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# ON THE NELSON–SIEGEL–SVENSSON NO-ARBITRAGE YIELD CURVE MODELS

It is shown that the requirement to satisfy the no-arbitrage conditions specifies the Nelson–Siegel–Svensson model in the sense that gives for coefficients of this model the obvious economic sense: the free coefficient should be function on term to maturity, and other coefficients should depend on the market state variables which, in turn, are selective values of stochastic processes at time point for which the time structure is designed. It is shown that the model is the member of family of affine yield models and is generated by two-dimensional model of a short-term interest rate for Nelson–Siegel model or four-dimensional model of a short-term interest rate for Nelson–Siegel–Svensson model.

**Keywords:** term structure; yield curve; factor model; affine no-arbitrage models; Nelson–Siegel yield model.

The term structure of yield interest rates is one of the most demanded characteristics of financial assets. At mathematical modeling of term structure are used no-arbitrage affine models as they imply possibility of derivation of decisions in an analytical form more often. For derivation of model of term structure is usual proceed from description as the financial market state variables evolve. It is accepted more often that the market state variables is a stochastic process of diffusion type.

Let's consider that for  $n$ -factor affine yield model it is supposed that the vector of a state variables of financial market  $X(t) = (X_1, X_2, \dots, X_n)^T$  follows to homogeneous for time Markov process generated by the stochastic differential equation

$$dX(t) = \mu(X(t)) dt + \sigma(X(t)) dW(t) \quad (1)$$

with a drift  $n$ -vector  $\mu(x)$ ,  $(n \times m)$ -matrix volatility  $\sigma(\xi)$ , and  $m$ -vector  $W(t)$  of independent standard Wiener processes.

In one-factor model the short-term interest rate is accepted that a market state variable  $r(t)$  evolves as the stochastic process with property of returning to a stationary average  $\theta$ . In this case there are three basic approaches for the description of stochastic process of the short-term interest rates, developed by the American authors:

1) the Vasicek model [1], when the rate volatility  $\sigma = \sqrt{2kD}$  – the determined constant ( $D$  – a stationary variance of process  $r(t)$ )

$$dr(t) = k(\theta - r(t)) dt + \sqrt{2kD} dW(t); \quad (2)$$

2) the CIR model [2], in which the rate volatility – the non-negative stochastic process proportional to a square root of the interest rate

$$dr(t) = k(\theta - r(t)) dt + \sqrt{\frac{2kD}{\theta} r(t)} dW(t), r(0) > 0; \quad (3)$$

3) the Duffie–Kan model [3], in which frameworks the interest rate volatility – random, proportional to excess of the interest rate over the bottom (unattainable) border  $r_{\inf}$ :

$$dr(t) = k(\theta - r(t)) dt + \sqrt{\frac{2kD}{\theta - r_{\inf}} (r(t) - r_{\inf})} dW(t), r(0) > r_{\inf}. \quad (4)$$

Generally speaking, the Duffie–Kan model is more general model than early two because at zero bottom border of the interest rate it turns to CIR model, and at moving of the bottom border of the interest rate to a minus infinity (i.e. at  $r_{\inf} \rightarrow -\infty$ ) model becomes the Vasicek model.

The term structure of yield interest rates is determined as dependence the yield interest rate (or the prices) zero coupon bonds at some currently time  $t$  on term to maturity of this bond.

For models (2)–(4) price of zero coupon bonds  $P(t, r; T)$  with maturity date  $T$  provided that at time point  $t$  the short-term interest rate  $r(t)$  has accepted value  $r$ , i.e.  $r(t) = r$ , is calculated by the formula

$$P(t, r; T) \equiv P(t, r; t + \tau) = P(t, r; T) \exp \{A(\tau) - rB(\tau)\}, \quad (5)$$

where  $\tau = T - t$  – term to maturity, is usual is supposed that  $P(T, r; T) = 1$  that does not limit an analysis generality. Functions  $A(\tau)$  and  $B(\tau)$  is usual are called as functions of term structure. At derivation of the formula (5) conditions of absence of arbitrage have been considered. For such function of the price the yield to maturity (or simply yield)  $y(\tau, r)$  is affine function  $r$  and is calculated by the formula

$$y(\tau, r) = \frac{-\ln P(t, r; T)}{T - t} = \frac{rB(\tau) - A(\tau)}{\tau}. \quad (6)$$

Functions  $A(\tau)$  and  $B(\tau)$  are calculated in an explicit analytical form though for their derivation it is necessary to solve the differential equations of Riccati. Dependence  $y(\tau, r)$  on  $\tau$  also determines a term structure of yield.

The yield to maturity  $y(\tau, r)$  is an *average* characteristic for the time period of duration  $\tau$ . At the same time the practitioners are interested in the question what will be a *short-term rate* of yield in the end of this period. Such rate is named the instant forward rate  $f(\tau, r)$  and it is functionally connected with yield to maturity  $y(\tau, r)$  by relation

$$y(\tau, r) = \frac{1}{\tau} \int_0^\tau f(s, r) ds, \quad f(\tau, r) = y(\tau, r) + \tau \frac{\partial y(\tau, r)}{\partial \tau}. \quad (7)$$

In frameworks no-arbitrage affine models the forward rate  $f(\tau, r)$  is calculated by the formula

$$f(\tau, r) = \frac{-\partial \ln P(t, r; T)}{\partial T} = r \frac{\partial B(\tau)}{\partial \tau} - \frac{\partial A(\tau)}{\partial \tau}. \quad (8)$$

Usually the analytical properties of function  $f(\tau, r)$  turn out more simple, than properties  $y(\tau, r)$ . Both these functions – yield curve  $y(\tau, r)$  and a forward curve  $f(\tau, r)$  are equally interesting for financial analysts.

It appears that depending on a market variables (for one-factorial model it is values  $r$ ) the function  $y(\tau, r)$  can belong usually only to one of four types of curves: monotonously increasing to some final limiting value ("normal" yield curve), monotonously decreasing to some final limiting value ("inverse" yield curve), yield curve with a maximum (a "humped" curve) and flat yield curve.

Such classification of curves proves to be true on real financial markets. At the same time real market yield though have such functional appearance, but on size often enough strongly differ from received on the basis of the resulted models. Therefore there are various modifications described above models, in particular it is considered that the dimension increase, i.e. transition to two-factor, three-factor etc. models, can raise their accuracy of the description of market yield. In a multidimensional case when the financial market variables are determined by a vector  $X(t)$  which evolution is described by diffusion process (1), formulas (6) and (8) for a yield curve and forward curve will be transformed to a form

$$y(\tau, r) = \frac{x B(\tau) - A(\tau)}{\tau}, \quad f(\tau, r) = x \frac{dB(\tau)}{d\tau} - \frac{dA(\tau)}{d\tau}, \quad (9)$$

provided that at point of time  $t$  the vector of a variables of financial market  $X(t)$  has turned out equal to  $x$ , i.e.  $X(t) = x$ ; in this case  $B(\tau)$  – the vector, and  $A(\tau)$  remains scalar function.

However the dimension increases essentially increases number of parameters of models, but accuracy of models increases poorly.

