ВЕСТНИК ТОМСКОГО ГОСУДАРСТВЕННОГО УНИВЕРСИТЕТА
2019 Управление, вычислительная техника и информатика № 48

УДК 62-50
DOI: 10.17223/19988605/48/4

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THE SENSITIVITY FUNCTIONALS IN THE BOLTS'S PROBLEM FOR MULTIVARIATE DYNAMIC SYSTEMS DESCRIBED BY INTEGRO-DIFFERENTIAL EQUATIONS WITH DELAY TIME

The variational method of calculation of sensitivity functionals (connecting first variation of quality functionals with variations of variable parameters) and sensitivity coefficients (components of vector gradient from the quality functional to constant parameters) for multivariate non-linear dynamic systems described by continuous vectorial Volterra's integro-differential equations of the second-kind with delay time is developed. The base of calculation is the decision of corresponding integro-differential conjugate equations for Lagrange's multipliers in the opposite direction of time. The presence of a discontinuity in an initial value of coordinates and dependence the initial and final instants and magnitude of delay time from parameters are taken into account also.
Keywords: variational method; sensitivity functional; sensitivity coefficient; integro-differential equation; conjugate equation; delay time.

The sensitivity functional (SF) connect the first variation of quality functional with variations of variable and constant parameters and the sensitivity coefficients (SC) are components of vector gradient from quality functional according to constant parameters. Sensitivity coefficients are components of SF.

The problem of calculation of SF and SC of dynamic systems is principal in the syntheses and analysis of control laws, of identification and optimization algorisms, in the stability criterions [1–27]. The first-order sensitivity characteristics are mostly used. Later on we shall examine only SC and SF of the first-order. The most difficult are the distributed objects which are described by the dynamic (differential, integral, integro-differential, difference) equations with delays and by the dynamic equations in partial derivatives [2, 10, 11, 13, 17, 18, 20–23, 27].

Consider a vector output $x(t)$ of dynamic object model under continuous time $t \in [t_0, t^1]$, implicitly depending on parameters vectors $\tilde{\alpha}(t), \bar{\alpha}$ and functional I constructed on $x(t)$ under $t \in [t_0, t^1]$. The first variation δI of functional I and variations $\delta\tilde{\alpha}(t)$ are connected with each other with the help of a single-line functional – SF with respect to variable parameters $\tilde{\alpha}(t)$: $\delta_{\tilde{\alpha}(t)} I = \int_{t_0}^{t^1} V(t)\delta\tilde{\alpha}(t)dt$. SC with respect to constant parameters $\bar{\alpha}$ are called a gradient of I on $\bar{\alpha}$: $(dI / d\bar{\alpha})^T \equiv \nabla_{\bar{\alpha}} I$. SC are a coefficients of single-line relationship between the first variation of functional δI and the variations $\delta\bar{\alpha}$ of constant parameters $\bar{\alpha}$: $\delta_{\bar{\alpha}} I = (\nabla_{\bar{\alpha}} I)^T \delta\bar{\alpha} = (dI / d\bar{\alpha})\delta\bar{\alpha} \equiv \sum_{j=1}^{m} \frac{\partial I}{\partial \bar{\alpha}_j} \delta\bar{\alpha}_j$.

The direct method of SC calculation (by means of the differentiation of quality functional with respect to constant parameters) inevitably requires a solution of cumbersome sensitivity equations to sensitivity functions $W(t)$. $W(t)$ is the matrix of single-line relationship of the first variation of dynamic model output with parameter variations: $\delta x(t) = W(t)\delta\bar{\alpha}$. For instance, for functional $I = \int_{t_0}^{t^1} f_0(x(t), \bar{\alpha}, t)dt$ we have following

SC vector (row vector): $dI / d\overline{\alpha} = \int_{t_0}^{t^1} [(\partial f_0 / \partial x) W(t) + \partial f_0 / \partial \overline{\alpha}] dt$. For obtaining the matrix $W(t)$ it is necessary to decide a bulky system equations – sensitivity equations. The *j*-th column of matrix $W(t)$ is made of the sensitivity functions $dx(t) / d\overline{\alpha}_j$ with respect to component $\overline{\alpha}_j$ of vector $\overline{\alpha}$. They satisfy a vector equation (if y is a vector) resulting from dynamic model (for x) by derivation on a parameter $\overline{\alpha}_j$.

To variable parameters such a method is inapplicable because the sensitivity functions exist with respect to constant parameters.

For relatively simply classes of dynamic systems it is shown that in the SC calculation it is possible to get rid of deciding the bulky sensitivity equations due to the passage of deciding the conjugate equations – conjugate with respect to dynamic equations of object. Method of receipt of conjugate equations (it was offered in 1962) is cumbersome, because it is based on the analysis of sensitivity equations, and it does not get its developments.

Variational method [7], ascending to Lagrange's, Hamilton's, Euler's memoirs, makes possible to simplify the process of determination of conjugate equations and formulas of account of SC and SF. On the basis of this method it is an extension of quality functional by means of inclusion into it an object dynamic equations by means of Lagrange's multipliers and obtaining the first variation of extended functional on phase coordinates of object and on interesting parameters. Dynamic equations for Lagrange's multipliers are obtained due to set equal to a zero (in the first variation of extended functional) the functions before the first variations of phase coordinates. Given simplification first variation of extended functional brings at presence in the right part only parameter variations, i.e. it is got the SF. If all parameters are constant that the parameters variations are carried out from corresponding integrals and at the final result in obtained functional variation the coefficients before parameters variations are the required SC. Given method was used in [21–23, 25–27] for calculation of sensitivity coefficients and sensitivity functionals in the Bolts's problem for multivariate dynamic systems described by the differential, integral, integro-differential ordinary equations and equations with delay time under various initial conditions.

In [21] the sensitivity coefficients for many-dimensional dynamic systems described by the continuous and discontinuous differential equations with delay time are calculated.

In [22. 23] for dynamic systems described by ordinary continuous Volterra's of the second-kind integral equations [22] with delay time and integro-differential equations with delay time [23] the SC are received. Calculation possibility also SF for variable parameters is noted.

In [25] the SF and SC for multivariate dynamic systems described by generalized ordinary integral equations are calculated.

In [26] the same problem for multivariate dynamic systems described by generalized ordinary integro-differential equations is solved.

In [27] the SF and SC for multivariate dynamic systems described by generalized integral equations with delay time are calculated.

In this paper the variational method of account of SF is developed to more general continuous many-dimensional non-linear dynamic systems circumscribed by the vectorial non-linear continuous Volterra's integro-differential equations of the second genus with delay time, with variable $\tilde{\alpha}(t)$ and constant $\overline{\alpha}$ parameters and with reviewing of generalised quality functional (the Bolts problem) and registration of dependencies: 1) disturbing actions of a object model from initial instant; 2) of initial t_0 and final t^1 instants and of dead time from constant parameters $\overline{\alpha}$.

