P.V. Danchev

COMMUTATIVE FEEBLY INVO-CLEAN GROUP RINGS

A commutative ring $R$ is called feebly invo-clean if any its element is of the form $v + e - f$, where $v$ is an involution and $e, f$ are idempotents. For every commutative unital ring $R$ and every abelian group $G$ we find a necessary and sufficient condition only in terms of $R, G$ and their sections when the group ring $R[G]$ is feebly invo-clean. Our result improves two recent own achievements about commutative invo-clean and weakly invo-clean group rings, published in Univ. J. Math. & Math. Sci. (2018) and Ural Math. J. (2019), respectively.

Keywords: invo-clean rings, weakly invo-clean rings, feebly invo-clean rings, group rings.

1. Introduction and Conventions

Throughout the current paper, we will assume that all groups $G$ are multiplicative abelian and all rings $R$ with Jacobson radical $J(R)$ are associative, containing the identity element 1 which differs from the zero element 0. The standard terminology and notation are mainly in agreement with [9 and 10], whereas the specific notion and notation shall be explained explicitly below. As usual, both objects $R$ and $G$ form the group ring $RG$ of $G$ over $R$.

The next concepts appeared in [1, 2, and 3], respectively.

**Definition 1.1.** A ring $R$ is said to be invo-clean if, for each $r \in R$, there exist an involution $v$ and an idempotent $e$ such that $r = v + e$. If $r = v + e$ or $r = v - e$, the ring is called weakly invo-clean.

The next necessary and sufficient condition for a commutative ring $R$ to be invo-clean was established in [1, 2], namely: A ring $R$ is invo-clean if, and only if, $R \cong R_1 \times R_2$, where $R_1$ is a nil-clean ring with $z^2 = 2z$ for all $z \in J(R_1)$, and $R_2$ is a ring of characteristic 3 whose elements satisfy the equation $x^3 = x$. Moreover, it was proved in [6] that a ring $R$ is weakly invo-clean $\iff$ either $R$ is invo-clean or $R$ can be decomposed as $R = K \times \mathbb{Z}_5$, where $K = \{0\}$ or $K$ is invo-clean.

The above two notions could be expanded as follows:

**Definition 1.2.** A ring $R$ is said to be feebly invo-clean if, for each $r \in R$, there exist an involution $v$ and idempotents $e, f$ such that $r = v + e - f$.

We will give up in the sequel an useful criterion for a commutative ring to be feebly invo-clean in order to be successfully applied to commutative group rings (compare with Proposition 2.2).
It was asked in [6] to find a suitable criterion only in terms of the commutative unital ring \( R \) and the abelian group \( G \) when the group ring \( R[G] \) is feebly invo-clean. So, the goal of this short article is to address that question in the affirmative. Some related results in this area can also be found in [4 and 7].

2. The Characterization Result

We begin here with the following key formula from [8] which will be freely used below without concrete citation: Suppose that \( R \) is a commutative ring and \( G \) is an abelian group. Then

\[
J(R[G]) = J(R)[G] + \left\{ r(g-1)|g \in G_p, pr \notin J(R) \right\},
\]

where \( G_p \) designates the \( p \)-primary component of \( G \).

The next two technicalities are crucial for our further considerations.

Lemma 2.1. Let \( K \) be a commutative ring of characteristic 5. Then \( K \) is feebly invo-clean \( \iff \) \( x^5 = x \) holds for any \( x \in K \).

Proof. The "left-to-right" implication is almost trivial as writing \( x = v + e - f \) with \( v^2 = 1, e^2 = e \) and \( f^2 = f \), we have that \( x^5 = (v + e - f)^5 = v^5 + e^5 - f^5 = v + e - f = x \), as asserted.

