UDC 512.53 DOI 10.17223/20710410/38/3

ON IRREDUCIBLE ALGEBRAIC SETS OVER LINEARLY ORDERED SEMILATTICES II[1]

A. N. Shevlyakov

Sobolev Institute of Mathematics, Omsk, Russia;
Omsk State Technical University, Omsk, Russia

Equations over finite linearly ordered semilattices are studied. It is assumed that the order of a semilattice is not less than the number of variables in an equation. For any equation $t(X) = s(X)$, we find irreducible components of its solution set. We also compute the average number $\overline{\mathrm{Irr}}(n)$ of irreducible components for all equations in n variables. It turns out that $\overline{\mathrm{Irr}}(n)$ and the function $\frac{4}{9}n!$ are asymptotically equivalent.

Keywords: *irreducible components, algebraic sets, semilattices.*

Introduction

This paper is the sequel of [1], and we recall below the general problems studied in both papers.

Following [2], one can define a notion of an equation over a linearly ordered semilattice $L_l = \{a_1, a_2, \ldots, a_l\}$ (the formal definition of an equation is given below in the paper). A set Y is *algebraic* if it is the solution set for a system of equations over L_l. Let us consider an equation $t(X) = s(X)$ in n variables over L_l, and let Y be the solution set for $t(X) = s(X)$. One can find algebraic sets $Y_1, Y_2, \ldots, Y_m$ such that $Y = \bigcup_{i=1}^{m} Y_i$. One can decompose each Y_i into a union of other algebraic sets, etc. This process terminates after a finite number of steps and gives a decomposition of Y into a union of *irreducible* algebraic sets Y_i (the sets Y_i are called the *irreducible components* of Y). Roughly speaking, irreducible algebraic sets are "atoms" which form any algebraic set. The size and the number of such "atoms" are important characteristics of the semilattice L_l, since there are connections between irreducible algebraic sets and universal theory of linearly ordered semilattices [2]. Moreover, the number of irreducible components was involved in the estimation of lower bounds of algorithm complexity (see [3] for more details).

In the previous paper [1], we studied equations $t(X) = s(X)$ with $n > l$, i.e. the number of variables occurring in $t(X) = s(X)$ is more than the order of the semilattice L_l. In [1], we also studied algebraic sets and irreducible components and computed the average number of irreducible components of the solution sets for equations in n variables.

In this paper, we assume $n \leqslant l$ (i.e. the order of the semilattice L_l is not less than the number of variables in $t(X) = s(X)$) and study the similar problems. Precisely, for any equation $t(X) = s(X)$ in n variables, we study the number and properties of its solution set irreducible components, and for all equations in n variables, we count the average number $\overline{\mathrm{Irr}}(n)$ of irreducible components of the solution sets.

Note that the cases $n > l$ and $n \leqslant l$ need a completely different techniques, and we can not directly use the results of [1] in the current paper. Moreover, almost all the results of [1] do not hold for the current case.

[1]The author was supported by the RSF-grant 17-11-01117.

1. Main definitions

Let $L_l = \{a_1, a_2, \ldots, a_l\}$ be the linearly ordered semilattice of l elements and $a_1 < a_2 < \ldots < a_l$. The multiplication in L_l is defined by $a_i \cdot a_j = a_{\min(i,j)}$. Obviously, the linear order on L_l can be expressed by the multiplication as follows

$$a_i \leqslant a_j \Leftrightarrow a_i a_j = a_i.$$

A *term* $t(X)$ *in variables from* $X = \{x_1, x_2, \ldots, x_n\}$ is a commutative word in letters x_i.

