ВЕСТНИК ТОМСКОГО ГОСУДАРСТВЕННОГО УНИВЕРСИТЕТА
2018 Математика и механика № 56

УДК 519.81, 519.21 MSC: 60H10, 60G44, 60J65
DOI 10.17223/19988621/56/3

A.A. Shishkova

THE HEDGING STRATEGY FOR ASIAN OPTION[1]

The article deals with the problem of portfolio investment in the Black-Scholes model with several risky assets. The hedging strategy for Asian option is found using the martingale method. The analytical properties (differentiability) of the densities of exponential random variables are studied.

Keywords: *hedging strategy*, *Asian option*, *stochastic differential equations*, *Brownian motion*, *Black and Scholes model.*

Introduction

The world financial system is continuously developing, which causes its ever increasing fragmentation. The derivatives market as one of the elements of the system develops even more rapidly. The first derivatives were currency futures and forwards, emerged in the early 70's a little later there were options [1].

After the first option transaction, which took place in 1973 on the Chicago Board of Options Exchange [2], a revolution in the development of option trading began. By the end of the 1970's options were well studied on stock exchange and then new exotic options appeared. In the late 1980's and early 1990's exotic options became more in demand and their trade became more active in the over-the-counter market. Soon in the commodity and currency markets Asian options are becoming popular.

Geman and Yor have considered Asian options in their work [3], such derivatives are based on the average prices of underlying assets. Using the Bessel processes authors found the value of the Asian option. Moreover, applying simple probabilistic methods they obtained the following results about these options: calculated moments of all orders of the arithmetic average of the geometric Brownian motion; obtained simple, closed form expression of the Asian option price when the option is "in the money".

The exact pricing of fixed-strike Asian options is a difficult task, since the distribution of the average arithmetic of asset prices is unknown when its prices are distributed lognormally. The study of this problem is divided into three groups. A large number of works are connected with the numerical approach. Kemna and Vorst were among the first who solved the task [4]. In their work the pricing strategy includes Monte Carlo simulation with elements of dispersion reduction and improves the pricing method. Furthermore, the authors showed that the price of an option with an average value will always be lower than of a standard European option. Carverhill and Clewlow [5] used a fast Fourier transform to calculate the density of the sum of random variables, as convolution of individual densities. Then the payoff function is numerically integrated against the density. In this direction other authors continued to work, applying to the calculations improved methods of numerical simulation [6–8]. Unfortunately, these methods do not provide information on the hedging portfolio.

---

[1] This work was supported by the Ministry of Education and Science of the Russian Federation, state assignment No.2.3208.2017/4.6.

The second approach, used by Ruttiens [9] and Vorst [10], is to change the geometric average price of the option. The third approach proposed by Levy [11] and accepted by some practitioners, suggests that the distribution of the arithmetic average is well approximated at least in some markets by a lognormal distribution, and therefore the problem is reduced to determining the necessary parameters. This problem is less complicated since the first two moments of the arithmetic mean are relatively simple.

For a trader or an investor the main task is not only the saving but also the multiplication of its capital. Many risks can be avoided with the help of one popular and very effective technique – hedging. The option is hedged to protect its value from the risk of price movement of the underlying asset in an unfavorable direction. To solve the hedging problem stochastic calculus methods are used which became a powerful tool used in practice in the financial world. Stochastic calculus is a well-developed branch of modern mathematics with a "correct" approach to analyzing complex phenomena occurring on world stock markets. Book of A.N. Shiryaev [12] provides a complete and systemic view of ideas and techniques in stochastic finance. It should be noted that the task of options pricing and the construction of a hedging strategy is well studied for American and European options, for such derivatives there is a so-called delta strategy. But this technique is not developed for Asian options.

In this paper we consider the financial portfolio with several risky assets. Based on the results presented in the article [14] we have solved the hedging problem for the Asian option using martingale methods. The main result of the paper is the formulas for the hedging strategy $\gamma(t)=(\gamma_1(t),...,\gamma_d(t))$ which are obtained with the help of the martingale representation and Ito's formula and is defined as

$$\gamma_i(t)=\partial_{y_i}G(t,\xi(t),S(t)),\quad i=\overline{1,d}\qquad \xi(t),S(t)\in\mathbb{R}^d, \tag{1}$$

where
$$G(t,x,y)=\mathbf{E}\left(\frac{\sum_{i=1}^d x_i+\sum_{i=1}^d y_i\tilde{\eta}_i(v)}{d}-K\right)_+ . \tag{2}$$

In solving the problem we found the densities for the following random exponential variables

$$\tilde{\eta}_i(v)=\int_0^v \exp\left\{\sigma_i W_i(u)-u\sigma_i^2/2\right\}du,\qquad v=1-t. \tag{3}$$

Also we proved that the function $G(t,x,y)$ is a unique solution of the elliptic differential equation and has continuous derivatives with respect to all variables.

The article is structured as follows. In section 1 we define the model, construct the strategy, find the densities for random variables (3). In section 2 we prove uniqueness of the solution $G(t,x,y)$. In section 3 we prove the theorem on the differentiability of the function (2). In section 4 we formulate the auxiliary theorems.

