Y_2 = V(x_1 \leq x_3 \leq x_2), \quad Y_3 = V(x_2 = x_3 \leq x_1).
\]

The irreducible components $Y_1, Y_2, Y_3$ define the following permutations:

\[
\sigma_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}.
\]

Moreover, $\sigma_1$ and $\sigma_2$ are permutations of the first kind, whereas $\sigma_3$ is of the second kind.

A permutation $\sigma$ defines an algebraic set $Y_\sigma$ as follows:

\[
Y_\sigma = V \left( \bigcup_{i=1}^{n-1} \{x_{\sigma(i)} \leq x_{\sigma(i+1)}\} \right) \quad (4)
\]
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if \( \chi(\sigma) = 1 \), and

\[
Y_\sigma = V\left( \{x_{\sigma(1)} = x_{\sigma(2)}\} \cup \{x_{\sigma(i)} \leq x_{\sigma(i+1)}\} \right)
\]

(5)

if \( \chi(\sigma) = 2 \).

**Example 2.** Let \( \sigma_1, \sigma_2, \sigma_3 \) be permutations from Example 1. Obviously, the sets \( Y_{\sigma_1}, Y_{\sigma_2}, Y_{\sigma_3} \) defined by (4) and (5) coincide with the sets \( Y_1, Y_2, Y_3 \) respectively.

**Lemma 1.** Let \( \chi(\sigma) \in \{1, 2\} \), then the set \( Y_\sigma \) is irreducible and, moreover,

\[
\Gamma(Y_\sigma) \cong \begin{cases} 
L_n, & \text{if } \chi(\sigma) = 1, \\
L_{n-1}, & \text{if } \chi(\sigma) = 2.
\end{cases}
\]

(6)

**Proof.** By the definition of a coordinate semilattice, \( \Gamma(Y_\sigma) \) is generated by the elements of \( \{x_1, x_2, \ldots, x_n\} \) and has the following defined relations:

\[
x_{\sigma(1)} \leq x_{\sigma(2)} \leq \ldots x_{\sigma(n)} \quad \text{if } \chi(Y_\sigma) = 1
\]

and

\[
x_{\sigma(1)} = x_{\sigma(2)} \leq \ldots x_{\sigma(n)} \quad \text{if } \chi(Y_\sigma) = 2.
\]

Thus, \( \Gamma(Y_\sigma) \) is a linearly ordered semilattice, and (6) holds. By Proposition 2, the set \( Y_\sigma \) is irreducible. \( \blacksquare \)

The following lemma gives the irreducible decomposition of an algebraic set \( Y = \bigcup_{\sigma \in \{1, 2\}} Y_\sigma \).

**Lemma 2.** An algebraic set \( Y = V(t(X) = s(X)) \) is a union

\[
Y = \bigcup_{\chi(\sigma) \in \{1, 2\}} Y_\sigma.
\]

(7)

**Proof.** Suppose \( P = (p_1, p_2, \ldots, p_n) \in Y \). Let us sort \( p_i \) in the ascending order

\[
p_{\sigma(1)} \leq p_{\sigma(2)} \leq \ldots \leq p_{\sigma(n)},
\]

where \( \sigma \) is a permutation of the set \( \{1, 2, \ldots, n\} \). We have that \( \sigma \) induces the sorting of the variable set \( X \). Obviously, we may assume that \( x_{\sigma(1)} \in \text{Var}(t) \), otherwise the properties of \( L_t \) provide the existence of a variable \( x_{\sigma(i)} \in \text{Var}(t) \) such that \( p_{\sigma(i)} = p_{\sigma(1)} \), and we can swap the values \( \sigma(1) \) and \( \sigma(i) \).

For example, the point \( P = (a_2, a_1, a_1) \in V(x_1x_2 = x_1x_3) \) defines \( \sigma(1) = 2, \sigma(2) = 3, \sigma(3) = 1 \) (the permutation obtained equals \( \sigma_3 \) from Example 1, so the point \( (a_2, a_1, a_1) \) belongs to the set \( Y_3 \)).

Since \( \sigma \) is defined by the inequalities between the coordinates \( p_i \), it follows \( P \in Y_\sigma \).