## 1. Nelson – Sigel – Svensson model

In this connection there was other method of improvement of model of the yield term structure, based on a small variety of types yield curve. As their only four, there was an idea – to enter some standard functional dependences and by them to build the combinations approximating yield curve. Nelson and Sigel [4] have offered as such standard functions for designing forward curve three simple functions:  $\alpha_1(\tau) = 1$ ,  $\alpha_2(\tau) = \exp(-\gamma\tau)$ ,  $\alpha_3(\tau) = \gamma\tau \exp(-\gamma\tau)$ . Function  $\alpha_1(\tau)$  is intended for approximation of a long-term segment of a

curve, function  $\alpha_2(\tau)$  – for approximation of a short-term segment and function  $\alpha_3(\tau)$  should approximate medium-term of yield. Use of such approach leads to following result

$$f(\tau, r) = \beta_1 \alpha_{1f}(\tau) + \beta_2 \exp(-\gamma\tau) + \beta_3 \gamma\tau \exp(-\gamma\tau). \quad (10)$$

Evidently, function  $f(\tau, r)$  is determined very simply, but for its final identification it is necessary to find four parameters  $\gamma > 0$ ,  $\beta_1 > 0$ ,  $\beta_2$  and  $\beta_3$ . By means of relations (7), (10) it is easy to receive and function  $y(\tau, r)$  in a form

$$y(\tau, r) = \beta_1 \alpha_{1y}(\tau) + \beta_2 \frac{1 - e^{-\gamma\tau}}{\gamma\tau} + \beta_3 \left( \frac{1 - e^{-\gamma\tau}}{\gamma\tau} - e^{-\gamma\tau} \right). \quad (11)$$

Note that the assumption about  $\alpha_1(\tau) = 1$ , as it will be shown later, is not be agreed to the requirement of absence of arbitrage opportunities and therefore it is more correctly to write the first item in the right parts (10) and (11) as  $\beta_1 \alpha_{1f}(\tau)$  and  $\beta_1 \alpha_{1y}(\tau)$  where between  $\alpha_{1f}(\tau)$  and  $\alpha_{1y}(\tau)$  there is a mutual relation which is determined by equality (7):

$$\alpha_{1y}(\tau) = \frac{1}{\tau} \int_0^\tau \alpha_{1f}(s) ds.$$

To increase the flexibility and improve the fit models to empirical data Svensson [5] extend Nelson – Sigel function by adding a fourth function  $\alpha_5(\tau) = \delta\tau \exp(-\delta\tau)$ ,  $\delta > 0$ , a second hump-shape with two additional parameters  $\beta_5$  and  $\delta \neq \gamma$ . So

$$f(\tau, r) = \beta_1 + \beta_2 \exp(-\gamma\tau) + \beta_3 \gamma\tau \exp(-\gamma\tau) + \beta_4 \exp(-\delta\tau) + \beta_5 \delta\tau \exp(-\delta\tau). \quad (12)$$

So the number of parameters of model has increased by two and has reached six. In equality (12) for a generality is entered another function  $\alpha_4(\tau) = \exp(-\delta\tau)$  but if to assume  $\beta_4 = 0$ , Svensson representation will turn out.

In June, 1996 in Bank for International Settlements (BIS, Basel) was accepted the agreement about that the central banks of Europe should to represent the data in BIS for calculations zero coupon yield curve and estimations of parameters of models. It was found out that the majority of banks of Europe is made use for modeling of yield curve the approach of Nelson – Sigel (Italy and Finland) or its Svensson modification (Belgium, Germany, Spain, Norway, France, Switzerland and Sweden) [6]. It, in particular, underlines importance of the analysis of this approach.

Unfortunately, the approach of Nelson – Sigel – Svensson (NSS) does not give recommendations how to determine parameters of model and in any way does not explain, whether such model is no-arbitrage. It makes sense to modify this model on purpose to enter it in the no-arbitrage class. We will notice in the beginning that no-arbitrage affine yields (6), (8) and (9) possess following general limit properties [7]:

$$\lim_{\tau \rightarrow 0} y_{NA}(\tau, r) = \lim_{\tau \rightarrow 0} f_{NA}(\tau, r) = x^T \phi = r, \quad \lim_{\tau \rightarrow \infty} y_{NA}(\tau, r) = \lim_{\tau \rightarrow \infty} f_{NA}(\tau, r) = y_\infty.$$

Here it is designated  $B'(0) = \phi$ ;  $y_\infty$  – the limiting long-term yield, which determined only in parameters of model and not dependent on state variables of the market  $x$ . Since it is general properties of yields, yields of model of Nelson – Sigel (NS) should possess by them too:

$$\lim_{\tau \rightarrow 0} y_{NS}(\tau, r) = \lim_{\tau \rightarrow 0} f_{NS}(\tau, r) = \beta_1 + \beta_2, \quad \lim_{\tau \rightarrow \infty} y_{NS}(\tau, r) = \lim_{\tau \rightarrow \infty} f_{NS}(\tau, r) = \beta_1.$$

Thus, the sense of parameters  $\beta_1$  and  $\beta_2$  in Nelson – Sigel model is cleared up:  $\beta_1 = y_\infty$ ,  $\beta_2 = r - y_\infty$ .

Comparing (10)–(12) with (9) it is possible to conclude that yields of model NSS are similar to no-arbitrage affine yields in case of four-factor model when the market condition is described by a four-dimensional vector  $X(t) = (X_1(t), X_2(t), X_3(t), X_4(t))$ . In this case it is possible to consider that the market state is described by a vector  $(X_1(t) = \beta_2, X_2(t) = \beta_3, X_3(t) = \beta_4, X_4(t) = \beta_5)$  and in the following way compare details of these models:

$$\beta_1 \alpha_{1f}(\tau) \leftrightarrow -\frac{dA(\tau)}{d\tau}, \quad \alpha_2(\tau) \leftrightarrow \frac{dB_1(\tau)}{d\tau}, \quad \alpha_3(\tau) \leftrightarrow \frac{dB_2(\tau)}{d\tau}, \quad \alpha_4(\tau) \leftrightarrow \frac{dB_3(\tau)}{d\tau}, \quad \alpha_5(\tau) \leftrightarrow \frac{dB_4(\tau)}{d\tau}.$$

It means that parameters  $\beta_2, \beta_3, \beta_4$  and  $\beta_5$  are not constants, and are values of stochastic processes  $X_1(t), X_2(t), X_3(t), X_4(t)$  at the point of time  $t$  for which the term structure is determined. Function  $\beta_1 \alpha_{1f}(\tau)$  too not a constant and a determined function of  $\tau$ .

The stochastic differential equation for  $X(t) = (X_1(t), X_2(t), X_3(t), X_4(t))$  looks like (1). In order that yield was in frameworks of affine structure, components of the equation (1) should be specified as follows [8]:

$$\mu(x) = K(\theta - x), \sigma(x)\sigma(x)^T = \varphi + \sum_{i=1}^4 \omega_i x_i, \sigma(x)\lambda(x) = \xi + \sum_{i=1}^4 \eta_i x_i. \quad (13)$$

Here  $K, \varphi$  and  $\omega_i$  –  $(4 \times 4)$ -matrixes;  $\theta, \xi$  and  $\eta_i$  – 4-vectors,  $x_i$  – vector components of  $x$ , the vector function  $\lambda(x)$  determines the market prices of risk. It results [3] in the ordinary differential equations for function  $A(\tau)$  and components of vector  $B(\tau) = (B_1(\tau), B_2(\tau), B_3(\tau), B_4(\tau))$ :

$$A'(\tau) = (\xi - K\theta)^T B(\tau) + B(\tau)^T \varphi B(\tau)/2, A(0) = 0, \quad (14)$$

$$B_i'(\tau) = \phi_i - B(\tau)^T (\eta_i + K_i) - B(\tau)^T \omega_i B(\tau)/2, B_i(0) = 0. \quad (15)$$

In the equation for  $B_i(\tau)$  symbol  $K_i$  designates  $i$ -th column of matrix  $K$ ,  $1 \leq i \leq 4$ .