Let $\mathrm{Var}(t)$ be the set of all variables occurring in a term $t(X)$. Following [2], an *equation* is an equality of some terms $t(X) = s(X)$. Below we consider inequalities $t(X) \leqslant s(X)$ as equations, since $t(X) \leqslant s(X)$ is the short form of $t(X)s(X) = t(X)$. Notice that we consider equations as *ordered pairs* of terms, i.e. the expressions $t(X) = s(X)$ and $s(X) = t(X)$ are *different* equations. Let $Eq(n)$ denote the set of all equations in variables from $X = \{x_1, x_2, \ldots, x_n\}$. We assume that each equation $t(X) = s(X)$ in $Eq(n)$ contains the occurrences of all variables $x_1, x_2, \ldots, x_n$. An equation $t(X) = s(X)$ in $Eq(n)$ is said to be a (k_1, k_2)*-equation* if $|\mathrm{Var}(t) \setminus \mathrm{Var}(s)| = k_1$ and $|\mathrm{Var}(s) \setminus \mathrm{Var}(t)| = k_2$. For example, $x_1x_2 = x_1x_3x_4$ is a $(1, 2)$-equation. Let $Eq(k_1, k_2, n)$ be the set of all (k_1, k_2)-equations in $Eq(n)$. Obviously,

$$Eq(n) = \bigcup_{(k_1,k_2)\in K_n} Eq(k_1, k_2, n), \tag{1}$$

where $K_n = \{(k_1, k_2) : k_1 + k_2 \leqslant n\} \setminus \{(0, n), (n, 0)\}$.

Each equation $t(X) = s(X)$ in $Eq(k_1, k_2, n)$ is uniquely defined by k_1 variables in the left part and by k_2 other variables in the right part (the other $n - k_1 - k_2$ variables should occur in both parts of the equation). Thus,

$$\#Eq(k_1, k_2, n) = \binom{n}{k_1}\binom{n-k_1}{k_2}.$$

By (1), one can compute that $\#Eq(n) = 3^n - 2$.

Remark 1. Recall that we consider only equations $t(X) = s(X)$ with $n \leqslant l$, i.e. the number of variables occurring in $t(X) = s(X)$ is not more than the order of the semilattice L_l.

A point $P \in L_l^n$ is a *solution* of an equation $t(X) = s(X)$ if $t(P)$ and $s(P)$ define the same element in the semilattice L_l. By the properties of linearly ordered semilattices, a point $P = (p_1, p_2, \ldots, p_n)$ is a solution of $t(X) = s(X)$ iff there exist variables x_i in $\mathrm{Var}(t)$ and x_j in $\mathrm{Var}(s)$ such that $p_i = p_j$ and $p_i \leqslant p_k$ for all k, $1 \leqslant k \leqslant n$. The set of all solutions of an equation $t(X) = s(X)$ is denoted by $\mathrm{V}(t(X) = s(X))$.

An arbitrary set of equations is called a *system*. The set $\mathrm{V}(\mathbf{S})$ of all solutions of a system $\mathbf{S} = \{t_i(X) = s_i(X) : i \in I\}$ is defined as $\bigcap_{i\in I} \mathrm{V}(t_i(X) = s_i(X))$. A subset Y of the set L_l^n is called *algebraic over* L_l if there exists a system $\mathbf{S}$ in n variables with $\mathrm{V}(\mathbf{S}) = Y$. An algebraic set Y is *irreducible* if Y is not a proper finite union of other algebraic sets.

Proposition 1 [1, Proposition 2.2]. Any algebraic set Y over L_l is a finite union of irreducible sets, that is,

$$Y = Y_1 \cup Y_2 \cup \ldots \cup Y_m, \tag{2}$$

where $Y_i \not\subseteq Y_j$ for all i and j such that $i \neq j$, and this decomposition is unique up to a permutation of components.

The subsets Y_i from the union (2) are called the *irreducible components* of Y.

Let Y be an algebraic set over L_l defined by a system $\mathbf{S}(X)$. One can define an equivalence relation $\sim_Y$ over the set of all terms in variables X as follows

$$t(X) \sim_Y s(X) \Leftrightarrow t(P) = s(P) \text{ for any point } P \in Y.$$

The set of all $\sim_Y$-equivalence classes is called *the coordinate semilattice of* Y and denoted by $\Gamma(Y)$ (see [2] for more details). The following statement describes the coordinate semilattices of irreducible algebraic sets.