As for the "right-to-left" implication, we process like this: Given an arbitrary non-identity element \( x \) in \( K \). Then the subring, \( S \), generated by 1 and \( x \) will have the same property, namely its characteristic is again 5 and \( y^5 = y \) for all \( y \in S \). So, with no harm of generality, we may replace \( K \) by this subring \( S \), and thus it needs to prove the wanted representation property in \( S \) only. To that purpose, we claim that \( S \) is isomorphic to a quotient of the factor-ring \( \mathbb{Z}_5[X]/(X^5 - X) \cong \mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_5 \) of the polynomial ring \( \mathbb{Z}_5[X] \) over \( \mathbb{Z}_5 \). In fact, we just consider the map \( \mathbb{Z}_5[X] \rightarrow S \), defined by mapping \( X \rightarrow x \), which is elementary checked to be a surjective homomorphism with kernel which contains the ideal generated by \( X^5 - X \), and henceforth the classical Homomorphism Theorem works to get the desired claim. Working now in the direct product of five copies of the five-element field \( \mathbb{Z}_5 = \{0,1,2,3,4\}5 = 0\), a plain technical argument gives our wanted initial assertion that \( S \) and hence \( K \) are both feebly invo-clean. This is subsumed by the presentations \( 0 = 1 + 0 - 1, 1 = 1 + 0 - 0, 2 = 1 + 1 - 0, 3 = 4 + 0 - 1 \) and \( 4 = 4 + 0 - 0 \), where \( 4^2 = 1, l^2 = 1 \) and \( 0^2 = 0 \).

Proposition 2.2. A commutative ring \( R \) is feebly invo-clean \( \iff \) \( R = P \times K \) for two rings \( P, K \), where \( P = \{0\} \) or \( P \) is invo-clean, and \( K = \{0\} \) or \( K \) possesses characteristic 5 such that \( x^5 = x \), \( \forall x \in K \).

Proof. "\( \Rightarrow \)". It follows from the corresponding characterization method used in [3, Theorem 2.6].

"\( \Leftarrow \)". Firstly, it needs to show that \( K \) is feebly invo-clean. This, however, follows directly from Lemma 2.1. Furthermore, one suffices to observe again with [3, Theorem 2.6] at hand that the direct product of such a ring \( K \) with an invo-clean ring remains a feebly invo-clean ring, thus getting resultantly that \( R \) is feebly invo-clean, as expected.

We are now ready to proceed by proving the following preliminary statement (see [5] as well).
Proposition 2.3. Suppose $R$ is a non-zero commutative ring and $G$ is an abelian group. Then $R[G]$ is invo-clean if, and only if, $R$ is invo-clean having the decomposition $R = R_1 \times R_2$ such that precisely one of the next three items holds:

1. $G = \{1\}$
2. $|G| > 2$, $G^2 = \{1\}$, $R_1 = \{0\}$ or $R_1$ is a ring of char $(R_1) = 2$, and $R_2 = \{0\}$, or $R_2$ is a ring of char $(R_2) = 3$
3. $|G| = 2$, $2r_1^2 = 2r_1$ for all $r_1 \in R_1$ (in addition $4 = 0$ in $R_1$), and $R_2 = \{0\}$ or $R_2$ is a ring of char $(R_2) = 3$.

Proof. If $G$ is the trivial i.e., the identity group, there is nothing to do, so we shall assume hereafter that $G$ is non-identity.

"Necessity." Since there is an epimorphism $R[G] \to R$, and an epimorphic image of an invo-clean ring is obviously an invo-clean ring (see, e.g., [1]), it follows at once that $R$ is again an invo-clean ring. According to the criterion for invo-cleanness alluded to above, one writes that $R = R_1 \times R_2$, where $R_1$ is a nil-clean ring with $a^2 = 2a$ for all $a \in J(R_1)$ and $R_2$ is a ring whose elements satisfy the equation $x^3 = x$. Therefore, it must be that $R[G] \cong R_1[G] \times R_2[G]$, where it is not too hard to verify by [1] that both $R_1[G]$ and $R_2[G]$ are invo-clean rings.

First, we shall deal with the second direct factor $R_2[G]$ being invo-clean. Since char $(R_2) = 3$, it follows immediately that char $(R_2[G]) = 3$ too. Thus an application of an assemble of facts from [1, 2] allows us to deduce that all elements in $R_2[G]$ also satisfy the equation $y^3 = y$. So, given $g \in G \subseteq R[G]$, it follows that $g^3 = g$, that is, $g^2 = 1$.