In order that the NSS model was also affine no-arbitrage model, decisions of the equations (15)  $B_1(\tau), B_2(\tau), B_3(\tau), B_4(\tau)$  should be agreed in appropriate way with functions  $\alpha_2(\tau), \alpha_3(\tau), \alpha_4(\tau), \alpha_5(\tau)$ .

By determination of function  $\alpha_2(\tau), \alpha_3(\tau), \alpha_4(\tau), \alpha_5(\tau)$  are either exponents or their combinations. Such decisions of the equation (15) can be only when the equations (15) are linear. Hence, nonlinear items in the equations (15) should be absent, i.e.  $\omega_i = 0, 1 \leq i \leq 4$ . And it means that according to representation (13) volatility matrix  $\sigma(\xi)$  should not depend on market state variables  $x$ , i.e. volatilities should be the determinate constants. From here also follows that function of market price of risk  $\lambda(x)$  should be a vector consisting of constants, i.e. according to representation (13)  $\eta_i = 0, 1 \leq i \leq 4$ . This considerably simplifies a task of determination of vector  $B(\tau) = (B_1(\tau), B_2(\tau), B_3(\tau), B_4(\tau))$  as instead of the equations (15) the linear system of the differential equations turns out

$$B'(\tau) = \phi - K^T B(\tau), B(0) = 0. \quad (16)$$

We define the matrix  $K$  in the form

$$K = \begin{pmatrix} \gamma & -\gamma & 0 & 0 \\ 0 & \gamma & 0 & 0 \\ 0 & 0 & \delta & -\delta \\ 0 & 0 & 0 & \delta \end{pmatrix}. \quad (17)$$

Then the decision of system (16) will be the following

$$B_1(\tau) = \phi_1 \frac{1 - e^{-\gamma\tau}}{\gamma}, B_2(\tau) = (\phi_1 + \phi_2) \frac{1 - e^{-\gamma\tau}}{\gamma} - \phi_1 \tau e^{-\gamma\tau}, \quad (18)$$

$$B_3(\tau) = \phi_3 \frac{1 - e^{-\delta\tau}}{\delta}, B_4(\tau) = (\phi_3 + \phi_4) \frac{1 - e^{-\delta\tau}}{\delta} - \phi_3 \tau e^{-\delta\tau}.$$

It means that no-arbitrage yield curves  $y_{NA}(\tau, r)$  and  $f_{NA}(\tau, r)$  according to ratios (9) are determined by formulas

$$\begin{aligned} y_{NA}(\tau, r) &= x_1 \phi_1 \frac{1 - e^{-\gamma\tau}}{\gamma\tau} + x_2 (\phi_1 + \phi_2) \frac{1 - e^{-\gamma\tau}}{\gamma\tau} - x_2 \phi_1 e^{-\gamma\tau} + \\ &+ x_3 \phi_3 \frac{1 - e^{-\delta\tau}}{\delta\tau} + x_4 (\phi_3 + \phi_4) \frac{1 - e^{-\delta\tau}}{\delta\tau} - x_4 \phi_3 e^{-\delta\tau} - \frac{A(\tau)}{\tau} = \\ &= (x_1 \phi_1 + x_2 (\phi_1 + \phi_2)) \frac{1 - e^{-\gamma\tau}}{\gamma\tau} - x_2 \phi_1 e^{-\gamma\tau} + (x_3 \phi_3 + x_4 (\phi_3 + \phi_4)) \frac{1 - e^{-\delta\tau}}{\delta\tau} - x_4 \phi_3 e^{-\delta\tau} - \frac{A(\tau)}{\tau}, \end{aligned} \quad (19)$$

$$\begin{aligned} f_{NA}(\tau, r) &= x_1 \phi_1 e^{-\gamma\tau} + x_2 \phi_2 e^{-\gamma\tau} + x_2 \phi_1 \gamma\tau e^{-\gamma\tau} + x_3 \phi_3 e^{-\delta\tau} + x_4 \phi_4 e^{-\delta\tau} + x_4 \phi_3 \delta\tau e^{-\delta\tau} - \frac{dA(\tau)}{d\tau} = \\ &= (x_1 \phi_1 + x_2 \phi_2) e^{-\gamma\tau} + x_2 \phi_1 \gamma\tau e^{-\gamma\tau} + (x_3 \phi_3 + x_4 \phi_4) e^{-\delta\tau} + x_4 \phi_3 \delta\tau e^{-\delta\tau} - \frac{dA(\tau)}{d\tau}. \end{aligned} \quad (20)$$

In these expressions the function  $A(\tau)$  in an explicit form is not presented, as it has a cumbersome form though its computation by the formula (14) does not represent difficulties when functions  $\{B_i(\tau)\}$  are already determined since the formula (14) is not the equation, and for determination  $A(\tau)$  it is enough to integrate the right part of equality (14).

Let's notice that limiting values of functions  $\{B_i(\tau)\}$  at  $\tau \rightarrow \infty$  are finite quantities:

$$\lim_{\tau \rightarrow \infty} B_1(\tau) = \phi_1 / \gamma, \quad \lim_{\tau \rightarrow \infty} B_2(\tau) = (\phi_1 + \phi_2) / \gamma, \quad \lim_{\tau \rightarrow \infty} B_3(\tau) = \phi_3 / \delta, \quad \lim_{\tau \rightarrow \infty} B_4(\tau) = (\phi_3 + \phi_4) / \delta.$$

It allows by means of (9) and (14) to find limiting long-term yields in the explicit form

$$y_\infty = \lim_{\tau \rightarrow \infty} y_{NA}(\tau, r) = \lim_{\tau \rightarrow \infty} f_{NA}(\tau, r) = - \lim_{\tau \rightarrow \infty} \frac{dA(\tau)}{d\tau} = (\sigma\lambda - K\theta)^T B(\infty) + B(\infty)^T \sigma \sigma^T B(\infty) / 2.$$

Let's remind that vector components  $\phi$  determine the weight coefficients of influence a component of a vector of the market states  $X(t)$  on yield size (through a short-term rate  $r$ ). If  $\phi_3 = \phi_4 = 0$ , then components  $X_3(t)$  and  $X_4(t)$  do not influence on yield, and considered NSS model turns to usual NS model. The affine no-arbitrage model in this case provides representations

$$f_{NA}(\tau, r) = - \frac{dA(\tau)}{d\tau} + (x_1\phi_1 + x_2\phi_2) e^{-\gamma\tau} + x_2\phi_1 \gamma \tau e^{-\gamma\tau}. \quad (21)$$

$$\begin{aligned} y_{NA}(\tau, r) &= - \frac{A(\tau)}{\tau} + (x_1\phi_1 + x_2(\phi_1 + \phi_2)) \frac{1 - e^{-\gamma\tau}}{\gamma\tau} - x_2\phi_1 e^{-\gamma\tau} = \\ &= - \frac{A(\tau)}{\tau} + (x_1\phi_1 + x_2\phi_2) \frac{1 - e^{-\gamma\tau}}{\gamma\tau} + x_2\phi_1 \left( \frac{1 - e^{-\gamma\tau}}{\gamma\tau} - e^{-\gamma\tau} \right). \end{aligned} \quad (22)$$