1. Problem statement

We suppose that the dynamic object is described by system of non-linear continuous Volterra's of the second-kind integro-differential equations (I-DE) with delay time τ

$$\dot{x}(t) = f(x(t), x(t-\tau), y(t), y(t-\tau), \tilde{\alpha}(t), \overline{\alpha}, t),\ \ t_0 < t \le t^1,\ \ 0 < \tau, \tag{1}$$

$$y(t) = r(\tilde{\alpha}(t), \overline{\alpha}, t_0, t) + \int_{t_0}^{t} K(t, x(s), x(s-\tau), y(s), y(s-\tau), \tilde{\alpha}(s), \overline{\alpha}, s)\, ds,\ \ t_0 \le t \le t^1,$$

$$t_0 = t_0(\overline{\alpha}),\ t^1 = t^1(\overline{\alpha}),\ \tau = \tau(\overline{\alpha}),\ \ x(t) = \psi_x(\tilde{\alpha}(t), \overline{\alpha}, t),\ t \in [t_0 - \tau, t_0),\ x(t_0) = x_0(\overline{\alpha}, t_0),$$

$$y(t) = \psi_y(\tilde{\alpha}(t), \overline{\alpha}, t),\ t \in [t_0 - \tau, t_0).$$

Here: the magnitudes of initial t_0 and final t^1 instants and also dead time τ and initial values $x(t_0)$ are known functions from constant parameters $\overline{\alpha}$: $t_0 = t_0(\overline{\alpha}),\ t^1 = t^1(\overline{\alpha}),\ \tau = \tau(\overline{\alpha})$, $x(t_0) = x_0(\overline{\alpha}, t_0)$; x , y – vector-columns of phase coordinates; $\tilde{\alpha}(t), \overline{\alpha}$ – vector-columns of interesting variable and constant parameters; $f(\cdot)$, $\psi_x(\cdot)$, $r(\cdot)$, $K(\cdot)$, $\psi_y(\cdot)$, $t_0(\overline{\alpha})$, $t^1(\overline{\alpha})$, $\tau(\overline{\alpha})$, $x_0(\cdot)$ – known continuously differentiated limited vector-functions. The phase coordinates x , y in an index point t_0 makes a discontinuity if:

$$x^+(t_0) \equiv x(t_0 + 0) \equiv x_0(\alpha, t_0) \ne x(t_0 - 0) \equiv x^-(t_0) \equiv \psi_x(\tilde{\alpha}(t_0), \overline{\alpha}, t_0),$$

$$y^+(t_0) \equiv y(t_0 + 0) \equiv r(\tilde{\alpha}(t_0), \overline{\alpha}, t_0, t_0) \ne y(t_0 - 0) \equiv y^-(t_0) \equiv \psi_y(\tilde{\alpha}(t_0), \overline{\alpha}, t_0).$$

But at the expense of an integration in a model (1) phase coordinates become continuous in instants $t_0 + n\tau,\ n = 1, 2, \cdots$. Here is designated: $x^+(t_0) \equiv x(t_0 + 0)$ – value of a phase coordinate to the right of a point t_0 and accordingly $x^-(t_0) \equiv x(t_0 - 0)$ – to the left of a point t_0 .

The model of a measuring device is given as

$$\eta(t) = \eta(x(t), y(t), \tilde{\alpha}(t), \overline{\alpha}, t),\ t \in [t_0, t^1], \tag{2}$$

where $\eta(\cdot)$ is also continuous, differentiated, limited (together with the first derivatives) vector-function. The required parameters $\tilde{\alpha}(t), \overline{\alpha}$ are inserted also in (2). Dimensionality of vectors x , y and η in general event can be different.

Let's notice that models (1), (2) are more the general in comparison with used in [17. P. 88].

On the basis of output coordinates of a measuring device η the quality functional of a dynamic system is constructed:

$$I(\alpha) = \int_{t_0}^{t^1} f_0(\eta(t), \tilde{\alpha}(t), \overline{\alpha}, t)\, dt + I_1(\eta(t^1), \overline{\alpha}, t^1)\ . \tag{3}$$

Conditions for functions $f_0(\cdot), I_1(\cdot)$ are the same as for $f(\cdot)$, $r(\cdot)$, $K(\cdot)$, $t_0(\cdot)$, $t^1(\cdot)$, $\tau(\cdot)$, $\psi_x(\cdot)$, $x_0(\cdot)$, $\psi_y(\cdot)$, etc.

With use of a functional (3) in the optimization problem (in the theory of optimal control) known as the Bolts's problem. From it as the individual variants follows: Lagrange's problem (when there is only integrated component) and Mayer's problem (when there is only second component – function from phase coordinates at a finishing point).

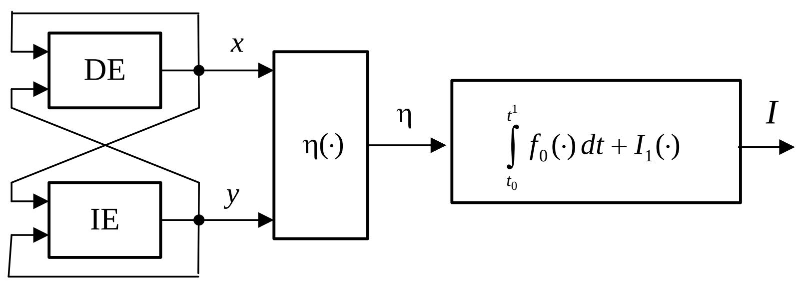

Fig. 1. The scheme of interaction between variables in (1)–(3)

With the purpose of simplification of appropriate deductions with preservation of a generality in all transformations (1)–(3) there are a two vectors of parameters $\tilde{\alpha}(t), \overline{\alpha}$. If in the equations (1)–(3) the parameters are different then it is possible formally to unite them in one vector $\tilde{\alpha}(t)$ or $\overline{\alpha}$, to use obtained outcomes and then, taking into account a structure of a vectors $\tilde{\alpha}(t), \overline{\alpha}$, to make appropriate simplifications.

The scheme of interaction between variables of object model, measuring device and quality functional is shown in fig. 1.

By obtaining of results the obvious designations:

$$f(t) \equiv f(x(t), x(t-\tau), y(t), y(t-\tau), \tilde{\alpha}(t), \overline{\alpha}, t), \ r(t) \equiv r(\tilde{\alpha}(t), \overline{\alpha}, t_0, t),$$
$$K(t,s) \equiv K(t, x(s), x(s-\tau), y(s), y(s-\tau), \tilde{\alpha}(t), \overline{\alpha}, s), \ \eta(t) \equiv \eta(x(t), y(t), \tilde{\alpha}(t), \overline{\alpha}, t),$$
$$f_0(t) \equiv f_0(\eta(t), \tilde{\alpha}(t), \overline{\alpha}, t), \ I_1(t^1) \equiv I_1(\eta(t^1), \overline{\alpha}, t^1)$$

are used.

The convenience of a integro-differential model consists in its structural universality. At simplification of a model it is enough in final results to convert in a zero appropriate addends. This reception we shall apply in a next paper.

Let's pass from the I-DE to integral equations (IE).

Is shown also that the variation method without basic modifications allows to receive SF $\delta I(\alpha) = \int_{t_0-\tau}^{t^1} V(t)\delta\tilde{\alpha}(t)dt + (dI(\alpha)/d\tilde{\alpha}(t^1))\delta\tilde{\alpha}(t^1) + (dI(\alpha)/d\overline{\alpha})\,\delta\overline{\alpha}$ in relation to variable and constant parameters.