Now we prove that \( Y_\sigma \subseteq Y \) for each \( \sigma \). Suppose \( P = (p_1, p_2, \ldots, p_n) \in Y_\sigma \). If \( \chi(Y_\sigma) = 1 \), then

\[
x_{\sigma(1)} \in \text{Var}(t) \cap \text{Var}(s) \Rightarrow t(P) = s(P) = p_{\sigma(1)} \Rightarrow P \in V(t(X) = s(X)).
\]

Otherwise \( (\chi(Y_\sigma) = 2) \), \( t(P) = p_{\sigma(1)} \), \( s(P) = p_{\sigma(2)} \), and (5) gives \( p_{\sigma(1)} = p_{\sigma(2)} \). Therefore \( P \in V(t(X) = s(X)) \). \( \blacksquare \)

**Lemma 3.** For distinct permutations \( \sigma \) and \( \sigma' \), we have \( Y_\sigma \nsubseteq Y_{\sigma'} \) in (7).
Proof. Let \( \sigma \) be a permutation of the first or second kind, and \( P_\sigma \) denote the following point:

\[
p_{\sigma(i)} = a_i \text{ if } \chi(\sigma) = 1,
\]

and

\[
p_{\sigma(i)} = \begin{cases} a_i, & 2 \leq i \leq n, \\ a_2, & i = 1, \end{cases} \text{ if } \chi(\sigma) = 2.
\]

For example, the permutations \( \sigma_1, \sigma_2, \sigma_3 \) from Example 1 define the points \( P_1 = (a_1, a_2, a_3), P_2 = (a_1, a_3, a_2), P_3 = (a_3, a_2, a_2) \), respectively.

Since \( P_\sigma \) preserves the order of variables, we have \( P_\sigma \in Y_\sigma \).

Now we can show that \( P_\sigma \notin Y_{\sigma'} \) for every \( \sigma' \neq \sigma \) (for example, each of the points \( P_1, P_2, P_3 \) above belongs to a unique irreducible component from Example 1:

\[
P_1 \in Y_1 \setminus (Y_2 \cup Y_3), \quad P_2 \in Y_2 \setminus (Y_1 \cup Y_3), \quad P_3 \in Y_3 \setminus (Y_1 \cup Y_2).
\]

There exist numbers \( i \) and \( j \) such that \( i < j, i = \sigma(\alpha), j = \sigma(\beta) \) with \( \alpha < \beta \) and \( i = \sigma'(\alpha'), j = \sigma'(\beta') \) with \( \alpha' > \beta' \). Hence, the inequality \( x_i \leq x_j \) holds in \( Y_\sigma \), and the inequality \( x_j \leq x_i \) holds in \( Y_{\sigma'} \). Let us consider the two possible cases:

1) If \( \chi(\sigma) = 1 \), then \( p_i < p_j \) in \( P_\sigma \), and we immediately obtain \( P_\sigma \notin Y_{\sigma'} \).

2) Suppose \( \chi(\sigma) = 2 \). Assume that \( p_i = p_j = a_2 \) (if \( p_i < p_j \), we immediately obtain \( P_\sigma \notin Y_{\sigma'} \)). Then \( \alpha = 1, \beta = 2 \) and \( i = \sigma(1), j = \sigma(2) \) (one can similarly consider the case \( i = \sigma(2), j = \sigma(1) \)). Hence, \( x_i \in \Var(t) \setminus \Var(s), x_j \in \Var(s) \setminus \Var(t) \). By the definition of a permutation of the second kind, \( \sigma'(1) = k \neq j \), and the inequality \( x_k \leq x_j \) holds in \( Y_{\sigma'} \). Let \( \sigma(\gamma) = k \). Since \( \alpha = 1, \beta = 2 \), we have \( \gamma > 2 \). Then \( p_k = a_\gamma \) and \( p_j < p_k \) for \( P_\sigma \). Thus, \( P \notin Y_{\sigma'} \).

The Lemma 3 is proved. \( \blacksquare \)

According to Lemmas 1–3, we obtain the following statement.