1. Statement of the problem and main results

Consider a market consisting of d risky assets with the price process $S(t)=(S_1(t),...,S_d(t))$ driven by the following system of SDEs

$$dS_i(t)=\sigma_i S_i(t)\,dW_i(t),\qquad i=1,...,d,\qquad 0\le t\le 1, \tag{4}$$

where $W = (W_1, ..., W_d)$ is a d-dimensional standard Brownian process with correlation matrix $R = (\rho_{i,j})_{d\times d}$,where $\rho_{i,j} = \chi_{\{i=j\}}$.

We assume that the riskless asset is a constant over time, i.e.

$$B(t) = 1.$$

The payoff function f_1 is given by

$$f_1 = \left(\frac{1}{Td} \int_0^T \sum_{i=1}^d S_i(t)\, dt - K \right)_+, \qquad T = 1.$$

Let $(\mathcal{F}_t)_{0\le t\le 1}$ be the natural filtration generated by all random sources, i.e. $\mathcal{F}_t = \sigma\{W(s), s \le t\}$ and denote by $(\Omega, \mathcal{F}_1, (\mathcal{F}_t)_{0\le t\le T}, \mathbf{P})$ the corresponding probability space which represents market fundamental elements.

Remark that the asset processes $S_i(t)$, $i = 1, ..., d$, admit the following explicit form

$$S_i(t) = S_i(0)\exp\{\sigma_i W_i(t) - t\sigma_i^2/2\}, \qquad i = 1, ..., d.$$

The procedure for obtaining the strategy is the same as in the article [14]. To construct a hedging strategy in the case of model (4) apply the representation Theorem 2 to the following martingale

$$M(t) = \mathbf{E}(f_1 \mid \mathcal{F}_t).$$

To this end is we will find square integrable processes $(\alpha_i(t))_{0\le t\le 1}$ adapted w.r.t. $\mathcal{F}_t$ such that for all $t \in [0,1]$

$$M(t) = M(0) + \sum_{i=1}^d \int_0^t \alpha_i(s)\, dW_i(s).$$

Clearly that

$$dM(t) = \sum_{i=1}^d \alpha_i(t) dW_i(t). \tag{5}$$

For coefficients $\alpha_i(t)$ we use the following formula

$$\langle M, W_i \rangle_t = \int_0^t \alpha_i(s)\, ds.$$

Therefore

$$\alpha_i(t) = \frac{d}{dt} \langle M, W_i \rangle_t . \tag{6}$$

Also the portfolio value satisfies the equality

$$dX(t) = \sum_{i=1}^d \gamma_i(t)\, dS_i(t) = \sum_{i=1}^d \gamma_i(t)\, \sigma_i S_i(t) dW_i(t). \tag{7}$$

Then equating the equalities (5) and (7) we obtain the strategy given by on the formulas

$$\gamma_i(t) = \frac{\alpha_i(t)}{\sigma_i S_i(t)},$$

$$\beta(t) = M(0) + \sum_{i=1}^{d} \int_0^t \alpha_i(s)\, dW_i(s) - \sum_{i=1}^{d} \gamma_i(t) S_i(t),$$

where $\gamma_i(t)$ and $\beta(t)$ are the quantities of risky assets and riskless asset respectively. In our case the martingale has the following form

$$M(t) = \mathbf{E}\left(\left(\frac{1}{d} \int_0^1 \sum_{i=1}^{d} S_i(v)\, dv - K \right)_+ \middle| \mathcal{F}_t \right). \tag{8}$$

If $t \le v \le 1$ then we have

$$S_i(v) = S_i(t) \exp\left\{ \sigma_i \left(W_i(v) - W_i(t) \right) - \frac{\sigma_i^2}{2}(v - t) \right\}.$$

We obtain

$$\sum_{i=1}^{d} \int_0^1 S_i(v)\, dv = \sum_{i=1}^{d} \int_0^t S_i(v)\, dv + \sum_{i=1}^{d} S_i(t) \int_t^1 \exp\left\{ \sigma_i \left(W_i(v) - W_i(t) \right) - \frac{\sigma_i^2}{2}(v - t) \right\} dv.$$

It means that we can represent the integral in the equality (8) as

$$\frac{1}{d} \int_0^1 \sum_{i=1}^{d} S_i(v)\, dv = \frac{\sum_{i=1}^{d} \xi_i(t) + \sum_{i=1}^{d} S_i(t) \eta_i(t)}{d},$$

where

$$\xi_i(t) = \int_0^t S_i(v)\, dv$$

and

$$\eta_i(t) = \int_t^1 \exp\left\{ \sigma_i \left(W_i(v) - W_i(t) \right) - \frac{\sigma_i^2}{2}(v - t) \right\} dv.$$