Proposition 2 [1, Proposition 2.3]. A set Y is irreducible over L_l iff $\Gamma(Y)$ is embedded into L_l.

There are different algebraic sets over L_l with isomorphic coordinate semilattices. Such sets are called *isomorphic*. For example, the following sets

$$Y_1 = \mathrm{V}(\{x_1 \leqslant x_2 \leqslant x_3\}),\ Y_2 = \mathrm{V}(\{x_3 \leqslant x_2 \leqslant x_1\})$$

have the isomorphic coordinate semilattices

$$\Gamma(Y_1) = \langle x_1, x_2, x_3 \mid x_1 \leqslant x_2 \leqslant x_3\rangle \cong L_3,$$
$$\Gamma(Y_2) = \langle x_1, x_2, x_3 \mid x_3 \leqslant x_2 \leqslant x_1\rangle \cong L_3.$$

Thus, Y_1 and Y_2 are isomorphic.

2. Example

Let $n = 3$, $l = 3$. We have exactly $Eq(3) = 3^3 - 2 = 25$ equations in three variables over L_3. The Table on the page 52 contains the information about such equations over L_3. The second column contains systems which define irreducible components of the solution set for an equation in the first column. A cell of the table contains $\uparrow$ if an information in this cell is similar to the cell above.

Notice that $\mathrm{V}(x_1 = x_2 \leqslant x_3)$ does not define an irreducible component for $Y = \mathrm{V}(x_1x_2 = x_1x_3)$, since $\mathrm{V}(x_1 = x_2 \leqslant x_3)$ is included into the solution set of another irreducible component $\mathrm{V}(x_1 \leqslant x_2 \leqslant x_3)$. Similarly, $\mathrm{V}(x_3 = x_1 \leqslant x_2)$ is not an irreducible component for Y, since it is contained in the irreducible component $\mathrm{V}(x_1 \leqslant x_3 \leqslant x_2)$.

It turns out that the number of irreducible components does not depend on the semilattice order l. One can directly compute the average number of irreducible components of algebraic sets defined by equations in three variables:

$$\overline{\mathrm{Irr}}(3) = \frac{6 + 2(2+2+2+2+2+2+3+3+3+4+4+4)}{25} = \frac{72}{25} = 2.88. \qquad (3)$$

Equations	Irreducible components (IC)	Number of IC
$x_1x_2x_3 = x_1x_2x_3$	$x_1 \leqslant x_2 \leqslant x_3 \cup x_1 \leqslant x_3 \leqslant x_2 \cup$ $\cup x_2 \leqslant x_1 \leqslant x_3 \cup x_2 \leqslant x_3 \leqslant x_2 \cup$ $\cup x_3 \leqslant x_1 \leqslant x_2 \cup x_3 \leqslant x_2 \leqslant x_1$	6
$x_1 = x_1x_2x_3,$ $x_1x_2x_3 = x_1$	$x_1 \leqslant x_2 \leqslant x_3 \cup x_1 \leqslant x_3 \leqslant x_1$	2
$x_2 = x_1x_2x_3,$ $x_1x_2x_3 = x_2$	$\uparrow$	2
$x_3 = x_1x_2x_3,$ $x_1x_2x_3 = x_3$	$\uparrow$	2
$x_1 = x_2x_3,$ $x_2x_3 = x_1$	$x_1 = x_2 \leqslant x_3 \cup x_1 = x_3 \leqslant x_2$	2
$x_2 = x_1x_3,$ $x_1x_3 = x_2$	$\uparrow$	2
$x_3 = x_1x_2,$ $x_1x_2 = x_3$	$\uparrow$	2
$x_1x_2 = x_1x_3,$ $x_1x_3 = x_1x_2$	$x_1 \leqslant x_2 \leqslant x_3 \cup x_1 \leqslant x_3 \leqslant x_2 \cup$ $\cup x_2 = x_3 \leqslant x_1$	3
$x_1x_2 = x_2x_3,$ $x_2x_3 = x_1x_2$	$\uparrow$	3
$x_1x_3 = x_2x_3,$ $x_2x_3 = x_1x_3$	$\uparrow$	3
$x_1x_2 = x_1x_2x_3,$ $x_1x_2x_3 = x_1x_2$	$x_1 \leqslant x_2 \leqslant x_3 \cup x_1 \leqslant x_3 \leqslant x_2 \cup$ $\cup x_2 \leqslant x_1 \leqslant x_3 \cup x_2 \leqslant x_3 \leqslant x_1$	4
$x_1x_3 = x_1x_2x_3,$ $x_1x_2x_3 = x_1x_3$	$\uparrow$	4
$x_2x_3 = x_1x_2x_3,$ $x_1x_2x_3 = x_2x_3$	$\uparrow$	4