2. Passage to IE

In I-DE (1) the differential equations we write in the integral form

$$x(t) = x_0(\overline{\alpha}, t_0) + \int_{t_0}^{t} f(x(s), x(s-\tau), y(s), y(s-\tau), \tilde{\alpha}(s), \overline{\alpha}, s)ds, \ \ t_0 \le t \le t^1 \tag{4}$$

and use notations

$$\tilde{y}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \ \tilde{r}(\tilde{\alpha}(t), \overline{\alpha}, t_0, t) = \begin{pmatrix} x_0(\overline{\alpha}, t_0) \\ r(\tilde{\alpha}(t), \overline{\alpha}, t_0, t) \end{pmatrix} \equiv \begin{pmatrix} x_0(\overline{\alpha}, t_0) \\ r(t) \end{pmatrix} \equiv \tilde{r}(t), \tag{5}$$

$$\tilde{K}(t, \tilde{y}(s), \tilde{y}(s-\tau), \tilde{\alpha}(s), \overline{\alpha}, s) = \begin{pmatrix} f(x(s), x(s-\tau), y(s), y(s-\tau), \tilde{\alpha}(s), \overline{\alpha}, s) \\ K(t, x(s), x(s-\tau), y(s), y(s-\tau), \tilde{\alpha}(s), \overline{\alpha}, s) \end{pmatrix} \equiv \begin{pmatrix} f(s) \\ K(t,s) \end{pmatrix} \equiv \tilde{K}(t,s),$$

$$\psi(\tilde{\alpha}(t), \overline{\alpha}, t) = \begin{pmatrix} \psi_x(\tilde{\alpha}(t), \overline{\alpha}, t) \\ \psi_y(\tilde{\alpha}(t), \overline{\alpha}, t) \end{pmatrix}.$$

Then we obtain IE

$$\tilde{y}(t) = \tilde{r}(\tilde{\alpha}(t), \overline{\alpha}, t_0, t) + \int_{t_0}^{t} \tilde{K}(t, \tilde{y}(s), \tilde{y}(s-\tau), \tilde{\alpha}(s), \overline{\alpha}, s)ds, \ \ t_0 \le t \le t^1, \tag{6}$$
$$\tilde{y}(t) = \psi(\tilde{\alpha}(t), \overline{\alpha}, t), \ t \in [t_0 - \tau, t_0).$$

In further also a notation

$$\eta(t) \equiv \eta(\tilde{y}(t), \tilde{\alpha}(t), \overline{\alpha}, t) \tag{7}$$

is used for a model of a measuring device.

3. SF with use of a models (6), (7) and a quality functional (3)

We use results of paper [27] for models (6), (7), (3) with variables and constant parameters $\tilde{\alpha}(t), \overline{\alpha}$ and write out in the beginning the conjugate equations for basic Lagrange's multipliers $\gamma(t), \overline{\gamma}(t)$:

$$\gamma^T(t)=\Phi(t^1)\frac{\partial\tilde K(t^1,t)}{\partial\tilde y(t)}+\frac{\partial f_0(t)}{\partial\eta(t)}\frac{\partial\eta(t)}{\partial\tilde y(t)}+\int_t^{t^1}\gamma^T(s)\frac{\partial\tilde K(s,t)}{\partial\tilde y(t)}ds+ \tag{8}$$

$$+1(t^1-\tau-t)[\Phi(t^1)\frac{\partial\tilde K(t^1,t+\tau)}{\partial\tilde y(t)}+\int_{t+\tau}^{t^1}\gamma^T(s)\frac{\partial\tilde K(s,t+\tau)}{\partial\tilde y(t)}ds],\ t_0\le t\le t^1,$$

$$\bar\gamma^T(t)=1(t^1-\tau-t)[\Phi(t^1)\frac{\partial\tilde K(t^1,t+\tau)}{\partial\tilde y(t)}+\int_{t+\tau}^{t^1}\gamma^T(s)\frac{\partial\tilde K(s,t+\tau)}{\partial\tilde y(t)}ds],\ t_0-\tau\le t\le t_0. \tag{9}$$

Here: $\gamma(t),\bar\gamma(t)$ are column vectors; $\Phi(t^1)\equiv\frac{\partial I_1(t^1)}{\partial\eta(t^1)}\frac{\partial\eta(t^1)}{\partial\tilde y(t^1)}$; $1(z)$ is single function: it is equal to zero under negative values of argument and is equal to unit under positive values z. These conjugate equations are decided in the opposite direction of time (from t^1).

From the conjugate equations (8), (9) it is possible to remove single function and to add them a customary aspect.

If $t_0\le t^1-\tau\le t^1$, i.e. length of an interval $[t_0,\ t^1]$ transcends magnitude of a delay time τ, then:

$$\gamma^T(t)=\Phi(t^1)\frac{\partial\tilde K(t^1,t)}{\partial\tilde y(t)}+\frac{\partial f_0(t)}{\partial\eta(t)}\frac{\partial\eta(t)}{\partial\tilde y(t)}+\int_t^{t^1}\gamma^T(s)\frac{\partial\tilde K(s,t)}{\partial\tilde y(t)}ds\quad\text{for } t^1-\tau\le t\le t^1,$$

$$\gamma^T(t)=\Phi(t^1)\frac{\partial\tilde K(t^1,t)}{\partial\tilde y(t)}+\frac{\partial f_0(t)}{\partial\eta(t)}\frac{\partial\eta(t)}{\partial\tilde y(t)}+\int_t^{t^1}\gamma^T(s)\frac{\partial\tilde K(s,t)}{\partial\tilde y(t)}ds+$$

$$+\Phi(t^1)\frac{\partial\tilde K(t^1,t+\tau)}{\partial\tilde y(t)}+\int_{t+\tau}^{t^1}\gamma^T(s)\frac{\partial\tilde K(s,t+\tau)}{\partial\tilde y(t)}ds\quad\text{for } t_0\le t\le t^1-\tau,$$

$$\bar\gamma^T(t)=\Phi(t^1)\frac{\partial\tilde K(t^1,t+\tau)}{\partial\tilde y(t)}+\int_{t+\tau}^{t^1}\gamma^T(s)\frac{\partial\tilde K(s,t+\tau)}{\partial\tilde y(t)}ds\quad\text{for } t_0-\tau\le t\le t_0.$$

If $t^1-\tau\le t_0$, i.e. the magnitude of delay τ transcends length of an interval $[t_0,t^1]$, (in this case magnitude $t_0+\tau$ exceeds t^1 – goes out for an interval of object work):

$$\gamma^T(t)=\Phi(t^1)\frac{\partial\tilde K(t^1,t)}{\partial\tilde y(t)}+\frac{\partial f_0(t)}{\partial\eta(t)}\frac{\partial\eta(t)}{\partial\tilde y(t)}+\int_t^{t^1}\gamma^T(s)\frac{\partial\tilde K(s,t)}{\partial\tilde y(t)}ds\ \text{ for } t_0\le t\le t^1,\ \ \bar\gamma^T(t)=0\ \text{ for } t^1-\tau\le t\le t_0,$$

$$\bar\gamma^T(t)=\Phi(t^1)\frac{\partial\tilde K(t^1,t+\tau)}{\partial\tilde y(t)}+\int_{t+\tau}^{t^1}\gamma^T(s)\frac{\partial\tilde K(s,t+\tau)}{\partial\tilde y(t)}ds\ \text{ for } t_0-\tau\le t\le t^1-\tau.$$