Note that $\xi_i(t)$ are measurable w.r.t. $\mathcal{F}_t$ and $\eta_i(t)$ are independent on $\mathcal{F}_t$. Hence

$$M(t) = G(t, \xi(t), S(t)).$$

Here $\xi(t) = (\xi_1(t), \ldots, \xi_d(t))$, $S(t) = (S_1(t), \ldots, S_d(t))$, and

$$G(t, x, y) = \mathbf{E}\left(\frac{\sum_{i=1}^{d} x_i + \sum_{i=1}^{d} y_i \eta_i(t)}{d} - K \right)_+,$$

where $x = (x_1, \ldots, x_d)$ and $y = (y_1, \ldots, y_d)$. After some transformation we come to the random variables

$$\tilde{\eta}_i(v) = \int_0^v \exp\left\{ \sigma_i W_i(u) - \frac{\sigma_i^2}{2} u \right\} du, \qquad v = 1 - t.$$

Since $\tilde{\eta}_i(v)$ are independent random variables therefore the following proposition holds

Proposition 1. For all t the random variables $\tilde{\eta}_i(v)$, $i=1,...,d$, have the following distribution densities

$$q_{\tilde{\eta}_i(v)}(t,z_i)=\mathbf{E}\frac{\varphi_{0,1}(a_i)}{K_i(t,a_i(t,z_i))},\qquad i=\overline{1,d}, \tag{10}$$

where

$$z_i=F_i(t,a_i(t,z_i))=\int_0^v \exp\{\sigma_i\tilde{B}_i(u)+\sigma_i u a_i\}du,\quad \tilde{B}_i(u)=W_i(u)-uW_i(1)-\frac{\sigma_i^2 u}{2},$$

$$K_i(t,a_i(t,z_i))=\frac{dF_i(t,a_i)}{da_i},\qquad \varphi_{0,1}(\cdot)\sim N(0,1).$$

The proof is carried out in the same way as in the article [14].

Represent martingale $M(t)=G(t,\xi(t),S(t))$ by Ito's formula. The existence of the continuous derivatives of the function $G(t,x,y)$ is proved in section 3. In our case processes $\xi_i(t)$ and $S_i(t)$ have the following stochastic differentials

$$d\xi_i(t)=S_i(t)dt,$$
$$dS_i(t)=\sigma_i S_i(t)dW_i(t).$$

By Theorem 4 we have

$$dG(t,\xi(t),S(t))=$$
$$\left[G_t'(t,\xi(t),S(t))+\sum_{i=1}^d G_{x_i}'(t,\xi(t),S(t))S_i(t)+\frac{1}{2}\sum_{i=1}^d G_{y_iy_i}''(t,\xi(t),S(t))\sigma_i^2S_i^2(t)\right]dt+$$
$$+\sum_{i=1}^d G_{y_i}'(t,\xi(t),S(t))\sigma_i S_i(t)dW_i(t).$$

Then
$$M(t)=f_t+\tilde{M}_t,$$
where

$$f_t=M(0)+\int_0^t\left[G_t'(v,\xi(v),S(v))+\sum_{i=1}^d G_{x_i}'(v,\xi(v),S(v))S_i(v)+\right.$$
$$\left.+\frac{1}{2}\sum_{i=1}^d G_{y_iy_i}''(v,\xi(v),S(v))\sigma_i^2S_i^2(v)\right]dv$$

and
$$\tilde{M}_t=\sum_{i=1}^d\hat{M}_j(t),\ \hat{M}_j(t)=\int_0^t G_{y_i}'(v,\xi(v),S(v))\sigma_iS_i(v)dW_j(v).$$

Since f_t is continuous martingale with finite variation, then

$$\int_0^t\left[G_t'(v,\xi(v),S(v))+\sum_{i=1}^d G_{x_i}'(v,\xi(v),S(v))S_i(v)+\right.$$
$$\left.+\frac{1}{2}\sum_{i=1}^d G_{y_iy_i}''(v,\xi(v),S(v))\sigma_i^2S_i^2(v)\right]dv=0.$$

So far as $\langle \hat{M}_j, W_i \rangle_t = 0$ when $i \neq j$ then

$$\langle M, W_i \rangle_t = \langle \tilde{M}, W_i \rangle_t = \sum_{i=1}^{d} \langle \hat{M}_j, W_i \rangle_t = \langle \hat{M}_i, W_i \rangle_t = \int_0^t G'_{y_i}(v, \xi(v), S(v)) \sigma_i S_i(v) dv.$$

Hence, we obtain that

$$\langle M, W_i \rangle_t = \int_0^t G'_{y_i}(v, \xi(v), S(v)) \sigma_i S_i(v) dv.$$

So, remark that the function $G(t,x,y)$ is a solution of the following equation

$$\begin{cases} G_t(t,x,y) + \sum_{i=1}^{d} G_{x_i}(t,x,y) y_i + \frac{1}{2} \sum_{i=1}^{d} \sigma_i^2 y_i^2 G''_{y_i y_i}(t,x,y) = 0, \\ G(1,x,y) = (x-K)_+ . \end{cases} \quad (13)$$