3. Decompositions of algebraic sets

Let Y denote the solution set for an equation $t(X) = s(X)$ over the semilattice $L_l = \{a_1, a_2, \dots, a_l\}$. The table on the page 52 shows that any irreducible component sorts the variables X into some order. The following definition formalizes this property of irreducible components.

Let σ be a permutation of the set $\{1, 2, \dots, n\}$; σ sorts the set X as follows: $\{x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)}\}$, i.e. $\sigma(i)$ is the i-th variable in the sorted set X. A permutation σ is called a *permutation of the first* (*second*) *kind* if $x_{\sigma(1)} \in \mathrm{Var}(t) \cap \mathrm{Var}(s)$ (respectively, $x_{\sigma(2)} \in \mathrm{Var}(t) \setminus \mathrm{Var}(s)$, $x_{\sigma(1)} \in \mathrm{Var}(s) \setminus \mathrm{Var}(t)$). Let $\chi(\sigma) \in \{1, 2\}$ denote the kind of a permutation σ.

Example 1. Let us consider an algebraic set $Y_0 = \mathrm{V}(x_1x_2 = x_1x_3)$. By the table, Y_0 is the union of the following irreducible components:

$$Y_1 = \mathrm{V}(x_1 \leqslant x_2 \leqslant x_3),\ Y_2 = \mathrm{V}(x_1 \leqslant x_3 \leqslant x_2),\ Y_3 = \mathrm{V}(x_2 = x_3 \leqslant x_1).$$

The irreducible components Y_1, Y_2, Y_3 define the following permutations:

$$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix},\ \sigma_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix},\ \sigma_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}.$$

Moreover, σ_1 and σ_2 are permutations of the first kind, whereas σ_3 is of the second kind.

A permutation σ defines an algebraic set Y_σ as follows:

$$Y_\sigma = \mathrm{V}\left(\bigcup_{i=1}^{n-1} \{x_{\sigma(i)} \leqslant x_{\sigma(i+1)}\}\right) \tag{4}$$

if $\chi(\sigma) = 1$, and

$$Y_\sigma = \mathrm{V}\left(\{x_{\sigma(1)} = x_{\sigma(2)}\} \bigcup_{i=2}^{n-1} \{x_{\sigma(i)} \leqslant x_{\sigma(i+1)}\}\right) \tag{5}$$

if $\chi(\sigma) = 2$.

Example 2. Let $\sigma_1, \sigma_2, \sigma_3$ be permutations from Example 1. Obviously, the sets $Y_{\sigma_1}, Y_{\sigma_2}, Y_{\sigma_3}$ defined by (4) and (5) coincide with the sets Y_1, Y_2, Y_3 respectively.