SF for a models (6), (7), (3) has the form:

$$\delta I=\delta_{\tilde\alpha(t)}I+\delta_{\tilde\alpha(t^1)}I+\delta_{\bar\alpha}I,\ \delta_{\tilde\alpha(t)}I=\int_{t_0}^{t^1}[\frac{\partial f_0(t)}{\partial\eta(t)}\frac{\partial\eta(t)}{\partial\tilde\alpha(t)}+\frac{\partial f_0(t)}{\partial\tilde\alpha(t)}+\gamma^T(t)\frac{\partial\tilde r(t)}{\partial\tilde\alpha(t)}+ \tag{10}$$

$$+\Phi(t^1)\frac{\partial\tilde K(t^1,t)}{\partial\tilde\alpha(t)}+\int_t^{t^1}\gamma^T(s)\frac{\partial\tilde K(s,t)}{\partial\tilde\alpha(t)}ds]\delta\tilde\alpha(t)\,dt+\int_{t_0-\tau}^{t_0}\bar\gamma^T(t)\frac{\partial\psi(t)}{\partial\tilde\alpha(t)}\delta\tilde\alpha(t)\,dt;$$

$$\delta_{\tilde\alpha(t^1)}I=[\frac{\partial I_1(t^1)}{\partial\eta(t^1)}\frac{\partial\eta(t^1)}{\partial\tilde\alpha(t^1)}+\Phi(t^1)\frac{\partial\tilde r(t^1)}{\partial\tilde\alpha(t^1)}]\delta\tilde\alpha(t^1);$$

$$\delta_{\bar\alpha}I=\left\{\frac{\partial I_1(t^1)}{\partial\eta(t^1)}\frac{\partial\eta(t^1)}{\partial\bar\alpha}+\frac{\partial I_1(t^1)}{\partial\bar\alpha}+\Phi(t^1)\left[\frac{\partial\tilde r(t^1)}{\partial\bar\alpha}+\int_{t_0}^{t^1}\frac{\partial\tilde K(t^1,s)}{\partial\bar\alpha}ds\right]+\right.$$

$$+\int_{t_0}^{t^1}[\frac{\partial f_0(t)}{\partial\eta(t)}\frac{\partial\eta(t)}{\partial\bar\alpha}+\frac{\partial f_0(t)}{\partial\bar\alpha}+\gamma^T(t)\frac{\partial\tilde r(t)}{\partial\bar\alpha}+\int_t^{t^1}\gamma^T(s)\frac{\partial\tilde K(s,t)}{\partial\bar\alpha}ds]dt+\int_{t_0-\tau}^{t_0}\bar\gamma^T(t)\frac{\partial\psi(t)}{\partial\bar\alpha}dt+$$

$$+\left[\Phi(t^1)[\frac{\partial\tilde r(t^1)}{\partial t_0}-\tilde K(t^1,t_0)+1(t^1-t_0-\tau)(\tilde K(t^1,t_0+\tau-0)-\tilde K(t^1,t_0+\tau+0))]-f_0(t_0)+\right.$$

$$\left.+\int_{t_0}^{t^1}\gamma^T(t)(\frac{\partial\tilde r(t)}{\partial t_0}-\tilde K(t,t_0))dt+1(t^1-t_0-\tau)\int_{t_0+\tau}^{t^1}\gamma^T(t)[\tilde K(t,t_0+\tau-0)-\tilde K(t,t_0+\tau+0)]dt\right]\frac{dt_0}{d\bar\alpha}+$$

$$+\left[\Phi(t^1)[\frac{\partial\tilde r(t^1)}{\partial t^1}+\tilde K(t^1,t^1)+\int_{t_0}^{t^1}\frac{\partial\tilde K(t^1,s)}{\partial t^1}ds]+\frac{\partial I_1(t^1)}{\partial\eta(t^1)}\frac{\partial\eta(t^1)}{\partial t^1}+\frac{\partial I_1(t^1)}{\partial t^1}+f_0(t^1)\right]\frac{dt^1}{d\bar\alpha}+$$

$$+\left[\Phi(t^1)[1(t^1-t_0-\tau)(\tilde K(t^1,t_0+\tau-0)-\tilde K(t^1,t_0+\tau+0))-\int_{t_0}^{t^1}\frac{\partial\tilde K(t^1,s)}{\partial\tilde y(s-\tau)}\frac{d\tilde y(s-\tau)}{d(s-\tau)}ds]+\right.$$

$$+1(t^1-t_0-\tau)\int_{t_0+\tau}^{t^1}\gamma^T(t)[\tilde K(t,t_0+\tau-0)-\tilde K(t,t_0+\tau+0)]dt-$$

$$\left.\left.-\int_{t_0}^{t^1}\gamma^T(t)\int_{t_0}^{t}\frac{\partial\tilde K(t,s)}{\partial\tilde y(s-\tau)}\frac{d\tilde y(s-\tau)}{d(s-\tau)}ds\,dt\right]\frac{d\tau}{d\bar\alpha}\right\}d\bar\alpha\equiv\frac{dI}{d\bar\alpha}d\bar\alpha\,.$$

It is necessary in (8)–(10) to fulfil matrix transformations (differentiation, multiplication) with the registration earlier entered notations (5), and also

$$\gamma(t)=\begin{pmatrix}\gamma_x(t)\\ \gamma_y(t)\end{pmatrix},\ \gamma^T(t)=\left(\gamma_x^T(t);\quad\gamma_y^T(t)\right),\ \bar\gamma(t)=\begin{pmatrix}\bar\gamma_x(t)\\ \bar\gamma_y(t)\end{pmatrix},\ \bar\gamma^T(t)=\left(\bar\gamma_x^T(t);\quad\bar\gamma_y^T(t)\right),\tag{11}$$

i.e. $\Phi(t^1)\equiv\frac{\partial I_1(t^1)}{\partial\eta(t^1)}\frac{\partial\eta(t^1)}{\partial\tilde y(t^1)}=\left(\frac{\partial I_1(t^1)}{\partial\eta(t^1)}\frac{\partial\eta(t^1)}{\partial x(t^1)};\quad\frac{\partial I_1(t^1)}{\partial\eta(t^1)}\frac{\partial\eta(t^1)}{\partial y(t^1)}\right)\equiv\left(\Phi_x(t^1);\quad\Phi_y(t^1)\right)$,