The equation (13) has the unique solution, see proof in the section 2. Then the coefficients in (6) are equal to

$$\alpha_i(t) = G'_{y_i}(t, \xi(t), S(t)) \sigma_i S_i(t).$$

Thus the option can be hedged by the strategy

$$\gamma = (\gamma_1(t), ..., \gamma_d(t)),$$

with the components

$$\gamma_i(t) = \partial_{y_i} G(t, \xi(t), S(t)), \qquad i = 1, ..., d.$$

2. Uniqueness of the solution $G(t,x,y)$

Definition 1. A $[0,T) \times \mathbb{R}_+^d \times \mathbb{R}_+^d \to \mathbb{R}$ function G is called of uniform polynomial growth if there exist some constants $m > 0$ and $L > 0$ such that

$$\sup_{0 \le t \le T} |G(t,x,y)| \le L\left(1 + |x|^m + |y|^m\right).$$

Theorem 1. The equation

$$\begin{cases} G_t(t,x,y) + \sum_{i=1}^{d} G_{x_i}(t,x,y) y_i + \frac{1}{2} \sum_{i=1}^{d} \sigma_i^2 y_i^2 G''_{y_i y_i}(t,x,y) = 0, \\ G(1,x,y) = \varphi(x) \end{cases} \quad (15)$$

has a unique solution in the class of uniform polynomial growth functions.

Remark 1. The equation (15) is a particular form of the following equation

$$\begin{cases} u_t(t,x,y) + u_x(t,x,y) y + \frac{a(y)}{2} u_{yy}(t,x,y) = 0, \\ u(1,x,y) = \varphi(x). \end{cases} \quad (16)$$

In the theory of elliptic equations in order to prove that (16) has a unique solution it is necessary that the following condition holds

$$\inf_y |a(y)| > 0.$$

In our case this condition does not hold since $a(y)=\sigma^2 y^2$ and $\inf_y |\sigma^2 y^2| = 0$. We obtain the degeneracy of the coefficient, thus u_{yy} disappears and there is a singularity. The theory of differential equations does not answer the question of the uniqueness of a solution of such an equation. Therefore, for proving this fact we will use a probability representation of the function G.

Proof. Without loss of generality we give the proof for $d=1$.We will prove by the rule of contraries. Suppose that there exists another solution $\tilde{G}(t,x,y)$ which satisfies the equation (15) that is

$$\begin{cases}\tilde{G}_t(t,x,y)+\tilde{G}_x(t,x,y)y+\dfrac{\sigma^2}{2}y^2\tilde{G}_{yy}(t,x,y)=0,\\ \tilde{G}(1,x,y)=\varphi(x).\end{cases}$$

We will prove uniqueness in the class of classical solutions. Represent $\tilde{G}(u,\xi_u,S_u)$ by Ito's formula

$$d\tilde{G}(u,\xi_u,S_u)=\left[\tilde{G}_u'(u,\xi_u,S_u)+\tilde{G}_x'(u,\xi_u,S_u)S_u+\frac{\sigma^2}{2}S_u^2\tilde{G}_{yy}''(u,\xi_u,S_u)\right]du$$
$$+\sigma S_u\tilde{G}_y'(u,\xi_u,S_u)dW_u.$$

And first of all we seek a solution in the class where the derivative w.r.t. y is bounded. Next we apply the Ito formula on the interval $t\le u\le 1$

$$\tilde{G}(1,\xi_1,S_1)=\tilde{G}(t,\xi_t,S_t)$$
$$+\int_t^1\left[\tilde{G}_u'(u,\xi_u,S_u)+\tilde{G}_x'(u,\xi_u,S_u)S_u+\frac{\sigma^2}{2}S_u^2\tilde{G}_{yy}''(u,\xi_u,S_u)\right]du$$
$$+\sigma\int_t^1 S_u\tilde{G}_y'(u,\xi_u,S_u)dW_u.$$

The second term is equal to zero by the assumption of the theorem. Since we suppose that $\tilde{G}_y'$ is bounded then $M_u=\int_t^1 S_u\tilde{G}_y'(u,\xi_u,S_u)dW_u$ is square integrable martingale. Hence

$$\tilde{G}(1,\xi_1,S_1)=\tilde{G}(t,\xi_t,S_t)+\sigma M_u\ . \tag{17}$$

Next consider the conditional expectation which on the one hand is equal to

$$\mathbf{E}\left(\tilde{G}(1,\xi_1,S_1)\,|\,\xi_t=x,S_t=y\right)=\mathbf{E}\left(\varphi(\xi_1)\,|\,\xi_t=x,S_t=y\right).$$

On the other hand we have by (17) the equality

$$\mathbf{E}\left(\tilde{G}(1,\xi_1,S_1)\,|\,\xi_t=x,S_t=y\right)$$
$$=\mathbf{E}\left(\tilde{G}(t,\xi_t,S_t)\,|\,\xi_t=x,S_t=y\right)+\mathbf{E}\left(\sigma\int_t^1 S_u\tilde{G}_y'(u,\xi_u,S_u)dW_u\,|\,\xi_t=x,S_t=y\right),$$

where $\mathbf{E}\left(\sigma\int_t^1 S_u\tilde{G}_y'(u,\xi_u,S_u)\,dW_u \mid \xi_t = x, S_t = y\right) = 0$, because M_u is a martingale. Then we equate the right parts of both equalities

$$\mathbf{E}(\varphi(\xi_1) \mid \xi_t = x, S_t = y) = \mathbf{E}\left(\tilde{G}(t,\xi_t,S_t) \mid \xi_t = x, S_t = y\right) = \tilde{G}(t,x,y).$$