Comparison of representations (10) with (21) and (11) with (22) shows that no-arbitrage NS model is usual affine no-arbitrage model, coefficients  $\beta_1, \beta_2, \beta_3$  of which are determined as follows

$$\beta_{1y}(\tau) = - \frac{A(\tau)}{\tau}, \quad \beta_2 = x_1\phi_1 + x_2\phi_2 = r, \quad \beta_3 = x_2\phi_1. \quad (23)$$

In contrast to determination of these coefficients by Nelson and Sigel here  $\beta_1, \beta_2$  and  $\beta_3$  are not constants:  $\beta_{1y}(\tau) \equiv \beta_1\alpha_{1y}(\tau)$  – function  $\tau$ , and  $\beta_2$  and  $\beta_3$  – functions of variables  $x_1 = X_1(t)$  and  $x_2 = X_2(t)$ , the market state at point time  $t$  for which the term structure is designed, and consequently are essentially a sample of random variables. Note that by determination the coefficient  $\beta_2$  is equal to value of a short-term interest rate at the specified point time  $t$ ,  $\beta_2 = r(t) = r$ . Since for obtaining the affine no-arbitrage NS models in the equation (1) the function of drift  $\mu(x)$  should be linear with respect to  $x$ , and a volatility matrix  $\sigma(x)$  – a constant (not dependent on  $x$ ) then the equation (1) generates stochastic process with normal distribution so the coefficients  $\beta_2$  and  $\beta_3$  in affine no-arbitrage NS models are normally distributed random variables correlated between themselves.

Let's consider, in what the modification of NS model offered by Svensson results. In this case  $\phi_3 \neq 0, \phi_4 \neq 0$  and yield curves are determined by expressions (19) and (20). Comparison of representations (12) and (19) gives the following interpretation of coefficients of NSS model

$$\beta_{1y}(\tau) = - \frac{A(\tau)}{\tau}, \quad \beta_2 = x_1\phi_1 + x_2\phi_2, \quad \beta_3 = x_2\phi_1, \quad \beta_4 = x_3\phi_3 + x_4\phi_4, \quad \beta_5 = x_4\phi_3. \quad (24)$$

Unlike the NS model here  $\beta_2 \neq r$ , but  $\beta_2 + \beta_4 = r$ .

Thus, the affine no-arbitrage NSS model proves to be a special case of four-dimensional affine no-arbitrage model with a constant matrix of volatility and matrix  $K$  of coefficients of the influence, looking like (17). Accordingly, the NS model is a special case two-dimensional affine no-arbitrage models with a constant matrix of volatility and matrix  $K$  of coefficients of the influence, which has the form

$$K = \begin{pmatrix} \gamma & -\gamma \\ 0 & \gamma \end{pmatrix}. \quad (25)$$

In other words, models NS and NSS are special cases of multidimensional model of Vasiček (2) if by this to understand multidimensional model with a constant volatility matrix and linear drift. It is possible to assume that matrixes (17) and (25) serve as signs of models NS and NSS. Then it is possible to assume that if in model (1), (13) matrix  $K$  is specified in the form of (17) or (25) yield curve will have term structure NS or NSS. Whether so it?

Let's consider for an example two-dimensional model CIR (3) with a matrix of coefficients of influence (25). For this purpose we will set a matrix and a vector in relations (13) as follows

$$\sigma(x) = \begin{pmatrix} \sigma_1 \sqrt{x_1} & 0 \\ 0 & \sigma_2 \sqrt{x_2} \end{pmatrix}, \lambda(x) = \begin{pmatrix} \sigma_1 \lambda_1 \sqrt{x_1} \\ \sigma_2 \lambda_2 \sqrt{x_2} \end{pmatrix}.$$

Here it is for brevity designated  $\sigma_i = \sqrt{2k_i D_i / \theta_i}$ ,  $i = 1, 2$ . It gives in expressions (13):

$$\varphi = 0, \omega_1 = \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & 0 \end{pmatrix}, \omega_2 = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_2^2 \end{pmatrix}, \xi = 0, \eta_1 = \begin{pmatrix} \sigma_1^2 \lambda_1 \\ 0 \end{pmatrix}, \eta_2 = \begin{pmatrix} 0 \\ \sigma_2^2 \lambda_2 \end{pmatrix}.$$

The equations (14)–(15) become

$$\begin{aligned} A'(\tau) &= B(\tau)^T K \theta, \quad A(0) = 0, \\ B_1'(\tau) &= \phi_1 - (\lambda_1 \sigma_1^2 + \gamma) B_1(\tau) - \sigma_1^2 B_1(\tau)^2 / 2, \quad B_1(0) = 0, \\ B_2'(\tau) &= \phi_2 + \gamma B_1(\tau) - (\lambda_2 \sigma_2^2 + \gamma) B_2(\tau) - \sigma_2^2 B_2(\tau)^2 / 2, \quad B_2(0) = 0. \end{aligned}$$

In this case the equations have turned out nonlinear. The second and third equations are the equations of Riccati. And the equation for  $B_1(\tau)$  – the equation with constant coefficients also can be solved in the explicit analytical form. The equation for  $B_2(\tau)$  – the equation of Riccati with variable free coefficient and in the explicit analytical form does not be resolved.

Let's designate for brevity  $g = \sqrt{(\gamma + \sigma_1^2 \lambda_1)^2 + 2\phi_1 \sigma_1^2}$ . Then the solution of the equation for  $B_1(\tau)$  has a form

$$B_1(\tau) = \frac{2\phi_1}{\gamma + \sigma_1^2 \lambda_1 + g + \frac{2g}{e^{g\tau} - 1}}, \quad (26)$$

that essentially differs from function  $B_1(\tau)$  in NS model (18). The same it is possible to tell and about function  $B_2(\tau)$  which can be found only numerically. Similar results are fair and for two-dimensional model of Duffie–Kan (4) in which function  $B_1(\tau)$  has form like (26), but its parameters  $g$  and  $\sigma_1$  are determined in another way:

$$g_{DK} = \sqrt{(\gamma + \sigma_{1,DK}^2 \lambda_1)^2 + 2\phi_1 \sigma_{1,DK}^2}, \quad \sigma_{1,DK} = \sqrt{2k_1 D_1 / (\theta_1 - x_{1,inf})}.$$

It turns out that the NS model – no-arbitrage only when in the equation (1) function of drift  $\mu(x)$  is linear with regard to  $x$ , and the volatility matrix  $\sigma(x)$  does not depend on  $x$ .