$$\frac{\partial\tilde K(t^1,t)}{\partial\tilde y(t)}=\begin{pmatrix}\frac{\partial f(t)}{\partial x(t)}; & \frac{\partial f(t)}{\partial y(t)}\\ \frac{\partial K(t^1,t)}{\partial x(t)}; & \frac{\partial K(t^1,t)}{\partial y(t)}\end{pmatrix},\ \frac{\partial\tilde r(t^1)}{\partial\bar\alpha}=\begin{pmatrix}\frac{\partial x_0(\bar\alpha,t_0)}{\partial\bar\alpha}\\ \frac{\partial r(t^1)}{\partial\bar\alpha}\end{pmatrix},$$

$$\Phi(t^1)\frac{\partial\tilde K(t^1,t)}{\partial\tilde y(t)}=\left(\Phi_x(t^1)\frac{\partial f(t)}{\partial x(t)}+\Phi_y(t^1)\frac{\partial K(t^1,t)}{\partial x(t)};\quad\Phi_x(t^1)\frac{\partial f(t)}{\partial y(t)}+\Phi_y(t^1)\frac{\partial K(t^1,t)}{\partial y(t)}\right),$$

$$\frac{\partial f_0(t)}{\partial\eta(t)}\frac{\partial\eta(t)}{\partial\tilde y(t)}=\left(\frac{\partial f_0(t)}{\partial\eta(t)}\frac{\partial\eta(t)}{\partial x(t)};\quad\frac{\partial f_0(t)}{\partial\eta(t)}\frac{\partial\eta(t)}{\partial y(t)}\right),$$

$$\gamma^T(s)\frac{\partial\tilde K(s,t)}{\partial\tilde y(t)}=\left(\gamma_x^T(s)\frac{\partial f(t)}{\partial x(t)}+\gamma_y^T(s)\frac{\partial K(s,t)}{\partial x(t)};\ \gamma_x^T(s)\frac{\partial f(t)}{\partial y(t)}+\gamma_y^T(s)\frac{\partial K(s,t)}{\partial y(t)}\right),$$

$$\Phi(t^1)\frac{\partial\tilde r(t^1)}{\partial\bar\alpha}=\Phi_x(t^1)\frac{\partial x_0(\bar\alpha,t_0)}{\partial\bar\alpha}+\Phi_y(t^1)\frac{\partial r(t^1)}{\partial\bar\alpha},\ \gamma^T(t)\frac{\partial\tilde r(t)}{\partial\bar\alpha}=\gamma_x^T(t)\frac{\partial x_0(\bar\alpha,t_0)}{\partial\bar\alpha}+\gamma_y^T(t)\frac{\partial r(t)}{\partial\bar\alpha},$$

$$\Phi(t^1)\frac{\partial\tilde K(t^1,t)}{\partial\bar\alpha}=\Phi_x(t^1)\frac{\partial f(t)}{\partial\bar\alpha}+\Phi_y(t^1)\frac{\partial K(t^1,t)}{\partial\bar\alpha},\ \int_t^{t^1}\gamma^T(s)\frac{\partial\tilde K(s,t)}{\partial\bar\alpha}ds=\int_t^{t^1}[\gamma_x^T(s)\frac{\partial f(t)}{\partial\bar\alpha}+\gamma_y^T(s)\frac{\partial K(s,t)}{\partial\bar\alpha}]ds\text{, etc.}$$

In the total we obtain the conjugate equations for Lagrange's multipliers

$$\gamma_x^T(t)=\Phi_x(t^1)\frac{\partial f(t)}{\partial x(t)}+\Phi_y(t^1)\frac{\partial K(t^1,t)}{\partial x(t)}+\frac{\partial f_0(t)}{\partial\eta(t)}\frac{\partial\eta(t)}{\partial x(t)}+\int_t^{t^1}[\gamma_x^T(s)\frac{\partial f(t)}{\partial x(t)}+\gamma_y^T(s)\frac{\partial K(s,t)}{\partial x(t)}]ds+\tag{12}$$

$$+1(t^1-\tau-t)[\Phi_x(t^1)\frac{\partial f(t+\tau)}{\partial x(t)}+\Phi_y(t^1)\frac{\partial K(t^1,t+\tau)}{\partial x(t)}+\int_{t+\tau}^{t^1}[\gamma_x^T(s)\frac{\partial f(t+\tau)}{\partial x(t)}+\gamma_y^T(s)\frac{\partial K(s,t+\tau)}{\partial x(t)}]ds\ ],$$

$$\gamma_y^T(t)=\Phi_x(t^1)\frac{\partial f(t)}{\partial y(t)}+\Phi_y(t^1)\frac{\partial K(t^1,t)}{\partial y(t)}+\frac{\partial f_0(t)}{\partial \eta(t)}\frac{\partial \eta(t)}{\partial y(t)}+\int_t^{t^1}[\gamma_x^T(s)\frac{\partial f(t)}{\partial y(t)}+\gamma_y^T(s)\frac{\partial K(s,t)}{\partial y(t)}]ds+$$

$$+1(t^1-\tau-t)[\Phi_x(t^1)\frac{\partial f(t+\tau)}{\partial y(t)}+\Phi_y(t^1)\frac{\partial K(t^1,t+\tau)}{\partial y(t)}+\int_{t+\tau}^{t^1}[\gamma_x^T(s)\frac{\partial f(t+\tau)}{\partial y}+\gamma_y^T(s)\frac{\partial K(s,t+\tau)}{\partial y}]ds\ ],\ t_0\le t\le t^1;$$

$$\bar\gamma_x^T(t)=1(t^1-\tau-t)[\Phi_x(t^1)\frac{\partial f(t+\tau)}{\partial x(t)}+\Phi_y(t^1)\frac{\partial K(t^1,t+\tau)}{\partial x(t)}+\int_{t+\tau}^{t^1}[\gamma_x^T(s)\frac{\partial f(t+\tau)}{\partial x(t)}+\gamma_y^T(s)\frac{\partial K(s,t+\tau)}{\partial x(t)}]ds\ ],$$

$$\bar\gamma_y^T(t)=1(t^1-\tau-t)[\Phi_x(t^1)\frac{\partial f(t+\tau)}{\partial y(t)}+\Phi_y(t^1)\frac{\partial K(t^1,t+\tau)}{\partial y(t)}+$$

$$+\int_{t+\tau}^{t^1}[\gamma_x^T(s)\frac{\partial f(t+\tau)}{\partial y(t)}+\gamma_y^T(s)\frac{\partial K(s,t+\tau)}{\partial y(t)}]ds\ ],\ t_0-\tau\le t\le t_0.$$