Finally we obtain

$$\tilde{G}(t,x,y) = \mathbf{E}(\varphi(\xi_1) \mid \xi_t = x, S_t = y) = G(t,x,y).$$

Now we will study the case when $G_y(t,x,y)$ is not bounded. Introduce the stopping time

$$\tau_n = \inf\left\{t \le u \le 1 : \int_t^u \theta_s^2\,ds \ge n\right\},$$

where $\theta_u = \sigma S_u \tilde{G}_y'(u,\xi_u,S_u)$. Consider

$$\tilde{G}\left(\tau_n,\xi_{\tau_n},S_{\tau_n}\right) = \tilde{G}(t,\xi_t,S_t) + \int_t^{\tau_n}\theta_u\,dW_u .$$

Since θ_u is a continuous function, it is bounded and $\tau_n \to 1$. In this case $\mathbf{E}\int_t^{\tau_n}\theta_u^2\,du \le n$ and hence $\mathbf{E}\left(\int_t^{\tau_n}\theta_u^2\,dW_u \mid \mathcal{F}_t\right) = 0$. Then

$$\mathbf{E}\left(\tilde{G}\left(\tau_n,\xi_{\tau_n},S_{\tau_n}\right) \mid \mathcal{F}_t\right) = G(t,x,y), \qquad \forall n \ge 1.$$

Now we need to go to the limit. Suppose that function $\tilde{G}(t,x,y)$ belongs to the class of function of polynomial growth, that is

$$\left|\tilde{G}(t,x,y)\right| \le L\left(1 + x^m + y^m\right), \qquad \forall m \ge 1.$$

Therefore, we can write

$$\tilde{G}(t,x,y) = \lim_{n\to\infty}\mathbf{E}\left(\tilde{G}\left(\tau_n,\xi_{\tau_n},S_{\tau_n}\right) \mid \mathcal{F}_t\right).$$

One can prove that

$$\mathbf{E}\left(\xi_{\tau_n}\right)^m \le \mathbf{E}\left(\int_0^1 S_u\,du\right)^m \le \mathbf{E}\int_0^1 S_u^m\,du = \int_0^1 \mathbf{E}S_u^m\,du.$$

Since ξ_t is a monotone function then $\mathbf{E}\sup_{0\le t\le 1} |\xi_t|^m < +\infty$. Also we have

$$\mathbf{E}\sup_{0\le t\le 1} S_t^m \le S_0^m\,\mathbf{E}\sup_{0\le t\le 1}\exp\left\{m\sigma W_t - m\sigma^2 t/2\right\} \le \mathbf{E}\exp\left\{\sigma\sup_{0\le t\le 1} W_t\right\} < +\infty.$$

The last inequality was proved in [14]. Then we obtain

$$\left|\tilde{G}\left(\tau_n,\xi_{\tau_n},S_{\tau_n}\right)\right| \le \sup_{0\le t\le 1}\left|\tilde{G}(t,\xi_t,S_t)\right| \le L\left(1 + \sup_{0\le t\le 1}\xi_t^m + \sup_{0\le t\le 1} S_t^m\right) \le L\left(1 + \xi_1^m + \sup_{0\le t\le 1} S_t^m\right).$$

Denoting

$$\psi_n = \tilde{G}\left(\tau_n,\xi_{\tau_n},S_{\tau_n}\right) \text{ and } \psi^* = \left|\sup_{0\le t\le 1}\tilde{G}(t,\xi_t,S_t)\right|$$

we get

$$|\psi_n| \le \psi^*, \quad \forall n \ \text{ and } \mathbf{E}\psi^* < +\infty .$$

Next we apply Lebesgue's theorem on majorized dominanted convergence to obtain

$$\tilde{G}(t,x,y) = \lim_{n\to\infty} \mathbf{E}(\psi_n \mid \xi_t = x, S_t = y) = \mathbf{E}\left(\lim_{n\to\infty} \psi_n \mid \xi_t = x, S_t = y\right) =$$
$$= \mathbf{E}(\varphi(\xi_1) \mid \xi_t = x, S_t = y) = G(t,x,y).$$

The theorem has been proved.

3. Investigation of the function $G(t,x,y)$ on differentiability

Earlier we used the fact that the function $G(t,x,y)$ is continuously differentiable w.r.t. all variables to represent the martingale by the Ito formula. This result is presented in the following theorem.