## 2. Latent factors

In [9] instead of the coefficients  $\beta_1, \beta_2, \beta_3$  of NS model it is offered to use depending on the current time  $t$  «dynamic processes»  $L_t, S_t$  and  $C_t$ , respectively, which are interpreted as the level, the slope and the curvature factors. According to the above analysis (see Equation (23)), these processes are in accordance with the following state variables of the financial market

$$\beta_1 \Rightarrow L_t = -\frac{A(\tau)}{\tau} \Big|_{\tau=T-t}, \quad \beta_2 \Rightarrow S_t = \phi_1 X_1(t) + \phi_2 X_2(t) = r(t), \quad \beta_3 \Rightarrow C_t = \phi_1 X_2(t). \quad (27)$$

So the factor of level  $L_t$  in the explicit form does not depend on current time, and depends only on term to maturity and is not random. The slope and curvature factors  $\{S_t, C_t\}$  are linear transformation of state variables  $\{X_1(t), X_2(t)\}$

$$\begin{pmatrix} S_t \\ C_t \end{pmatrix} = \begin{pmatrix} \phi_1 & \phi_2 \\ 0 & \phi_1 \end{pmatrix} \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix}. \quad (28)$$

Therefore  $\{S_t, C_t\}$  – two-dimensional normal random process with the mathematical expectations

$$\begin{pmatrix} E[S_t] \\ E[C_t] \end{pmatrix} = \begin{pmatrix} \phi_1 E[X_1(t)] + \phi_2 E[X_2(t)] \\ \phi_1 E[X_2(t)] \end{pmatrix}, \quad (29)$$

and the covariance matrix

$$\Sigma(t-t_0) = \begin{pmatrix} \phi_1 & \phi_2 \\ 0 & \phi_1 \end{pmatrix} \begin{pmatrix} \Omega_{11}(t-t_0) & \Omega_{12}(t-t_0) \\ \Omega_{21}(t-t_0) & \Omega_{22}(t-t_0) \end{pmatrix} \begin{pmatrix} \phi_1 & 0 \\ \phi_2 & \phi_1 \end{pmatrix}. \quad (30)$$

Elements of expressions (29) and (30) are easily calculated on the basis of explicit formulas for  $\{X_1(t), X_2(t)\}$  and for  $\Omega(t-t_0)$ .

Thus, at construction the forward curve and yield curve a problem of estimation of parameter  $\gamma$ , and values of processes  $L_t$ ,  $S_t$  and  $C_t$  in current time arises.

As the process of state variables  $X(t)$  satisfies to the stochastic differential equation (1) with drift and volatility (13) then process  $\{S_t, C_t\}$  is generated by the stochastic differential equation

$$\begin{pmatrix} dS_t \\ dC_t \end{pmatrix} = \begin{pmatrix} \phi_1 & \phi_2 \\ 0 & \phi_1 \end{pmatrix} \begin{pmatrix} \gamma & -\gamma \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} \theta_1 - (\phi_1 S_t - \phi_2 C_t) / \phi_1^2 \\ \theta_2 - C_t / \phi_1 \end{pmatrix} dt + \begin{pmatrix} \phi_1 & \phi_2 \\ 0 & \phi_1 \end{pmatrix} \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} dW(t)$$

or

$$\begin{pmatrix} dS_t \\ dC_t \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} \phi_1 & \phi_2 \\ 0 & \phi_1 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} - \begin{pmatrix} S_t \\ C_t \end{pmatrix} dt + \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} dW(t), \quad (31)$$

where

$$\begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} = \begin{pmatrix} \phi_1 & \phi_2 \\ 0 & \phi_1 \end{pmatrix} \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} = \begin{pmatrix} \phi_1 \sigma_{11} + \phi_2 \sigma_{21} & \phi_1 \sigma_{12} + \phi_2 \sigma_{22} \\ \phi_1 \sigma_{21} & \phi_1 \sigma_{22} \end{pmatrix}.$$

This equation is linear and can be solved in an explicit form. Thus the fundamental matrix of solutions  $\Theta(t-t_0)$  is the same as for process  $\{X_1(t), X_2(t)\}$ . Generally speaking, explicit dependence of factors  $\{S_t, C_t\}$  on time may be found from equality (28) if there to substitute explicit expressions of state variables  $\{X_1(t), X_2(t)\}$ . Let  $s$  – some point time, such that  $s < t$ , and  $\bar{S}$  and  $\bar{C}$  – stationary averages of processes  $\{S_t, C_t\}$ , accordingly,  $\bar{S} = \phi_1 \theta_1 + \phi_2 \theta_2$ ,  $\bar{C} = \phi_1 \theta_2$ . Then explicit expressions for these processes will be of the form

$$\begin{pmatrix} S_t \\ C_t \end{pmatrix} = \begin{pmatrix} S_s + \gamma(t-s)C_s \\ C_s \end{pmatrix} e^{-\gamma(t-s)} + \begin{pmatrix} (1 - e^{-(t-s)\gamma})\bar{S} - \gamma(t-s)e^{-(t-s)\gamma}\bar{C} \\ (1 - e^{-(t-s)\gamma})\bar{C} \end{pmatrix} + \zeta(t, s),$$

where  $\zeta(t, s)$  – the two-dimensional stochastic process determined by the relation

$$\begin{aligned} \zeta(t, s) &= \Theta(t-s) \int_s^t \begin{pmatrix} e^{\gamma(u-s)} & -\gamma(u-s)e^{\gamma(u-s)} \\ 0 & e^{\gamma(u-s)} \end{pmatrix} \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} dW(u) = \\ &= \int_s^t \begin{pmatrix} \rho_{11} + \gamma(t-u)\rho_{21} & \rho_{12} + \gamma(t-u)\rho_{22} \\ \rho_{21} & \rho_{22} \end{pmatrix} e^{-\gamma(t-u)} dW(u). \end{aligned}$$

### 3. The no-arbitrage conditions for Nelson – Sigel – Svensson model

Until now, it was thought that the state variables form a two-dimensional process  $\{X_1(t), X_2(t)\}$ , and two-dimensional process  $\{S_t, C_t\}$  is a process of the latent variables. Suppose now that the states variables are  $\{S_t, C_t\}$ , which are generated by the equation (31). The relationship of these variables with zero coupon yield  $y_t(\tau)$  is established with help of state variables  $\{X_1(t), X_2(t)\}$ . In order to in the market the arbitrage opportunities are absent it is necessary that zero coupon bond price  $P(t, T, S_t, C_t)$  satisfies the equation of term structure [1]

$$\begin{aligned} \frac{\partial P(t, T, x)}{\partial t} + \mu(x)^T \frac{\partial P(t, T, x)}{\partial x} + \frac{1}{2} \text{tr} \left( \sigma(x)^T \frac{\partial^2 P(t, T, x)}{\partial x^2} \sigma(x) \right) - r(t)P(t, T, x) = \\ = \lambda(t, x)^T \sigma(x)^T \frac{\partial P(t, T, x)}{\partial x}. \end{aligned}$$

This expression must be solved with the boundary condition  $P(T, T, S_T, C_T) = 1$  for all states  $\{S_T, C_T\}$ . Here, for the sake of brevity we use the notation:  $x \equiv \{S, C\}$ ;  $\mu(x)$  and  $\sigma(x)$  – accordingly a drift vector and a volatility matrix of the equations (31);  $r(t)$  – a short-term interest rate; the vector function  $\lambda(t, x)$  determines the market price of risk. As between the price of the bond and its yield there is a known relationship  $\ln P(t, T, S_t, C_t) = -\tau y_t(\tau | S_t, C_t)$  then this equation is more convenient to write for yield, rather than the bond price:

$$\frac{\partial[\tau y(\tau|x)]}{\partial \tau} - \frac{\partial[\tau y(\tau|x)]}{\partial x} [\mu(x) - \sigma(x)\lambda(x)] + \frac{e^{\tau y(\tau|x)}}{2} \text{tr} \left( \sigma(x)^T \frac{\partial^2 e^{-\tau y(\tau|x)}}{\partial x^2} \sigma(x) \right) = r. \quad (32)$$

Substituting into this equation the function  $y(\tau|S, C)$  in explicit form we can find conditions for the absence of arbitrage for Nelson–Siegel yield curve. Write down for the convenience the elements of the equation (32).