Next, we shall treat the invo-cleanness of the group ring $R_1[G]$. Since char $(R_1)$ is a power of 2 (see [1]), it follows the same for $R_1[G]$. Consequently, utilizing once again an assortment of results from [1, 2], we infer that $R_1[G]$ should be nil-clean, so that $z^2 = 2z$ for all $z \in J(R_1[G])$. That is why, invoking the criterion from [7], we have that $G$ is a 2-group. We claim that even $G^2 = 1$. In fact, for an arbitrary $g \in G$, we derive with the aid of the aforementioned formula from [8] that $1 - g \in J(R_1[G])$, because $2 \in J(R_1)$. Hence $(1 - g)^2 = 2(1 - g)$ which forces that $1 - 2g + g^2 = 2 - 2g$ and that $g^2 = 1$, as desired. We now assert that char $(R_1) = 2$ whenever $|G| > 2$. To that purpose, there are two nonidentity elements $g \neq h$ in $G$ with $g^2 = h^2 = 1$. Furthermore, again appealing to the formula from [8], the element $1 - g + 1 - h - 2g - h$ lies in $J(R_1[G])$, because $2 \in J(R_1)$. Thus $(2 - g - h)^2 = 2(2 - g - h)$ which yields that $2 - 2g - 2h + 2gh = 0$. Since $gh \neq 1$ as for otherwise $g = h^{-1} = h$, a contradiction, this record is in canonical form. This assures that $2 = 0$, as wanted.
However, in the case when $|G| = 2$, i.e. when $G = \{1, g|g^2 = 1\} = \langle g \rangle$, we can conclude that $2r^2 = 2r$ for any $r \in R_1$. Indeed, in view of the already cited formula from [8], the element $r(1-g)$ will always lie in $J(R_1[G])$, because $2 \in J(R_1)$. We therefore may write $[r(1-g)]^2 = 2r(1-g)$ which ensures that $2r^2 - 2r^2 = 2r - 2rg$ is canonically written on both sides. But this means that $2r^2 = 2r$, as pursued. Substituting $r = 2$, one obtains that $4 = 0$. Notice also that $2r^2 = 2r$ for all $r \in R_1$ and $a^2 = 2a$ for all $a \in J(R_1)$ will imply that $a^2 = 0$.

"Sufficiency." Foremost, assume that (1) is true. Since $R_1$ has characteristic $2$, whence it is nil-clean, and $G$ is a $2$-group, an appeal to [7] allows us to get that $R_1[G]$ is nil-clean as well. Since $z^2 = 2z = 0$ for every $z \in J(R_1)$, it is routinely checked that $\delta^2 = 2\delta = 0$ for each $\delta \in J(R_1[G])$, exploiting the formula from [8] for $J(R_1[G])$ and the fact that $R_1[G]$ is a modular group algebra of characteristic $2$. That is why, by a consultation with [1], one concludes that $R_1[G]$ is invo-clean, as expected. Further, by a new usage of [1], we derive that $R_2[G]$ is an invo-clean ring of characteristic $3$. To see that, given $x \in R_2[G]$, we write $x = \sum_{g \in G} r_g g$ with $r_g \in R_2$ satisfying $r_g^3 = r_g$.

Since $G^2 = 1$ will easily imply that $g^3 = g$, one obtains that $x^3 = (\sum_{g \in G} r_g g)^3 = \sum_{g \in G} r_g^3 g^3 = \sum_{g \in G} r_g g = x$, as needed. We finally conclude with the help of [1] that $R[G] \cong R_1[G] \times R_2[G]$ is invo-clean, as expected.

Let us now point (2) be fulfilled. Since $G^2 = 1$, similarly to (1), $R_2$ being invo-clean of characteristic $3$ implies that $R_2[G]$ is invo-clean, too. In order to prove that $R_1[G]$ is invo-clean, we observe that $R_1$ is nil-clean with $2 \in J(R_1)$. According to [7], the group ring $R_1[G]$ is also nil-clean. What remains to show is that for any element $\delta$ of $J(R_1[G])$ the equality $\delta^2 = 2\delta$ is valid. Since in conjunction with the explicit formula quoted above for the Jacobson radical, an arbitrary element in $J(R_1[G])$ has the form $j + j'g + r(1-g)$, where $j, j' \in J(R_1)$ and $r \in R_1$, we have that $[j + j'g + r(1-g)]^2 \in (J(R_1)^2 + 2J(R_1))[G] + r^2 (1-g)^2$. However, using the given conditions, $z^2 = 2z = 2z^2$ and thus $z^2 = 2z = 0$ for any $z \in J(R_1)$. Consequently, one checks that $[j + j'g + r(1-g)]^2 = r^2(1-g)^2 = 2r^2(1-g) = 2r(1-g) = 2[j + j'g + r(1-g)]$, because $2r^2 = 2r$, as required. Therefore, $R_1[G]$ is invo-clean with [1] at hand. Finally, again [1] gives that $R[G] \cong R_1[G] \times R_2[G]$ is invo-clean, as promised. □