Theorem 2. Let $x, y \in \mathbb{R}^d$ and $\tilde{\eta}_i(v) = \int_0^v \exp\{\sigma_i W_i(u) - \sigma_i^2 u/2\} du$. The function

$$G(t,x,y) = \mathbf{E}\left(\frac{\sum_{i=1}^d x_i + \sum_{i=1}^d y_i \tilde{\eta}_i(v)}{d} - K\right)_+$$

has the continuous derivatives

$$\frac{\partial}{\partial t} G(t,x,y), \quad \frac{\partial}{\partial x_i} G(t,x,y), \quad \frac{\partial}{\partial y_i} G(t,x,y), \quad \frac{\partial^2}{\partial y_i^2} G(t,x,y), \quad i = \overline{1,d}.$$

Proof. Represent function $G(t,x,y)$ as

$$G(t,x,y) = \int_{\mathbb{R}^d} \left(\sum_{i=1}^d x_i + \sum_{i=1}^d y_i z_i - Kd\right)_+ \prod_{i=1}^d q_i(t,z_i)\, dz_1 \dots dz_d,$$

where $q_i(t,z_i)$ are densities defined in Proposition 1. Denote $l(x) = Kd - \sum_{i=1}^d x_i$, then

$$G(t,x,y) = \int_{\mathbb{R}_+^d} \chi_{\{\sum_{i=1}^d y_i z_i > l(x)\}} \left(\sum_{i=1}^d y_i z_i - l(x)\right) \prod_{i=1}^d q_i(t,z_i)\, dz_1 \dots dz_d$$

$$= \int_{\mathbb{R}_+^{d-1}} \prod_{i=2}^d q_i(t,z_i) \left(\int_0^\infty \chi_{\{\sum_{i=1}^d y_i z_i > l(x)\}} \left(\sum_{i=1}^d y_i z_i - l(x)\right) q_1(t,z_1)\, dz_1\right) dz_2 \dots dz_d$$

$$= \int_{\mathbb{R}_+^{d-1}} \prod_{i=2}^d q_i(t,z_i) \left(\int_{a(x,y,z)}^\infty \left(\sum_{i=1}^d y_i z_i - l(x)\right) q_1(t,z_1)\, dz_1\right) dz_2 \dots dz_d,$$

where $a(x,y,z) = \left(\dfrac{Kd - \sum_{i=1}^d x_i - \sum_{i=2}^d y_i z_i}{y_1}\right)_+$.

By the Leibniz formula (18) we obtain

$$G'_{x_i}(t,x,y) = \int_{\mathbb{R}_+^{d-1}} \prod_{i=2}^d q_i(t,z_i)$$
$$\times \left(\int_{a(x,y,z)}^\infty q_1(t,z_1)\, dz_1 + q_1(t, a(x,y,z)) \left(y_1 a(x,y,z) + \sum_{i=1}^d y_i z_i - l(x)\right)\right) dz_2 \dots dz_d$$
$$= \int_{\mathbb{R}^{d-1}} \int_{a(x,y,z)}^\infty \prod_{i=2}^d q_i(t,z_i) dz_1 \dots dz_d.$$

Analogonsly we find derivatives w.r.t. y_i

$$G'_{y_i}(t,x,y)=\int_{\mathbb{R}_+^{d-1}}\prod_{i=2}^{d}q_i(t,z_i)$$
$$\times\left(\int_{a(x,y,z)}^{\infty}z_i q_1(t,z_1)\,dz_1-f\left(a(x,y,z),x,y\right)a'_{y_i}(x,y,z)\right)dz_2...dz_d.$$

We use the function $f(z_1,x,y)=\left(\sum_{i=1}^{d}y_i z_i-l(x)\right)q_1(t,z_1)$ which is equal to zero when $z_1=a(x,y,z)$. Hence

$$G'_{y_i}(t,x,y)=\int_{\mathbb{R}_+^{d-1}}\prod_{i=2}^{d}q_i(t,z_i)\left(\int_{a(x,y,z)}^{\infty}z_i q_1(t,z_1)\,dz_1\right)dz_2...dz_d.$$

Note that

$$a'_{y_i}(x,y,z)=\begin{cases}-\dfrac{l(x)-\sum_{i=2}^{d}y_i z_i}{y_1^2}, & \text{when } i=1,\\ -z_i, \quad \text{when} & i=2...d.\end{cases}$$

Then

$$G''_{y_iy_i}(t,x,y)=\int_{\mathbb{R}_+^{d-1}}\prod_{i=2}^{d}q_i(t,z_i)\left(-a(x,y,z)\,q_1(t,a(x,y,z))\,a'_{y_i}(x,y,z)\right)dz_2...dz_d.$$

Analyzing the obtained derivatives we conclude about its continuity.