$$\begin{aligned} \frac{\partial[\tau y(\tau|x)]}{\partial \tau} &= \tau \frac{dL(\tau)}{d\tau} + L(\tau) + S e^{-\gamma\tau} + C \gamma \tau e^{-\gamma\tau}, \\ \frac{\partial[\tau y(\tau|x)]}{\partial x} &= \left( \frac{\partial[\tau y(\tau|S, C)]}{\partial S}, \frac{\partial[\tau y(\tau|S, C)]}{\partial C} \right) = \left( \frac{1 - e^{-\gamma\tau}}{\gamma}, \frac{1 - e^{-\gamma\tau}}{\gamma} - \tau e^{-\gamma\tau} \right), \\ \mu(x) - \sigma(x)\lambda(x) &= \begin{pmatrix} \gamma & -\gamma \\ 0 & \gamma \end{pmatrix} \begin{bmatrix} \bar{S} \\ \bar{C} \end{bmatrix} - \begin{pmatrix} S \\ C \end{pmatrix} - \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}. \end{aligned}$$

Here, through  $\rho_{ij}$  the elements of the volatility matrix of equation (31) are denoted. As has been obtained above to find the Nelson–Siegel yield curve it is necessary, that  $\sigma(x)$  and  $\lambda(x)$  were constants (did not depend on  $\{S, C\}$ ).

$$\begin{aligned} \frac{\partial[\tau y(\tau|x)]}{\partial x} [\mu(x) - \sigma\lambda] &= S(1 - e^{-\gamma\tau}) - C\gamma\tau e^{-\gamma\tau} - \bar{S}(1 - e^{-\gamma\tau}) + \bar{C}\gamma\tau e^{-\gamma\tau} - \\ &- \frac{1}{\gamma} [(\rho_{11}\lambda_1 + \rho_{12}\lambda_2)(1 - e^{-\gamma\tau}) + (\rho_{21}\lambda_1 + \rho_{22}\lambda_2)(1 - e^{-\gamma\tau} - \gamma\tau e^{-\gamma\tau})]. \end{aligned}$$

As  $y(\tau|S, C)$  linearly depends from  $S$  and  $C$ , then

$$e^{\tau y(\tau|x)} \frac{\partial^2 e^{-\tau y(\tau|x)}}{\partial x^2} = \left( \frac{\partial[\tau y(\tau|x)]}{\partial x} \right)^T \frac{\partial[\tau y(\tau|x)]}{\partial x}$$

does not depend on these variables, and the third item in the left side of the equation (32) can be presented in a form

$$\begin{aligned} \frac{1}{2} \text{tr} \left( \sigma(x)^T \frac{\partial^2 P(t, T, x)}{\partial x^2} \sigma(x) \right) &= \frac{1}{2} \frac{\partial[\tau y(\tau|x)]}{\partial x} \sigma \sigma^T \left( \frac{\partial[\tau y(\tau|x)]}{\partial x} \right)^T = \\ &= \frac{\tau^2}{2} \left[ \left( \frac{(1 - e^{-\gamma\tau})}{\gamma\tau} \rho_{11} + \left( \frac{1 - e^{-\gamma\tau}}{\gamma\tau} - e^{-\gamma\tau} \right) \rho_{21} \right)^2 + \left( \frac{(1 - e^{-\gamma\tau})}{\gamma\tau} \rho_{12} + \left( \frac{1 - e^{-\gamma\tau}}{\gamma\tau} - e^{-\gamma\tau} \right) \rho_{22} \right)^2 \right]. \end{aligned}$$

Thus, substitution in the equation (32) yield  $y(\tau|S, C)$ , determined by expression (39), transforms this equation to a form

$$\begin{aligned} \tau \frac{dL(\tau)}{d\tau} + L(\tau) + S - \bar{S}(1 - e^{-\gamma\tau}) + \bar{C}\gamma\tau e^{-\gamma\tau} + \\ + \frac{1}{\gamma} [(\rho_{11}\lambda_1 + \rho_{12}\lambda_2)(1 - e^{-\gamma\tau}) + (\rho_{21}\lambda_1 + \rho_{22}\lambda_2)(1 - e^{-\gamma\tau} - \gamma\tau e^{-\gamma\tau})] + \\ + \frac{1}{2} \left[ \left( \frac{(1 - e^{-\gamma\tau})}{\gamma} \rho_{11} + \left( \frac{1 - e^{-\gamma\tau}}{\gamma} - \tau e^{-\gamma\tau} \right) \rho_{21} \right)^2 + \left( \frac{(1 - e^{-\gamma\tau})}{\gamma} \rho_{12} + \left( \frac{1 - e^{-\gamma\tau}}{\gamma} - \tau e^{-\gamma\tau} \right) \rho_{22} \right)^2 \right] = r. \end{aligned} \quad (33)$$

Note that factor  $C$  has disappeared from the obtained expression, and it includes only average value  $\bar{C}$  of this factor. Equality (33) should be satisfied for any values of factors  $L(\tau)$  and  $S_t$ . In addition, note that only two values depend on the current time  $t$  in this expression:  $S_t$  and  $r(t)$ . Therefore, in order to the yield  $y(\tau|S, C)$ , which determined by expression (39), satisfies the equation (32) requires two conditions from which factors  $S_t$  and  $L(\tau)$  are uniquely determined: 1) factor  $S_t$  is simply equal to a short-term interest rate

$$S_t = r(t), \quad (34)$$

that will completely be agreed equalities (27), and 2) factor  $L(\tau)$  satisfies a following differential relation



$$\begin{aligned}
\frac{d[\tau L(\tau)]}{d\tau} &= \bar{S} (1 - e^{-\gamma\tau}) - \bar{C} \gamma \tau e^{-\gamma\tau} - \\
&- \frac{1}{\gamma} [(\rho_{11}\lambda_1 + \rho_{12}\lambda_2)(1 - e^{-\gamma\tau}) + (\rho_{21}\lambda_1 + \rho_{22}\lambda_2)(1 - e^{-\gamma\tau} - \gamma\tau e^{-\gamma\tau})] - \\
&- \frac{1}{2} \left[ \left( \frac{(1 - e^{-\gamma\tau})}{\gamma} \rho_{11} + \left( \frac{1 - e^{-\gamma\tau}}{\gamma} - \tau e^{-\gamma\tau} \right) \rho_{21} \right)^2 + \left( \frac{(1 - e^{-\gamma\tau})}{\gamma} \rho_{12} + \left( \frac{1 - e^{-\gamma\tau}}{\gamma} - \tau e^{-\gamma\tau} \right) \rho_{22} \right)^2 \right],
\end{aligned} \tag{35}$$

which is substantially even is not equation with respect to  $\tau L(\tau)$  and is simply an integral from it. The calculation of integral from (35) is simple, but leads to cumbersome result and is not included here.

So, conditions of absence of arbitrage in Nelson–Siegel model uniquely determine factors  $S_t$  and  $L(\tau)$  by means of equalities (34) and (35). Factor  $C_t$  at first sight remained uncertain, but also on it there are restrictions. We will consider them.