The first variation of a functionality I in relation to variable $\tilde\alpha(t)$ and constant $\tilde\alpha(t^1)$, $\bar\alpha$ parameters, has three components:

$$\delta I=\delta_{\tilde\alpha(t)}I+\delta_{\tilde\alpha(t^1)}I+\delta_{\bar\alpha}I\ ; \tag{13}$$

$$\delta_{\tilde\alpha(t)}I=\int_{t_0}^{t^1}\left[\frac{\partial f_0(t)}{\partial\eta(t)}\frac{\partial\eta(t)}{\partial\tilde\alpha(t)}+\frac{\partial f_0(t)}{\partial\tilde\alpha(t)}+\gamma_y^T(t)\frac{\partial r(t)}{\partial\tilde\alpha(t)}+\Phi_x(t^1)\frac{\partial f(t)}{\partial\tilde\alpha(t)}+\Phi_y(t^1)\frac{\partial K(t^1,t)}{\partial\tilde\alpha(t)}+\right.$$

$$\left.+\int_t^{t^1}[\gamma_x^T(s)\frac{\partial f(t)}{\partial\tilde\alpha(t)}+\gamma_y^T(s)\frac{\partial K(s,t)}{\partial\tilde\alpha(t)}]ds\right]\delta\tilde\alpha(t)\,dt+\int_{t_0-\tau}^{t_0}[\bar\gamma_x^T(t)\frac{\partial\psi_x(t)}{\partial\tilde\alpha(t)}+\bar\gamma_y^T(t)\frac{\partial\psi_y(t)}{\partial\tilde\alpha(t)}]\delta\tilde\alpha(t)\,dt\ ;$$

$$\delta_{\tilde\alpha(t^1)}I=[\frac{\partial I_1(t^1)}{\partial\eta(t^1)}\frac{\partial\eta(t^1)}{\partial\tilde\alpha(t^1)}+\Phi_y(t^1)\frac{\partial r(t^1)}{\partial\tilde\alpha(t^1)}]\delta\tilde\alpha(t^1)\ ;\ \delta_{\bar\alpha}I=\left\{\frac{\partial I_1(t^1)}{\partial\eta(t^1)}\frac{\partial\eta(t^1)}{\partial\bar\alpha}+\frac{\partial I_1(t^1)}{\partial\bar\alpha}+\right.$$

$$+\Phi_x(t^1)\left[\frac{\partial x_0(\bar\alpha,t_0)}{\partial\bar\alpha}+\int_{t_0}^{t^1}\frac{\partial f(s)}{\partial\bar\alpha}ds\right]+\Phi_y(t^1)\left[\frac{\partial r(t^1)}{\partial\bar\alpha}+\int_{t_0}^{t^1}\frac{\partial K(t^1,s)}{\partial\bar\alpha}ds\right]+$$

$$+\int_{t_0}^{t^1}\left[\frac{\partial f_0(t)}{\partial\eta(t)}\frac{\partial\eta(t)}{\partial\bar\alpha}+\frac{\partial f_0(t)}{\partial\bar\alpha}+\gamma_x^T(t)\frac{\partial x_0(\bar\alpha,t_0)}{\partial\bar\alpha}+\gamma_y^T(t)\frac{\partial r(t)}{\partial\bar\alpha}+\right.$$

$$\left.+\int_t^{t^1}[\gamma_x^T(s)\frac{\partial f(t)}{\partial\bar\alpha}+\gamma_y^T(s)\frac{\partial K(s,t)}{\partial\bar\alpha}]ds\right]dt+\int_{t_0-\tau}^{t_0}[\bar\gamma_x^T(t)\frac{\partial\psi_x(t)}{\partial\bar\alpha}+\bar\gamma_y^T(t)\frac{\partial\psi_y(t)}{\partial\bar\alpha}]dt+$$

$$+\left[\Phi_x(t^1)[\frac{\partial x_0(\bar\alpha,t_0)}{\partial t_0}-f(t_0)+1(t^1-t_0-\tau)(f(t_0+\tau-0)-f(t_0+\tau+0))]+\right.$$

$$+\Phi_y(t^1)[\frac{\partial r(t^1)}{\partial t_0}-K(t^1,t_0)+1(t^1-t_0-\tau)(K(t^1,t_0+\tau-0)-K(t^1,t_0+\tau+0))]-f_0(t_0)+$$

$$+\int_{t_0}^{t^1}\gamma_x^T(t)dt(\frac{\partial x_0(\bar\alpha,t_0)}{\partial t_0}-f(t_0))+\int_{t_0}^{t^1}\gamma_y^T(t)(\frac{\partial r(t)}{\partial t_0}-K(t,t_0))dt+$$

$$+1(t^1-t_0-\tau)\int_{t_0+\tau}^{t^1}\gamma_x^T(t)dt[f(t_0+\tau-0)-f(t_0+\tau+0)]+$$

$$+\ 1(t^1-t_0-\tau)\int_{t_0+\tau}^{t^1}\gamma_y^T(t)[K(t,t_0+\tau-0)-K(t,t_0+\tau+0)]dt\Bigg]\frac{dt_0}{d\alpha}+$$

$$+\left[\Phi_x(t^1)f(t^1)+\Phi_y(t^1)[\frac{\partial r(t^1)}{\partial t^1}+K(t^1,t^1)+\int_{t_0}^{t^1}\frac{\partial K(t^1,s)}{\partial t^1}ds]+\frac{\partial I_1(t^1)}{\partial\eta(t^1)}\frac{\partial\eta(t^1)}{\partial t^1}+\frac{\partial I_1(t^1)}{\partial t^1}+f_0(t^1)\right]\frac{dt^1}{d\alpha}+$$

$$+\Big[\Phi_x(t^1)[1(t^1-t_0-\tau)(f(t_0+\tau-0)-f(t_0+\tau+0))-\int_{t_0}^{t^1}(\frac{\partial f(s)}{\partial x(s-\tau)}\frac{dx(s-\tau)}{d(s-\tau)}+\frac{\partial f(s)}{\partial y(s-\tau)}\frac{dy(s-\tau)}{d(s-\tau)})ds]+$$

$$+\Phi_y(t^1)[1(t^1-t_0-\tau)(K(t^1,t_0+\tau-0)-K(t^1,t_0+\tau+0))-$$

$$-\int_{t_0}^{t^1}(\frac{\partial K(t^1,s)}{\partial x(s-\tau)}\frac{dx(s-\tau)}{d(s-\tau)}+\frac{\partial K(t^1,s)}{\partial y(s-\tau)}\frac{dy(s-\tau)}{d(s-\tau)})ds]+1(t^1-t_0-\tau)\int_{t_0+\tau}^{t^1}\gamma_x^T(t)dt[f(t_0+\tau-0)-f(t_0+\tau+0)]+$$

$$+1(t^1-t_0-\tau)\int_{t_0+\tau}^{t^1}\gamma_y^T(t)[K(t,t_0+\tau-0)-K(t,t_0+\tau+0)]dt-\int_{t_0}^{t^1}\gamma_x^T(t)\int_{t_0}^{t}\left(\frac{\partial f(s)}{\partial x(s-\tau)}\frac{dx(s-\tau)}{d(s-\tau)}+\frac{\partial f(s)}{\partial y(s-\tau)}\frac{dy(s-\tau)}{d(s-\tau)}\right)ds\,dt-$$

$$-\int_{t_0}^{t^1}\gamma_y^T(t)\int_{t_0}^{t}\left(\frac{\partial K(t,s)}{\partial x(s-\tau)}\frac{dx(s-\tau)}{d(s-\tau)}+\frac{\partial K(t,s)}{\partial y(s-\tau)}\frac{dy(s-\tau)}{d(s-\tau)}\right)ds\,dt\Bigg]\frac{d\tau}{d\alpha}\Bigg\}d\alpha\,.$$

It is expedient to add the conjugate equations for Lagrange's multipliers (12) too form of the integro-differential equations.