Now we consider the partial derivative w.r.t. t. Introduce the notation

$$H(x,y,z)=\left(\sum_{i=1}^{d}x_i+\sum_{i=1}^{d}y_i z_i-Kd\right)_+ \qquad \text{and} \qquad \rho(t,z)=\prod_{i=1}^{d}q_i(t,z_i).$$

Then $G(t,x,y)=\int_{\mathbb{R}_+^{d}}H(x,y,z)\rho(t,z)\,dz_1...dz_d$. By the definition of a derivative we can write

$$\frac{G(t+\delta,x,y)-G(t,x,y)}{\delta}=\int_{\mathbb{R}_+^{d}}H(x,y,z)\varsigma_\delta(t,z)\,dz_1...dz_d\,,$$

where

$$\varsigma_\delta(t,z)=\frac{\rho(t+\delta,z)-\rho(t,z)}{\delta}\to\frac{\partial}{\partial t}\rho(t,z)=\sum_{j=1}^{d}\left(\prod_{\substack{i=1\\ i\neq j}}^{d}q_i(t,z_i)\right)\frac{\partial}{\partial t}q_j(t,z_j).$$

Note, taking into account of Proposition 2 in [14], that

$$\frac{\partial}{\partial t}q_i(t,z_i)\le b(z_i) \qquad \text{and} \qquad q_i(t,z_i)\le b_i.$$

Then

$$|\varsigma_\delta(t,z)|=\frac{1}{\delta}\left|\int_t^{t+\delta}\frac{\partial}{\partial t}\rho(u,z)\,du\right|\le\frac{1}{\delta}\int_t^{t+\delta}\left|\frac{\partial}{\partial t}\rho(u,z)\right|du$$
$$\le\frac{1}{\delta}\int_t^{t+\delta}\sum_{j=1}^{d}\left(\prod_{\substack{i=1\\ i\neq j}}^{d}|q_i(u,z_i)|\right)\left|\frac{\partial}{\partial t}q_j(u,z_j)\right|du<+\infty.$$

Therefore, by Lebesgue's theorem

$$\frac{\partial}{\partial t}G(t,x,y)=\int_{\mathbb{R}_+^d}H(x,y,z)\lim_{\delta\to 0}\varsigma_\delta(t,z)\,dz_1...dz_d$$

$$=\int_{\mathbb{R}_+^d}H(x,y,z)\sum_{j=1}^d\left(\prod_{\substack{i=1\\ i\neq j}}^d q_i(t,z_i)\right)\frac{\partial}{\partial t}q_j(t,z_j)\,dz_1...dz_d.$$

The continuity of the derivative $G_t^{'}(t,x,y)$ follows from the smoothness of densities $q_i(t,z_i)$, which was proved in Proposition 2 of the article [14].

4. Appendix

4.1. Representation theorem

Theorem 3. If $W_t=(W_1(t),...,W_n(t))$ – n-dimensional Wiener process and $X=(x_t,\mathcal{F}_t)_{t\le T}$ is a quadratic integrable martingale with $\mathcal{F}_t=\sigma\{\omega:W_1(s),...,W_n(s),s\le t\}$ then

$$x_t=x_0+\sum_{i=1}^n\int_0^t\alpha_i(s,\omega)dW_i(s),$$

where $\alpha_i(s,\omega)-\mathcal{F}_t$ adapted and $\sum_{i=1}^n\int_0^T\mathbf{E}\alpha_i^2(s,\omega)ds<+\infty$.

4.2. Ito's formula. Multidimensional version

Let $\xi=(\xi(t),\mathcal{F}_t)_{t\le T}$ and $S=(S(t),\mathcal{F}_t)_{t\le T}$ are vector random processes $\xi(t)=(\xi_1(t),...,\xi_d(t))$ and $S(t)=(S_1(t),...,S_d(t))$ which have the following stochastic differentials

$$d\xi_i=a_i^{(x)}(t,\omega)dt+b_i^{(x)}(t,\omega)dW_i^{(x)}(t),\quad i=\overline{1,d},$$

$$dS_i=a_i^{(y)}(t,\omega)dt+b_i^{(y)}(t,\omega)dW_i^{(y)}(t),\quad i=\overline{1,d}.$$

Theorem 4. Let function $f(t,x_1,...,x_d,y_1,...,y_d)$ is continuous and has derivatives $f_t^{'},f_{x_i}^{'},f_{y_i}^{'},f_{x_ix_j}^{''},f_{x_iy_j}^{''},f_{y_iy_j}^{''}$. Then with probability 1 the process $f(t,\xi(t),S(t))$ has the following stochastic differential

$$df(t,\xi(t),S(t))=\Big[f_t^{'}(t,\xi(t),S(t))$$

$$+\sum_{i=1}^d f_{x_i}^{'}(t,\xi(t),S(t))a_i^{(x)}(t,\omega)+\sum_{i=1}^d f_{y_i}^{'}(t,\xi(t),S(t))a_i^{(y)}(t,\omega)$$

$$+\frac{1}{2}\sum_{i=1}^d f_{x_ix_i}^{''}(t,\xi(t),S(t))\left(b_i^{(x)}(t,\omega)\right)^2+\frac{1}{2}\sum_{i=1}^d f_{y_iy_i}^{''}(t,\xi(t),S(t))\left(b_i^{(y)}(t,\omega)\right)^2\Big]dt$$

$$+\sum_{i=1}^d f_{x_i}^{'}(t,\xi(t),S(t))b_i^{(x)}(t,\omega)dW_i^{(x)}(t)+\sum_{i=1}^d f_{y_i}^{'}(t,\xi(t),S(t))b_i^{(y)}(t,\omega)dW_i^{(y)}(t).$$