Let's return at first to initial model (1) for which yield is determined by relation (9) using the functions of term structure  $A(\tau)$ ,  $B_1(\tau)$ ,  $B_2(\tau)$  and factors  $X_1(t)$ ,  $X_2(t)$ . Conditions of absence of arbitrage in this model are reduced to that functions  $A(\tau)$ ,  $B_1(\tau)$ ,  $B_2(\tau)$  should satisfy to the equations (14)–(15) and equality  $\phi_1 X_1(t) + \phi_2 X_2(t) = r(t)$ , in which constant weight coefficients  $\phi_1$  and  $\phi_2$  carry matching factors  $\{X_1(t), X_2(t)\}$  with a short-term rate  $r(t)$ . Factors  $\{X_1(t), X_2(t)\}$  can be considered as some zero coupon yields [3], the combination of which is the rate  $r(t)$ . From relations (23) and (27) follows that  $S_t = \phi_1 X_1(t) + \phi_2 X_2(t) = r(t)$ ,  $C_t = \phi_1 X_2(t)$ . Therefore factor  $C_t$  is also uniquely determined when conditions of no-arbitrage.

Let's pay attention that the Nelson–Siegel model is a special case of two-dimensional model (1), when there drift function  $\mu(x)$  is affine concerning state variables (see (13)), and the volatility matrix is constant. If in this case to set  $\phi_1 = 1$ , and  $\phi_2 = 0$  we will receive that  $S_t = X_1(t)$ ,  $C_t = X_2(t)$  and it is had complete coincidence of Nelson–Siegel model with the affine version of two-dimensional model (1).

#### 4. The no-arbitrage conditions for the Svensson expansion

Let's consider now the no-arbitrage conditions for the Svensson expansion. As already it has been above told, in Svensson modification the forward curve can be calculated by the formula (12), which implies the following form of the yield curve

$$y(\tau) = \beta_0 + \beta_1 \frac{1 - e^{-\gamma\tau}}{\gamma\tau} + \beta_2 \left( \frac{1 - e^{-\gamma\tau}}{\gamma\tau} - e^{-\gamma\tau} \right) + \beta_3 \frac{1 - e^{-\delta\tau}}{\delta\tau} + \beta_4 \left( \frac{1 - e^{-\delta\tau}}{\delta\tau} - e^{-\delta\tau} \right). \tag{36}$$

As it has been shown above, from reasons of no arbitrage coefficients  $\beta_0, \beta_1, \beta_2, \beta_3, \beta_4$  cannot be constants. Accordingly redesignate them:  $\beta_0 = L(\tau)$ ,  $\beta_1 = S_t$ ,  $\beta_2 = C_t$ ,  $\beta_3 = G_t$ ,  $\beta_4 = H_t$ . So

$$y(\tau) = L(\tau) + S_t \frac{1 - e^{-\gamma\tau}}{\gamma\tau} + C_t \left( \frac{1 - e^{-\gamma\tau}}{\gamma\tau} - e^{-\gamma\tau} \right) + G_t \frac{1 - e^{-\delta\tau}}{\delta\tau} + H_t \left( \frac{1 - e^{-\delta\tau}}{\delta\tau} - e^{-\delta\tau} \right). \tag{37}$$

We use this redesignation in representation (36) and we will substitute the obtained expression (37) in the equation (32), in which state vector  $x$  – four-dimensional,  $x = (S_t, C_t, G_t, H_t)^T$ . Function of drift

$$\mu(x) = K(\theta - x) = \begin{pmatrix} \gamma & -\gamma & 0 & 0 \\ 0 & \gamma & 0 & 0 \\ 0 & 0 & \delta & -\delta \\ 0 & 0 & 0 & \delta \end{pmatrix} \begin{pmatrix} \bar{S} - S_t \\ \bar{C} - C_t \\ \bar{G} - G_t \\ \bar{H} - H_t \end{pmatrix},$$

where matrix  $K$  is determined by equality (17), and  $(\bar{S}, \bar{C}, \bar{G}, \bar{H})^T$  – a vector of stationary expectations random coefficients  $(S_t, C_t, G_t, H_t)$ . A volatility matrix –  $(4 \times 4)$ -matrix with constant elements,  $\sigma(x) = \sigma$ , and a vector of market prices of risk – a 4-vector with constant elements,  $\lambda(x) = \lambda$ .

Then the explicit form of elements of the equation (32) will have the form

$$\begin{aligned}
\frac{\partial[\tau y(\tau|x)]}{\partial \tau} &= \frac{d[\tau L(\tau)]}{d \tau} + S_t e^{-\gamma \tau} + C_t \gamma \tau e^{-\gamma \tau} + G_t e^{-\delta \tau} + H_t \delta \tau e^{-\delta \tau}, \\
\frac{\partial[\tau y(\tau|x)]}{\partial x} &= \left( \frac{1-e^{-\gamma \tau}}{\gamma} \frac{1-e^{-\gamma \tau}}{\gamma} - \tau e^{-\gamma \tau}, \frac{1-e^{-\delta \tau}}{\delta} \frac{1-e^{-\delta \tau}}{\delta} - \tau e^{-\delta \tau} \right), \\
\frac{\partial[\tau y(\tau|x)]}{\partial \tau} - \frac{\partial[\tau y(\tau|x)]}{\partial x} \mu(x) &= \\
= \frac{d[\tau L(\tau)]}{d \tau} - \bar{S}(1-e^{-\gamma \tau}) - \bar{G}(1-e^{-\delta \tau}) + \bar{C} \gamma \tau e^{-\gamma \tau} + \bar{H} \delta \tau e^{-\delta \tau} + S_t + G_t. \\
e^{\tau y(\tau|x)} \frac{\partial^2 e^{-\tau y(\tau|x)}}{\partial x^2} &= \left( \frac{\partial[\tau y(\tau|x)]}{\partial x} \right)^T \frac{\partial[\tau y(\tau|x)]}{\partial x}, \\
\frac{e^{\tau y(\tau|x)}}{2} \text{tr} \left( \sigma(x)^T \frac{\partial^2 e^{-\tau y(\tau|x)}}{\partial x^2} \sigma(x) \right) &= \frac{1}{2} \frac{\partial[\tau y(\tau|x)]}{\partial x} \sigma \sigma^T \left( \frac{\partial[\tau y(\tau|x)]}{\partial x} \right)^T.
\end{aligned} \tag{38}$$

Note that the vector  $\frac{\partial[\tau y(\tau|x)]}{\partial x}$  in view of linearity  $y(\tau|x)$  regarding the factors  $S_t, C_t, G_t, H_t$  does not

depend on these factors. In addition, the explicit form of equation (32) is in general not contain variables  $C_t$  and  $H_t$ . Thus as the equation should be carried out at any values of independent variables  $\tau, S_t, C_t, G_t, H_t$  it breaks up on two parts – the equations regarding the factors  $S_t$  and  $G_t$  and the equation regarding the function of variable  $\tau, L(\tau)$ . On current time  $t$  are dependent only  $r(t), S_t$  and  $G_t$ , other components of the equation (32) depend from  $\tau$ . Therefore from the equation (32) it is obtained two relations, determining the no-arbitrage conditions

$$\begin{aligned}
S_t + G_t &= r(t), \\
\frac{d[\tau L(\tau)]}{d \tau} &= \bar{S}(1-e^{-\gamma \tau}) + \bar{G}(1-e^{-\delta \tau}) - \bar{C} \gamma \tau e^{-\gamma \tau} - \bar{H} \delta \tau e^{-\delta \tau} - \\
&\quad - \frac{\partial[\tau y(\tau|x)]}{\partial x} \sigma \lambda - \frac{1}{2} \frac{\partial[\tau y(\tau|x)]}{\partial x} \sigma \sigma^T \left( \frac{\partial[\tau y(\tau|x)]}{\partial x} \right)^T.
\end{aligned} \tag{40}$$

The vector  $\frac{\partial[\tau y(\tau|x)]}{\partial x}$  depends only on a variable  $\tau$  and parameters  $\gamma, \delta$ . It is determined by equality

(38). Its explicit expression in the formula (40) is not used because of bulkiness.