We enter new variable $\Phi_x(t^1)+\int_t^{t^1}\gamma_x^T(s)ds=\lambda_x^T(t)$, either $\gamma_x^T(t)=-\dot\lambda_x^T(t)$, $\lambda_x^T(t^1)=\Phi_x(t^1)$, and change an order of integrating in double integral inside of triangular area (see paper [25]) $\left(\text{i.e. }\int_{t_0}^{t^1}\int_{t_0}^{t}A(t,s)\,ds\,dt=\int_{t_0}^{t^1}\int_{t}^{t^1}A(s,t)\,ds\,dt\right)$. Then conjugate equations (12) are noted as (4) and SF (see (13)) are calculated under the formula (5).

4. Basic result

Conjugate equations have the form

$$-\dot\lambda_x^T(t)=\lambda_x^T(t)\frac{\partial f(t)}{\partial x(t)}+\Phi_y(t^1)\frac{\partial K(t^1,t)}{\partial x(t)}+\frac{\partial f_0(t)}{\partial\eta(t)}\frac{\partial\eta(t)}{\partial x(t)}+\int_t^{t^1}\gamma_y^T(s)\frac{\partial K(s,t)}{\partial x(t)}ds+ \tag{14}$$

$$+1(t^1-\tau-t)[\lambda_x^T(t+\tau)\frac{\partial f(t+\tau)}{\partial x(t)}+\Phi_y(t^1)\frac{\partial K(t^1,t+\tau)}{\partial x(t)}+\int_{t+\tau}^{t^1}\gamma_y^T(s)\frac{\partial K(s,t+\tau)}{\partial x(t)}ds\ ],\ t\in[t_0,t^1)\,,$$

$$\lambda_x^T(t^1)=\Phi_x(t^1)\,,\ \gamma_y^T(t)=\lambda_x^T(t)\frac{\partial f(t)}{\partial y(t)}+\Phi_y(t^1)\frac{\partial K(t^1,t)}{\partial y(t)}+\frac{\partial f_0(t)}{\partial\eta(t)}\frac{\partial\eta(t)}{\partial y(t)}+\int_t^{t^1}\gamma_y^T(s)\frac{\partial K(s,t)}{\partial y(t)}ds+$$

$$+1(t^1-\tau-t)[\lambda_x^T(t+\tau)\frac{\partial f(t+\tau)}{\partial y(t)}+\Phi_y(t^1)\frac{\partial K(t^1,t+\tau)}{\partial y(t)}+\int_{t+\tau}^{t^1}\gamma_y^T(s)\frac{\partial K(s,t+\tau)}{\partial y(t)}ds\ ],\ t_0\le t\le t^1;$$

$$\bar\gamma_x^T(t)=1(t^1-\tau-t)[\lambda_x^T(t+\tau)\frac{\partial f(t+\tau)}{\partial x(t)}+\Phi_y(t^1)\frac{\partial K(t^1,t+\tau)}{\partial x(t)}+\int_{t+\tau}^{t^1}\gamma_y^T(s)\frac{\partial K(s,t+\tau)}{\partial x(t)}ds\ ],\ t_0-\tau\le t\le t_0\,,$$

$$\bar\gamma_y^T(t)=1(t^1-\tau-t)[\lambda_x^T(t+\tau)\frac{\partial f(t+\tau)}{\partial y(t)}+\Phi_y(t^1)\frac{\partial K(t^1,t+\tau)}{\partial y(t)}+\int_{t+\tau}^{t^1}\gamma_y^T(s)\frac{\partial K(s,t+\tau)}{\partial y(t)}ds\ ],\ t_0-\tau\le t\le t_0\,.$$

Here $\Phi_x(t^1) \equiv \frac{\partial I_1(t^1)}{\partial \eta(t^1)} \frac{\partial \eta(t^1)}{\partial x(t^1)}$, $\Phi_y(t^1) \equiv \frac{\partial I_1(t^1)}{\partial \eta(t^1)} \frac{\partial \eta(t^1)}{\partial y(t^1)}$.

SF are calculated under the formula:

$$\delta I = \delta_{\tilde{\alpha}(t)} I + \delta_{\tilde{\alpha}(t^1)} I + \delta_{\bar{\alpha}} I;\ \delta_{\tilde{\alpha}(t)} I = \int_{t_0}^{t^1} \Bigg[\frac{\partial f_0(t)}{\partial \eta(t)} \frac{\partial \eta(t)}{\partial \tilde{\alpha}(t)} + \frac{\partial f_0(t)}{\partial \tilde{\alpha}(t)} + \lambda_x^T(t) \frac{\partial f(t)}{\partial \tilde{\alpha}(t)} + \tag{15}$$

$$+\gamma_y^T(t) \frac{\partial r(t)}{\partial \tilde{\alpha}(t)} + \Phi_y(t^1) \frac{\partial K(t^1,t)}{\partial \tilde{\alpha}(t)} + \int_t^{t^1} \gamma_y^T(s) \frac{\partial K(s,t)}{\partial \tilde{\alpha}(t)} ds \Bigg] \delta\tilde{\alpha}(t)\, dt + \int_{t_0-\tau}^{t_0} [\bar{\gamma}_x^T(t) \frac{\partial \psi_x(t)}{\partial \tilde{\alpha}(t)} + \bar{\gamma}_y^T(t) \frac{\partial \psi_y(t)}{\partial \tilde{\alpha}(t)}] \delta\tilde{\alpha}(t)\, dt;$$

$$\delta_{\tilde{\alpha}(t^1)} I = [\frac{\partial I_1(t^1)}{\partial \eta(t^1)} \frac{\partial \eta(t^1)}{\partial \tilde{\alpha}(t^1)} + \Phi_y(t^1) \frac{\partial r(t^1)}{\partial \tilde{\alpha}(t^1)}] \delta\tilde{\alpha}(t^1);$$

$$\delta_{\bar{\alpha}} I = \Bigg\{ \frac{\partial I_1(t^1)}{\partial \eta(t^1)} \frac{\partial \eta(t^1)}{\partial \bar{\alpha}} + \frac{\partial I_1(t^1)}{\partial \bar{\alpha}} + \lambda_x^T(t_0) \frac{\partial x_0(\bar{\alpha}, t_0)}{\partial \bar{\alpha}} + \int_{t_0}^{t^1} \lambda_x^T(t) \frac{\partial f(t)}{\partial \bar{\alpha}} dt +$$

$$+ \Phi_y(t^1) \left[\frac{\partial r(t^1)}{\partial \bar{\alpha}} + \int_{t_0}^{t^1} \frac{\partial K(t^1,s)}{\partial \bar{\alpha}} ds \right] + + \int_{t_0}^{t^1} [\frac{\partial f_0(t)}{\partial \eta(t)} \frac{\partial \eta(t)}{\partial \bar{\alpha}} + \frac{\partial f_0(t)}{\partial \bar{\alpha}} + \gamma_y^T(t) \frac{\partial r(t)}{\partial \bar{\alpha}} + \int_t^{t^1} \gamma_y^T(s) \frac{\partial K(s,t)}{\partial \bar{\alpha}} ds] dt +$$