4.3. Leibniz's formula

Let a function $f(x,z)$ and its partial derivative $f_x^{'}(x,z)$ be continuous on $[\alpha,\beta]\times[c,d]$ (a segment $[\alpha,\beta]$ contains a set of values $a(x),b(x)$ and functions $a(x)$ and $b(x)$ are differentiable on $[c,d]$). Then the integral

$$I(x)=\int_{a(x)}^{b(x)} f(x,z)\,dz$$

is differentiable w.r.t. x on $[c,d]$ and the following equality holds

$$I_x^{'}(x)=\int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,z)\,dz+f(x,b(x))b^{'}(x)-f(x,a(x))a^{'}(x). \quad (18)$$

Conclusions

In the paper, the hedging problem for the Asian option on a multidimensional financial market was considered. The main result is the hedging strategy for the Asian option. For this we have obtained a differential equation of elliptic type have proved the uniqueness of its solution. In addition, we have established that the obtained solution is a continuous differentiable function. This property allows us to apply Ito's formula for finding the coefficients in the martingale representation (5).

REFERENCES

1. Kozhin K. (2002) Vse ob ekzoticheskikh optsionakh [All about exotic options]. *Rynok tsennykh bumag* [Stocks and bonds market]. 1(15). pp. 53–57. (In Russian).
2. Zang P.G. (1995) An introduction to exotic options. *European Financial Managment*. 1(1). pp. 87–95.
3. Geman H., Yor M. (1993) Bessel processes, Asian options, and perpetuities. *Mathematical Finance*. 3(4). pp. 35–38.
4. Kemna A.G.Z.,Vorst A.C.F. (1990) Pricing method for options based on average asset values. *Banking Finance*. 14. pp. 113–129.
5. Carverhill A.P., Clewlow L.J. (1990) Valuing average rate (Asian) options. *Risk*. 3. pp. 25–29.
6. Lapeyre B., Temam E. (2001) Competitive Monte Carlo methods for the pricing of Asian options. *J. Computational Finance*. 5(1). pp. 39–57. DOI: 10.21314/JCF.2001.061.
7. Fu M., Madan D., Wang T. (1998) Pricing continuous Asian options: a comparison of Monte Carlo and Laplace transform invertion methods. *J. Computational Finance*. 2(2). pp. 49–74 DOI: 10.21314/JCF.1998.024.
8. Seghiouer H., Lidouh A., Nqi F.Z. (2008) Pricing Asian options by Monte Carlo Method under MPI Environment. *Int. J. Math. Analysis*. 2(27). pp. 1301–1317.
9. Ruttiens A. (1990) Average rate options, classical replica. *Risk*. 3. pp. 33–36.
10. Vorst T. (1992) Prices and hedge ratios of average exchange rate options. *Int. Review of Financial Analysis*. 1(3). pp. 179–193. DOI: 10.1016/1057-5219(92)90003-M.
11. Levy E. (1992) Pricing European Average Rate Currency Options. *Int. Money Finance*. 11. pp. 474–491.
12. Shiryaev A.N. (1999) *Essentials of Stochastic Finance: Facts, Models, Theory.* Hackensack New Jersey: World Scientific Publishing Company. 834 p.
13. Liptser R.S., Shiryaev A.N. (2001) *Statistics of random processes*. 2nd rev. and exp. ed. Springer Verlag: Berlin. 425 p.
14. Shishkova A.A. (2018) Calculation of Asian options for the Black–Scholes model. *Vestnik Tomskogo gosudarstvennogo universiteta. Matematika i mekhanika* [Tomsk State University Journal of Mathematics and Mechanics]. 51. pp. 48–63. DOI 10.17223/19988621/51/5.

Received: July 4, 2018

Shishkova A.A. (2018) THE HEDGING STRATEGY FOR ASIAN OPTION. *Vestnik Tomskogo gosudarstvennogo universiteta. Matematika i mekhanika* [Tomsk State University Journal of Mathematics and Mechanics]. 56. pp. 29–41

Шишкова А.А. (2018) ХЕДЖИРУЮЩАЯ СТРАТЕГИЯ ДЛЯ АЗИАТСКОГО ОПЦИОНА. *Вестник Томского государственного университета. Математика и механика*. № 56. С. 29–41

DOI 10.17223/19988621/56/3

Ключевые слова: хеджирующая стратегия, Азиатский опцион, стохастические дифференциальные уравнения, Броуновское движение, модель Блэка – Шоулса.

Рассматривается задача портфельного инвестирования в модели Блэка – Шоулса с несколькими рисковыми активами. Хеджирующая стратегия для Азиатского опциона найдена с использованием мартингальных методов. Изучены аналитические свойства (дифференцируемость) плотности случайной экспоненциальной величины.

AMS Mathematical Subject Classification: 60H10, 60G44, 60J65

SHISHKOVA Alena Andreevna (Tomsk State University, Tomsk, Russian Federation). E-mail: alshishkovatomsk@gmail.com