Lemma 1. Let $\chi(\sigma) \in \{1, 2\}$, then the set Y_σ is irreducible and, moreover,

$$\Gamma(Y_\sigma) \cong \begin{cases} L_n, \text{ if } \chi(\sigma) = 1, \\ L_{n-1}, \text{ if } \chi(\sigma) = 2. \end{cases} \tag{6}$$

Proof. By the definition of a coordinate semilattice, $\Gamma(Y_\sigma)$ is generated by the elements of $\{x_1, x_2, \ldots, x_n\}$ and has the following defined relations:

$$x_{\sigma(1)} \leqslant x_{\sigma(2)} \leqslant \ldots x_{\sigma(n)} \text{ if } \chi(Y_\sigma) = 1$$

and

$$x_{\sigma(1)} = x_{\sigma(2)} \leqslant \ldots x_{\sigma(n)} \text{ if } \chi(Y_\sigma) = 2.$$

Thus, $\Gamma(Y_\sigma)$ is a linearly ordered semilattice, and (6) holds. By Proposition 2, the set Y_σ is irreducible. ∎

The following lemma gives the irreducible decomposition of an algebraic set $Y = \mathrm{V}(t(X) = s(X))$.

Lemma 2. An algebraic set $Y = \mathrm{V}(t(X) = s(X))$ is a union

$$Y = \bigcup_{\chi(\sigma)\in\{1,2\}} Y_\sigma. \tag{7}$$

Proof. Suppose $P = (p_1, p_2, \ldots, p_n) \in Y$. Let us sort p_i in the ascending order

$$p_{\sigma(1)} \leqslant p_{\sigma(2)} \leqslant \ldots \leqslant p_{\sigma(n)},$$

where σ is a permutation of the set $\{1, 2, \ldots, n\}$. We have that σ induces the sorting of the variable set X. Obviously, we may assume that $x_\sigma(1) \in \mathrm{Var}(t)$, otherwise the properties of L_l provide the existence of a variable $x_\sigma(i) \in \mathrm{Var}(t)$ such that $p_{\sigma(i)} = p_{\sigma(1)}$, and we can swap the values $\sigma(1)$ and $\sigma(i)$.

For example, the point $P = (a_2, a_1, a_1) \in \mathrm{V}(x_1x_2 = x_1x_3)$ defines $\sigma(1) = 2$, $\sigma(2) = 3$, $\sigma(3) = 1$ (the permutation obtained equals σ_3 from Example 1, so the point (a_2, a_1, a_1) belongs to the set Y_3).

Since σ is defined by the inequalities between the coordinates p_i, it follows $P \in Y_\sigma$.

Now we prove that $Y_\sigma \subseteq Y$ for each σ. Suppose $P = (p_1, p_2, \ldots, p_n) \in Y_\sigma$. If $\chi(Y_\sigma) = 1$, then

$$x_{\sigma(1)} \in \mathrm{Var}(t) \cap \mathrm{Var}(s) \Rightarrow t(P) = s(P) = p_{\sigma(1)} \Rightarrow P \in \mathrm{V}(t(X) = s(X)).$$

Otherwise ($\chi(Y_\sigma) = 2$), $t(P) = p_{\sigma(1)}$, $s(P) = p_{\sigma(2)}$, and (5) gives $p_{\sigma(1)} = p_{\sigma(2)}$. Therefore $P \in \mathrm{V}(t(X) = s(X))$. ∎

Lemma 3. For distinct permutations σ and σ', we have $Y_\sigma \nsubseteq Y_{\sigma'}$ in (7).

Proof. Let σ be a permutation of the first or second kind, and P_σ denote the following point:

$$p_{\sigma(i)} = a_i \text{ if } \chi(\sigma) = 1,$$

and

$$p_{\sigma(i)} = \begin{cases} a_i, \ 2 \leqslant i \leqslant n, \\ a_2, \ i = 1, \end{cases} \quad \text{if } \chi(\sigma) = 2.$$

For example, the permutations $\sigma_1, \sigma_2, \sigma_3$ from Example 1 define the points $P_1 = (a_1, a_2, a_3)$, $P_2 = (a_1, a_3, a_2)$, $P_3 = (a_3, a_2, a_2)$, respectively.