It is worthwhile noticing that concrete examples of an invo-clean ring of characteristic $4$, such that its elements are solutions of the equation $2r^2 = 2r$, are the rings $\mathbb{Z}_4$ and $\mathbb{Z}_4 \times \mathbb{Z}_4$. 
We thereby come to our main theorem which states the following:

**Theorem 2.4.** Let $G$ be an abelian group and let $R$ be a commutative non-zero ring. Then the group ring $R[G]$ is feebly invo-clean if, and only if, at most one of the next points is valid:

1. $G = \{1\}$ and $R$ is feebly invo-clean.
2. $G \neq \{1\}$ and $R \cong P \times K$, where $P \cong R_1 \times R_2$ is an invo-clean ring and either $K = \{0\}$ or $K$ is a ring of char($K$) = 5 which is a subdirect product of a family of copies of the field $\mathbb{Z}_5$ such that either

   - (2.1) $P = \{0\}$ and $G^4 = \{1\}$
   - or
   - (2.2) $|G| > 2$, $G^2 = \{1\}$, $P \neq \{0\}$ with $R_1 = \{0\}$ or $R_1$ is a ring of char($R_1$) = 2 and $R_2 = \{0\}$ or $R_2$ is a ring of char($R_2$) = 3
   - or
   - (2.3) $|G| = 2$, $P \neq \{0\}$ with $2r_1^2 = 2r_1$ for all $r_1 \in R_1$ (in addition $4 = 0$ in $R_1$) and $R_2 = \{0\}$ or $R_2$ is a ring of char($R_2$) = 3.

**Proof.** If $G$ is trivial, there is nothing to prove because of the validity of the isomorphism $R[G] \cong R$, so let us assume hereafter that $G$ is non-trivial.

"Necessity." As the feebly invo-cleanness of the group ring $R[G]$ implies the same property for $R$, utilizing Proposition 2.2 we come to the fact that $R[G] \cong P[G] \times K[G]$ will imply feebly invo-cleaness of both group rings $P[G]$ and $K[G]$ whence $P[G]$ is necessarily invo-clean whereas $K[G]$ is either zero or a subdirect product of a family of copies of the field $\mathbb{Z}_5$. After that, under the presence of $P[G] \neq \{0\}$, we just need apply Proposition 2.3 to deduce the described above things in points (2), (2.2) and (2.3). Letting now $P[G] = \{0\}$, we shall deal only with $K[G]$. To that goal, what we now assert is that the group ring $K[G]$ having the property $x^5 = x$ for all $x \in K[G]$ with char($K[G]$) = 5 yields that $K$ has the property $y^5 = y$ for all $y \in K$ with char($K$) = 5 and $G^4 = \{1\}$. Indeed, since $K \subseteq K[G]$ and $G \subseteq K[G]$, this can be extracted elementarily thus substantiating our initial statement after all.

"Sufficiency." Item (2) ensures that $R[G] \cong P[G] \times K[G]$ and so it is simple verified that the feebly invo-cleanness of both $P[G]$ and $K[G]$ will assure feebly invo-cleanness of $R[G]$ as well. That is why, we will be concentrated separately on these two group rings. Firstly, the stated above conditions are a guarantor with the aid of Proposition 2.3 that $P[G]$ is invo-clean. Secondly, it is pretty easily seen that as $y^5 = y$ and $g^5 = g$ for all $y \in K$ and $g \in G$, because $K$ is a subdirect product of copies of the field $\mathbb{Z}_5$ possessing characteristic 5 and $G^4 = \{1\}$, we may conclude that $x^5 = x$ holds in $K[G]$ too, as required. This substantiates our former assertion after all. $\square$
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DANCHEV Peter V. (Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Sofia, Bulgaria). E-mail: pvdanchev@yahoo.com, danchev@math.bas.bg

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