The relation (39) is practically a condition of absence of arbitrage – the sum of factors  $S_t$  and  $G_t$  should be equal to a short-term interest rate  $r(t)$  at any point  $t$ .

The relation (40) actually the equation is not, as function  $L(\tau)$  is found from it by simple integration.

As to factors  $C_t$  and  $H_t$  to clarify their values it is necessary to address again to initial model (1), (13) in its four-dimensional version. Then according to relations (24) it is obtained

$$S_t = \phi_1 X_1(t) + \phi_2 X_2(t), C_t = \phi_1 X_2(t), G_t = \phi_3 X_3(t) + \phi_4 X_4(t), H_t = \phi_3 X_4(t).$$

Thus one of the basic properties of yield proves to be true

$$\lim_{T \rightarrow t} y(t, T, r) = \phi_1 X_1(t) + \phi_2 X_2(t) + \phi_3 X_3(t) + \phi_4 X_4(t) = r(t).$$

Strictly speaking, representation (37) does not coincide with the Svensson expansion as there is "extra" term (with factor  $G_t$ ). Probably, availability of this term will improve approximation yield curve, it is a question opened. However, it should find out, what conditions of absence of arbitrage will be without this "extra" term, taking as an expression of the yield curve

$$y(\tau) = L(\tau) + S_t \frac{1-e^{-\gamma \tau}}{\gamma \tau} + C_t \left( \frac{1-e^{-\gamma \tau}}{\gamma \tau} - e^{-\gamma \tau} \right) + H_t \left( \frac{1-e^{-\delta \tau}}{\delta \tau} - e^{-\delta \tau} \right) \tag{41}$$

in accuracy corresponding to Svensson expansion. In this case we have three factors  $X(t) = (S_t, C_t, H_t)$ . Drift function will be such

$$\mu(x) = K(\theta - x) = \begin{pmatrix} \gamma & -\gamma & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & \delta \end{pmatrix} \begin{pmatrix} \bar{S} - S_t \\ \bar{C} - C_t \\ \bar{H} - H_t \end{pmatrix} = \begin{pmatrix} \gamma(\bar{S} - \bar{C} - S_t + C_t) \\ \gamma(\bar{C} - C_t) \\ \delta(\bar{H} - H_t) \end{pmatrix}.$$

Let's substitute these expressions in the equation (32) and calculate its components.

$$\begin{aligned} \frac{\partial[\tau y(\tau|x)]}{\partial \tau} &= \frac{d[\tau L(\tau)]}{d\tau} + S_t e^{-\gamma\tau} + C_t \gamma \tau e^{-\gamma\tau} + H_t \delta \tau e^{-\delta\tau}, \\ \frac{\partial[\tau y(\tau|x)]}{\partial x} &= \left( \frac{1-e^{-\gamma\tau}}{\gamma}, \frac{1-e^{-\gamma\tau}}{\gamma} - \tau e^{-\gamma\tau}, \frac{1-e^{-\delta\tau}}{\delta} - \tau e^{-\delta\tau} \right), \\ \frac{\partial[\tau y(\tau|x)]}{\partial \tau} - \frac{\partial[\tau y(\tau|x)]}{\partial x} \mu(x) &= \frac{d[\tau L(\tau)]}{d\tau} - \\ &- \bar{S}(1-e^{-\gamma\tau}) + \bar{C} \gamma \tau e^{-\gamma\tau} - \bar{H}(1-e^{-\delta\tau}(1+\delta\tau)) + S_t + H_t(1-e^{-\delta\tau}). \end{aligned}$$

It turns out that in this case the no arbitrage conditions are reduced to the following

$$S_t + H_t(1-e^{-\delta\tau}) = r(t), \quad (42)$$

$$\begin{aligned} \frac{d[\tau L(\tau)]}{d\tau} &= \bar{S}(1-e^{-\gamma\tau}) - \bar{C} \gamma \tau e^{-\gamma\tau} + \bar{H}(1-e^{-\delta\tau}(1+\delta\tau)) - \\ &- \frac{\partial[\tau y(\tau|x)]}{\partial x} \sigma \lambda - \frac{1}{2} \frac{\partial[\tau y(\tau|x)]}{\partial x} \sigma \sigma^T \left( \frac{\partial[\tau y(\tau|x)]}{\partial x} \right)^T. \end{aligned} \quad (43)$$

The relation (43) is already known, as was met earlier, but equality (42) is unusual.

From it turns out that the short-term rate depends on the term to maturity, but it of course can't be. Therefore, the Svensson expansion in its original form (41) (without the "extra" term) leads to unrealizable no arbitrage conditions.

## Conclusion

Thus, the requirement of accomplishment of conditions no-arbitrage specifies model of Nelson – Sigel – Svensson in the sense that gives to coefficients of this model the clear economic sense: the free coefficient should be function of term to maturity  $\tau$ , and other coefficients should depend on market state variables  $\{x_i\}$  which, in turn, are sample values of stochastic processes  $\{X_i(t)\}$  at time point  $t$  for which the term structure is designed, i.e. random variables. We will notice that the description of stochastic processes  $\{X_i(t)\}$  is produced under probability measure  $P$ , i.e. taking into account risk market prices  $\lambda(x)$ . The model is the representative of family of affine yield models and is generated by two-dimensional model of short-term interest rates for model of Nelson – Sigel or four-dimensional model of a short-term interest rates for model of Nelson – Sigel – Svensson. Stochastic processes  $\{X_i(t)\}$ , underlying NS- and NSS-models, are generated by the linear stochastic differential equations. In this connection the market state variables  $\{x_i\}$  have normal distribution and can accept with positive probability negative values. That is a certain lack of models NS and NSS.

Let's notice that dependence of coefficients of NS-model on current time was discussed in [9]. Communication of NS-model with affine безарбитражными the models based on three-factorial casual process at a risk neutral  $Q$ -measure, is considered in [10]. Application affine no-arbitrage NS-models to real tasks of dynamics of an exchange rate of currencies is described in [11].

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**О безарбитражных моделях доходности Нельсона – Сигеля – Свенссона.**

**Ключевые слова:** временная структура; кривая доходности; факторная модель; аффинные безарбитражные модели; модели доходности Нельсона–Сигеля.

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Показано, что требование выполнения условий отсутствия арбитража уточняет модель Нельсона–Сигеля–Свенссона в том смысле, что придает коэффициентам этой модели явный экономический смысл: свободный коэффициент должен быть функцией срока до погашения, а остальные коэффициенты должны зависеть от переменных состояния рынка, которые, в свою очередь, являются выборочными значениями случайных процессов в момент времени, для которого конструируется временная структура. Заметим, что описание случайных процессов производится при объективной вероятностной мере, т.е. с учетом рыночных цен риска. Показано, что сама модель является представителем семейства аффинных моделей доходности и порождается двухмерной моделью краткосрочной процентной ставки для модели Нельсона–Сигеля (NS) или четырехмерной моделью краткосрочной процентной ставки для модели Нельсона–Сигеля–Свенссона (NSS). Случайные процессы, лежащие в основе NS- и NSS-моделей, порождаются линейными стохастическими дифференциальными уравнениями, в связи с чем переменные состояния рынка имеют нормальное распределение и могут с положительной вероятностью принимать отрицательные значения, что является определенным недостатком моделей NS и NSS.

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