Since P_σ preserves the order of variables, we have $P_\sigma \in Y_\sigma$.

Now we can show that $P_\sigma \notin Y_{\sigma'}$ for every $\sigma' \neq \sigma$ (for example, each of the points P_1, P_2, P_3 above belongs to a unique irreducible component from Example 1:

$$P_1 \in Y_1 \setminus (Y_2 \cup Y_3), \ P_2 \in Y_2 \setminus (Y_1 \cup Y_3), \ P_3 \in Y_3 \setminus (Y_1 \cup Y_2)).$$

There exist numbers i and j such that $i < j$, $i = \sigma(\alpha)$, $j = \sigma(\beta)$ with $\alpha < \beta$ and $i = \sigma'(\alpha')$, $j = \sigma'(\beta')$ with $\alpha' > \beta'$. Hence, the inequality $x_i \leqslant x_j$ holds in Y_σ, and the inequality $x_j \leqslant x_i$ holds in $Y_{\sigma'}$. Let us consider the two possible cases:

1) If $\chi(\sigma) = 1$, then $p_i < p_j$ in P_σ, and we immediately obtain $P_\sigma \notin Y_{\sigma'}$.
2) Suppose $\chi(\sigma) = 2$. Assume that $p_i = p_j = a_2$ (if $p_i < p_j$, we immediately obtain $P_\sigma \notin Y_{\sigma'}$). Then $\alpha = 1$, $\beta = 2$ and $i = \sigma(1)$, $j = \sigma(2)$ (one can similarly consider the case $i = \sigma(2)$, $j = \sigma(1)$). Hence, $x_i \in \mathrm{Var}(t) \setminus \mathrm{Var}(s)$, $x_j \in \mathrm{Var}(s) \setminus \mathrm{Var}(t)$. By the definition of a permutation of the second kind, $\sigma'(1) = k \neq j$, and the inequality $x_k \leqslant x_j$ holds in $Y_{\sigma'}$. Let $\sigma(\gamma) = k$. Since $\alpha = 1$, $\beta = 2$, we have $\gamma > 2$. Then $p_k = a_\gamma$ and $p_j < p_k$ for P_σ. Thus, $P \notin Y_{\sigma'}$.

The Lemma 3 is proved. ■

According to Lemmas 1–3, we obtain the following statement.

Theorem 1. The union (7) is the irreducible decomposition of the set $Y = \mathrm{V}(t(X) = s(X))$. The number of irreducible components is equal to the number of permutations of the first and second kind.

4. Average number of irreducible components

One can directly compute that any (k_1, k_2)-equation admits

$$(n - k_1 - k_2)(n - 1)!$$

permutations of the first kind and

$$k_1 k_2 (n - 2)!$$

permutations of the second kind.

By Theorem 1, for a (k_1, k_2)-equation $t(X) = s(X)$ the number of its irreducible components equals

$$\mathrm{Irr}(k_1, k_2, n) = (n - k_1 - k_2)(n - 1)! + k_1 k_2 (n - 2)!$$

The average number of irreducible components of algebraic sets defined by equations from $Eq(n)$ is

$$\overline{\mathrm{Irr}}(n)=\frac{\sum\limits_{(k_1,k_2)\in K_n}\#Eq(k_1,k_2,n)\mathrm{Irr}(k_1,k_2,n)}{\#Eq(n)}=$$

$$=\frac{\sum\limits_{k_1=0}^{n-1}\sum\limits_{k_2=0}^{n-k_1}\#Eq(k_1,k_2,n)\mathrm{Irr}(k_1,k_2,n)-\#Eq(0,n,n)\mathrm{Irr}(0,n,n)}{\#Eq(n)}.$$