$$+ \int_{t_0-\tau}^{t_0} [\bar{\gamma}_x^T(t) \frac{\partial \psi_x(t)}{\partial \bar{\alpha}} + \bar{\gamma}_y^T(t) \frac{\partial \psi_y(t)}{\partial \bar{\alpha}}] dt + + \Bigg[\lambda_x^T(t_0) \left(\frac{\partial x_0(\bar{\alpha}, t_0)}{\partial t_0} - f(t_0) \right) +$$

$$+ 1(t^1 - t_0 - \tau) \lambda_x^T(t_0 + \tau)(f(t_0 + \tau - 0) - f(t_0 + \tau + 0)) +$$

$$+ \Phi_y(t^1)[\frac{\partial r(t^1)}{\partial t_0} - K(t^1, t_0) + 1(t^1 - t_0 - \tau)(K(t^1, t_0 + \tau - 0) - K(t^1, t_0 + \tau + 0))] -$$

$$- f_0(t_0) + \int_{t_0}^{t^1} \gamma_y^T(t) (\frac{\partial r(t)}{\partial t_0} - K(t, t_0)) dt + + 1(t^1 - t_0 - \tau) \int_{t_0+\tau}^{t^1} \gamma_y^T(t) [K(t, t_0 + \tau - 0) - K(t, t_0 + \tau + 0)] dt \Bigg] \frac{dt_0}{d\bar{\alpha}} +$$

$$+ \Bigg[\Phi_x(t^1) f(t^1) + \Phi_y(t^1)[\frac{\partial r(t^1)}{\partial t^1} + K(t^1, t^1) + \int_{t_0}^{t^1} \frac{\partial K(t^1, s)}{\partial t^1} ds] + + \frac{\partial I_1(t^1)}{\partial \eta(t^1)} \frac{\partial \eta(t^1)}{\partial t^1} + \frac{\partial I_1(t^1)}{\partial t^1} + f_0(t^1) \Bigg] \frac{dt^1}{d\bar{\alpha}} +$$

$$+ \Bigg[1(t^1 - t_0 - \tau) \lambda_x^T(t_0 + \tau)(f(t_0 + \tau - 0) - f(t_0 + \tau + 0)) - \int_{t_0}^{t^1} \lambda_x^T(t) \left(\frac{\partial f(t)}{\partial x(t-\tau)} \frac{dx(t-\tau)}{d(t-\tau)} + \frac{\partial f(t)}{\partial y(t-\tau)} \frac{dy(t-\tau)}{d(t-\tau)} \right) dt +$$

$$+ \Phi_y(t^1)[\ 1(t^1 - t_0 - \tau)(K(t^1, t_0 + \tau - 0) - K(t^1, t_0 + \tau + 0)) - \int_{t_0}^{t^1} \left(\frac{\partial K(t^1, s)}{\partial x(s-\tau)} \frac{dx(s-\tau)}{d(s-\tau)} + \frac{\partial K(t^1, s)}{\partial y(s-\tau)} \frac{dy(s-\tau)}{d(s-\tau)} \right) ds\ ] +$$

$$+ 1(t^1 - t_0 - \tau) \int_{t_0+\tau}^{t^1} \gamma_y^T(t)(K(t, t_0 + \tau - 0) - K(t, t_0 + \tau + 0)) dt -$$

$$- \int_{t_0}^{t^1} \gamma_y^T(t) \int_{t_0}^{t} \left(\frac{\partial K(t,s)}{\partial x(s-\tau)} \frac{dx(s-\tau)}{d(s-\tau)} + \frac{\partial K(t,s)}{\partial y(s-\tau)} \frac{dy(s-\tau)}{d(s-\tau)} \right) ds\, dt \Bigg] \frac{d\tau}{d\bar{\alpha}} \Bigg\} d\bar{\alpha} \equiv \frac{dI}{d\bar{\alpha}} d\bar{\alpha}.$$

Conclusion

In this paper the variational method of calculation SF and SC for the multivariate nonlinear dynamic systems described by general continuous vectorial Volterra’s integro-differential equations of the second genus with dead time is developed.

The variational method is based on invariant expansion of initial functional for system due to inclusion in it of the dynamic equations of object model and of measuring device model with the help of Lagrange's multipliers and on computation of the first variation expanded functional on phase coordinates of model and in required parameters. The equating with a zero of the functions facing to variations of phase coordinates gives the dynamic equations for Lagrange's multipliers. The simplified first variation represents required sensitivity functional.

Novelty and generality of results consists in a generality of dynamic object model, of the measuring device model and of quality functional. In models both variables and constant parameters are present. In a basis of calculation of sensitivity indexes the decision of the integro-differential equations of object model in a forward direction of time and obtained integro-differential equations for Lagrange's multipliers in the opposite direction of time lays.

Received in paper SF are more general in comparison with known in the scientific literature.

Results are applicable at the decision of problems of identification, adaptive optimal control and optimization of dynamic systems, i.e. they allow to create precision systems and devices.

This paper continues research in [17, 21–23. 25–27].

Integro-differential models structurally include separately differential and integrated models, and also 4 kinds of more simple integro-differential models which differ character of interaction of phase coordinates of integrated and differential parts. Examples of reception of these results will be presented in special paper.

Variational method of calculation of SC and SF allows to do a generalization on more complex dynamic system classes, described by: differential, integral and integro-differential equations with additional some dead times and different classes of discontinuous dynamic equations.

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Received: December 10, 2018

Rouban A.I. (2019) THE SENSITIVITY FUNCTIONALS IN THE BOLTS'S PROBLEM FOR MULTIVARIATE DYNAMIC SYSTEMS DESCRIBED BY INTEGRO-DIFFERENTIAL EQUATIONS WITH DELAY TIME. *Vestnik Tomskogo gosudarstvennogo universiteta. Upravlenie vychislitelnaja tehnika i informatika* [Tomsk State University Journal of Control and Computer Science]. 48. pp. 31–41

DOI: 10.17223/19988605/48/4

Рубан А.И. ФУНКЦИОНАЛЫ ЧУВСТВИТЕЛЬНОСТИ В ЗАДАЧЕ БОЛЬЦА ДЛЯ МНОГОМЕРНЫХ ДИНАМИЧЕСКИХ СИСТЕМ, ОПИСЫВАЕМЫХ ИНТЕГРО-ДИФФЕРЕНЦИАЛЬНЫМИ УРАВНЕНИЯМИ С ЗАПАЗДЫВАЮЩИМ АРГУМЕНТОМ. *Вестник Томского государственного университета. Управление, вычислительная техника и информатика.* 2019. № 48. С. 31–41

Вариационный метод применен для расчета функционалов чувствительности, которые связывают первую вариацию функционалов качества работы систем (функционалов Больца) с вариациями переменных и постоянных параметров, для многомерных нелинейных динамических систем, описываемых обобщенными интегро-дифференциальными уравнениями Вольтерра второго рода с запаздывающим аргументом.

Ключевые слова: вариационный метод; функционал чувствительности; интегро-дифференциальное уравнение с запаздывающим аргументом; функционал качества работы системы; задача Больца; сопряженное уравнение.

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