Theorem 1. The union (7) is the irreducible decomposition of the set \( Y = V(t(X) = s(X)) \). The number of irreducible components is equal to the number of permutations of the first and second kind.

4. Average number of irreducible components

One can directly compute that any \((k_1, k_2)\)-equation admits

\[(n - k_1 - k_2)(n - 1)!\]

permutations of the first kind and

\[k_1k_2(n - 2)!\]

permutations of the second kind.

By Theorem 1, for a \((k_1, k_2)\)-equation \( t(X) = s(X) \) the number of its irreducible components equals

\[\Irr(k_1, k_2, n) = (n - k_1 - k_2)(n - 1)! + k_1k_2(n - 2)!\]
The average number of irreducible components of algebraic sets defined by equations from $Eq(n)$ is

$$\overline{\text{Irr}}(n) = \frac{\sum_{(k_1, k_2) \in K_n} \#Eq(k_1, k_2, n) \text{Irr}(k_1, k_2, n)}{\#Eq(n)} = \frac{\sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-k_1} \#Eq(k_1, k_2, n) \text{Irr}(k_1, k_2, n) - \#Eq(0, n, n) \text{Irr}(0, n, n)}{\#Eq(n)}.$$

Since $\text{Irr}(0, n, n) = (n - 0 - n)(n - 1)! + 0n(n - 2)! = 0$, we obtain

$$\overline{\text{Irr}}(n) = \frac{\sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-k_1} \#Eq(k_1, k_2, n) \text{Irr}(k_1, k_2, n)}{\#Eq(n)}.$$

Below we compute $\overline{\text{Irr}}$ using the following notation:

1) $A^{(1)} B$: an expression $B$ is obtained from $A$ by the binomial identity

$$a\left(\frac{n}{a}\right) = n\left(\frac{n - 1}{a - 1}\right);$$

2) $A^{(2)} B$: an expression $B$ is obtained from $A$ by the following identity of binomial coefficients

$$\sum_{t=0}^{n} \binom{n}{t} t 2^t = 2n3^{n-1}. \tag{8}$$

Here is a proof of (8):

$$\sum_{t=0}^{n} \binom{n}{t} t 2^t = n \sum_{t=0}^{n} \binom{n-1}{t-1} 2^t = 2n \sum_{t=0}^{n} \binom{n-1}{t-1} 2^{t-1} = 2n \sum_{u=0}^{n-1} \binom{n-1}{u} 2 = 2n3^{n-1}.$$

Let us compute $\overline{\text{Irr}}(n)$. We have that

$$\sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-k_1} \#Eq(k_1, k_2, n) \text{Irr}(k_1, k_2, n) =$$

$$= \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-k_1} \binom{n}{k_1} \binom{n-k_1}{k_2} (n - k_1 - k_2)(n - 1)! + k_1 k_2 (n - 2)! =$$

$$= n! \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-k_1} \binom{n-k_1}{k_2} - (n - 1)! \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-k_1} \binom{n-k_1}{k_2} k_1 -$$

$$- (n - 1)! \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-k_1} \binom{n}{k_1} \binom{n-k_1}{k_2} k_1 + (n - 2)! \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-k_1} \binom{n-k_1}{k_2} k_1 k_2 = S_1 - S_2 - S_3 + S_4,$$

where

$$S_1 = n! \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-k_1} \binom{n}{k_1} \binom{n-k_1}{k_2} = n! \sum_{k_1=0}^{n-1} \binom{n}{k_1} 2^{n-k_1} = n! (3^n - 1),$$

$$S_2 = (n - 1)! \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-k_1} \binom{n}{k_1} \binom{n-k_1}{k_2} k_1 = (n - 1)! \sum_{k_1=0}^{n-1} \binom{n}{k_1} k_1 2^{n-k_1} \frac{(1)}{1}.$$
Notice that the final answer does not depend on \( n \) if \( l \leq n \). In particular, (9) gives

\[
\overline{\text{Irr}}(n) = \frac{4n!3^{n-2}}{3^n - 2} \sim 4\frac{4}{9}n!
\]

for \( n = 3 \), and (10) coincides with (3).

REFERENCES