Since $\mathrm{Irr}(0,n,n)=(n-0-n)(n-1)!+0n(n-2)!=0$, we obtain

$$\overline{\mathrm{Irr}}(n)=\frac{\sum\limits_{k_1=0}^{n-1}\sum\limits_{k_2=0}^{n-k_1}\#Eq(k_1,k_2,n)\mathrm{Irr}(k_1,k_2,n)}{\#Eq(n)}.$$

Below we compute $\overline{\mathrm{Irr}}$ using the following notation:

1) $A\overset{(1)}{=}B$: an expression B is obtained from A by the binomial identity

$$a\binom{n}{a}=n\binom{n-1}{a-1};$$

2) $A\overset{(2)}{=}B$: an expression B is obtained from A by the following identity of binomial coefficients

$$\sum_{t=0}^{n}\binom{n}{t}t2^t=2n3^{n-1}. \tag{8}$$

Here is a proof of (8):

$$\sum_{t=0}^{n}\binom{n}{t}t2^t\overset{(1)}{=}n\sum_{t=0}^{n}\binom{n-1}{t-1}2^t=2n\sum_{t=0}^{n}\binom{n-1}{t-1}2^{t-1}=2n\sum_{u=0}^{n-1}\binom{n-1}{u}2^u=2n3^{n-1}.$$

Let us compute $\overline{\mathrm{Irr}}(n)$. We have that

$$\sum_{k_1=0}^{n-1}\sum_{k_2=0}^{n-k_1}\#Eq(k_1,k_2,n)\mathrm{Irr}(k_1,k_2,n)=$$

$$=\sum_{k_1=0}^{n-1}\sum_{k_2=0}^{n-k_1}\binom{n}{k_1}\binom{n-k_1}{k_2}\left((n-k_1-k_2)(n-1)!+k_1k_2(n-2)!\right)=$$

$$=n!\sum_{k_1=0}^{n-1}\sum_{k_2=0}^{n-k_1}\binom{n}{k_1}\binom{n-k_1}{k_2}-(n-1)!\sum_{k_1=0}^{n-1}\sum_{k_2=0}^{n-k_1}\binom{n}{k_1}\binom{n-k_1}{k_2}k_1-$$

$$-(n-1)!\sum_{k_1=0}^{n-1}\sum_{k_2=0}^{n-k_1}\binom{n}{k_1}\binom{n-k_1}{k_2}k_2+(n-2)!\sum_{k_1=0}^{n-1}\sum_{k_2=0}^{n-k_1}\binom{n}{k_1}\binom{n-k_1}{k_2}k_1k_2=S_1-S_2-S_3+S_4,$$

where

$$S_1=n!\sum_{k_1=0}^{n-1}\sum_{k_2=0}^{n-k_1}\binom{n}{k_1}\binom{n-k_1}{k_2}=n!\sum_{k_1=0}^{n-1}\binom{n}{k_1}2^{n-k_1}=n!(3^n-1),$$

$$S_2=(n-1)!\sum_{k_1=0}^{n-1}\sum_{k_2=0}^{n-k_1}\binom{n}{k_1}\binom{n-k_1}{k_2}k_1=(n-1)!\sum_{k_1=0}^{n-1}\binom{n}{k_1}k_12^{n-k_1}\overset{(1)}{=}$$

$$\overset{(1)}{=} n!\sum_{k_1=0}^{n-1}\binom{n-1}{k_1-1}2^{n-k_1} = n!\sum_{t=0}^{n-2}\binom{n-1}{t}2^{n-1-t} = n!\left(\sum_{t=0}^{n-1}\binom{n-1}{t}2^{n-1-t}-1\right) = n!(3^{n-1}-1),$$

$$S_3 = (n-1)!\sum_{k_1=0}^{n-1}\sum_{k_2=0}^{n-k_1}\binom{n}{k_1}\binom{n-k_1}{k_2}k_2 \overset{(1)}{=} (n-1)!\sum_{k_1=0}^{n-1}\binom{n}{k_1}(n-k_1)\sum_{k_2=0}^{n-k_1}\binom{n-k_1-1}{k_2-1} =$$
$$= (n-1)!\sum_{k_1=0}^{n-1}\binom{n}{k_1}(n-k_1)2^{n-k_1-1} = (n-1)!\sum_{t=0}^{n}\binom{n}{t}t2^{t-1} = \frac{(n-1)!}{2}\sum_{t=0}^{n}\binom{n}{t}t2^t \overset{(2)}{=} n!3^{n-1},$$

$$S_4 = (n-2)!\sum_{k_1=0}^{n-1}\sum_{k_2=0}^{n-k_1}\binom{n}{k_1}\binom{n-k_1}{k_2}k_1k_2 \overset{(1)}{=} (n-2)!\sum_{k_1=0}^{n-1}\binom{n}{k_1}k_1(n-k_1)\sum_{k_2=0}^{n-k_1}\binom{n-k_1-1}{k_2-1} =$$
$$= (n-2)!\sum_{k_1=0}^{n-1}\binom{n}{k_1}k_1(n-k_1)2^{n-k_1-1} =$$
$$= \frac{(n-2)!}{2}\sum_{k_1=0}^{n}\binom{n}{k_1}k_1(n-k_1)2^{n-k_1} = \frac{(n-2)!}{2}\sum_{t=0}^{n}\binom{n}{t}t(n-t)2^t =$$
$$= \frac{(n-2)!}{2}\left(n\sum_{t=0}^{n}\binom{n}{k_1}t2^t - \sum_{t=0}^{n}\binom{n}{t}t^22^t\right) \overset{(2)}{=} \frac{(n-2)!}{2}\left(2n^23^{n-1}-S_5\right),$$

$$S_5 = \sum_{t=0}^{n}\binom{n}{k_1}t^22^t \overset{(1)}{=} n\sum_{t=0}^{n}\binom{n-1}{t-1}t2^t = n\left(\sum_{t=0}^{n}\binom{n-1}{t-1}(t-1)2^t + \sum_{t=0}^{n}\binom{n-1}{t-1}2^t\right) =$$
$$= n\left(2\sum_{t=0}^{n}\binom{n-1}{t-1}(t-1)2^{t-1} + \sum_{t=0}^{n}\binom{n-1}{t-1}2^t\right) \overset{(2)}{=} n\left(4(n-1)3^{n-2}+2\cdot 3^{n-1}\right).$$

Finally, we obtain that

$$S_1 - S_2 - S_3 + S_4 = n!(3^n-1) - n!(3^{n-1}-1) - n!3^{n-1}+$$
$$+\frac{(n-2)!}{2}\left(2n^23^{n-1} - n(4(n-1)3^{n-2}+2\cdot 3^{n-1})\right) =$$
$$= n!3^{n-1} + (n-2)!3^{n-2}n\left(3n-2(n-1)-3\right) = n!3^{n-1} + n!3^{n-2} = 4n!3^{n-2},$$

and

$$\overline{\mathrm{Irr}}(n) = \frac{4n!3^{n-2}}{3^n-2} \sim \frac{4}{9}n! \tag{9}$$

Notice that the final answer does not depend on l if $l \leqslant n$. In particular, (9) gives

$$\overline{\mathrm{Irr}}(3) = \frac{72}{25} = 2.88 \tag{10}$$

for $n = 3$, and (10) coincides with (3).

REFERENCES

1. *Shevlyakov A. N.* On irreducible algebraic sets over linearly ordered semilattices. Groups, Complexity, Cryptology, 2016, vol. 8, no. 2, pp. 187–196.
2. *Daniyarova E. Yu., Myasnikov A. G., and Remeslennikov V. N.* Algebraicheskaya geometriya nad algebraicheskimi sistemami [Algebraic Geometry over Algebraic Systems]. Novosibirsk, SB RAS Publ., 2016. 243 p. (in Russian)
3. *Ben-Or M.* Lower bounds for algebraic computation trees. 15th Ann. Symp. Theory Computing, 1983, pp. 80–86.