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ESTIMATION OF PRESENT VALUE OF WHOLE LIFE ANNUITY USING INFORMATION ABOUT EXPECTATION OF LIFE

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The paper deals with the estimation problem of the actuarial present value of the continuous whole life annuity using auxiliary information about the expectation of life. Nonparametric estimators of life annuity are constructed by individuals' death moments. It is shown that the usage of such auxiliary information can often provide the mean squared error (MSE) smaller than that of standard estimators. An adaptive estimator is also proposed. The asymptotic normality of all these estimators is proved.
Keywords: nonparametric estimation; whole life annuity; auxiliary information; mean squared error; asymptotic normality.

The idea of life annuity in accordance with [1. P. 170] is this: from the moment $t = 0$ an individual once a year begins to get a certain money, which we take as the unit of money, and payments are made only for the lifetime of an individual. It is known that the calculation of the characteristics of life annuity is based on the characteristics of the respective type of insurance. Thus, the average total cost of the present continuous annuity is defined by the following formula (see [1. P. 184]):

$$\bar{a}_x(\delta) = \frac{1-\bar{A}_x}{\delta},$$

where $\bar{A}_x$ is a net premium (the average of the present value of a single sum of money in the insurance lifetime at the age x), δ is a force of interest. Let x be an individual's age on the moment of payments start, X be his lifetime, $T_x = X - x$ be his future lifetime. Let us introduce the random variable

$$z(x) = \frac{1-e^{-\delta T_x}}{\delta}, T_x > 0. \quad (1)$$

Then, by averaging the random variable $z(x)$ (1), we get the formula of the whole life annuity (see [2–4]):

$$\bar{a}_x(\delta) = E(z(x)) = \frac{1}{\delta}\left(1 - \frac{\Phi(x,\delta)}{S(x)}\right), \quad (2)$$

where E is the symbol of the mathematical expectation, $S(x) = P(X > x)$ is a survival function,

$$\Phi(x,\delta) = e^{\delta x}\int_x^{\infty} e^{-\delta t} dF(t),$$

$F(x) = P(X \le x) = 1 - S(x)$ is a distribution function.

1. Estimation of Annuity

Suppose we have a random sample $X_1,...,X_N$ of N individuals' lifetimes. Now, separately estimate the numerator and denominator in (2). The substitution of unknown function $S(x)$ for its nonparametric estimator

$$S_N(x) = \frac{1}{N}\sum_{i=1}^{N} \mathrm{I}(X_i > x),$$

where I(A) is the indicator of an event A, gives us the following estimators of the whole life annuity:

$$\bar{a}_x^N(\delta) = \frac{1}{\delta}\left(1 - \frac{e^{\delta x}}{S_N(x)\cdot N}\sum_{i=1}^{N}\exp(-\delta X_i)\mathrm{I}(X_i > x)\right) = \frac{1}{\delta}\left(1 - \frac{\Phi_N(x,\delta)}{S_N(x)}\right), \tag{3}$$

$$\Phi_N(x,\delta) = \frac{e^{\delta x}}{N}\sum_{i=1}^{N}\exp(-\delta X_i)\mathrm{I}(X_i > x).$$

2. Bias and MSE of $\bar{a}_x^N(\delta)$

In this section, we will obtain the principal term of the asymptotic MSE and the bias convergence rate of the estimator (3). Now introduce the notation according to [5]: $t_N = (t_{1N}, t_{2N}, ..., t_{sN})^{\mathrm{T}}$ is an s-dimensional vector with components $t_{jN} = t_{jN}(x) = t_{jN}(x; X_1, ..., X_N)$, $j = \overline{1,s}$, $x \in R^{\alpha}$, R^{α} is the α-dimensional Euclidean space; $H(t): R^s \to R^1$ is a function, where $t = t(x) = (t_1(x), ..., t_s(x))^{\mathrm{T}}$ is an s-dimensional bounded vector function; $\mathrm{N}_s(\mu, \sigma)$ is the s-dimensional normally distributed random variable with a mean vector and covariance matrix $\sigma = \sigma(x)$; $\nabla H(t) = (H_1(t), ..., H_s(t))^{\mathrm{T}}$, $H_j(t) = \left.\frac{\partial H(z)}{\partial z_j}\right|_{z=t}$, $j = \overline{1,s}$, $\Rightarrow$ is the symbol of convergence in distribution (weak convergence); $\| x \|$ is the Euclidean norm of a vector x, $\mathfrak{R}$ is the set of natural numbers.

Definition 1. The function $H(t): R^s \to R^1$ and the sequence $\{H(t_N)\}$ are said to belong to class $\mathrm{N}_{\nu,s}(t;\gamma)$, provided that:

1) there exists an ε-neighborhood

$$\sigma = \left\{ z : | z_i - t_i | < \varepsilon, i = \overline{1,s} \right\},$$

in which the function $H(z)$ and all its partial derivatives up to order ν are continuous and bounded;

2) for any values of variables $X_1, ..., X_N$ the sequence $\{H(t_N)\}$ is dominated by a numerical sequence $C_0 d_N^{\gamma}$, such that $d_N \uparrow \infty$, as $N \to \infty$, and $0 \le \gamma < \infty$.

Theorem 1 [5]. Let the conditions

1) $H(z)$, $\{H(t_N)\} \in \mathrm{N}_{2,s}(t;\gamma)$,

2) $E\|t_N - t\|^i = O\left(d_N^{-i/2}\right)$

hold for all $i \in \mathfrak{R}$. Then, for every $k \in \mathfrak{R}$,

$$\left| E\left[H(t_N) - H(t)\right]^k - E\left[\nabla H(t)\cdot(t_N - t)\right]^k \right| = O\left(d_N^{-(k+1)/2}\right). \tag{4}$$

Note, if in formula (4) k = 1, we obtain the principal term of the bias for $H(t_n)$, and at k = 2, we have the principal term of the MSE.

Theorem 2. If $S(x) > 0$ and $S(t)$ is continuous at x, then

1) for the bias of (3), the following relation holds:

$$\left| b(\bar{a}_x^N(\delta)) \right| = \left| E(\bar{a}_x^N(\delta) - \bar{a}_x(\delta)) \right| = O\left(N^{-1}\right);$$

2) the MSE of (3) is given by the formula

$$u^2(\bar{a}_x^N(\delta)) = E(\bar{a}_x^N(\delta) - \bar{a}_x(\delta))^2 = \frac{\Phi(x, 2\delta) - \Phi^2(x,\delta)/S(x)}{N\delta^2 S^2(x)} + O\left(N^{-3/2}\right).$$

Proof. For the estimator $\bar{a}_x^N(\delta)$ (3) in the notation of Theorem 1, we have: $s = 2$;

$$t_N = (t_{1N}, t_{2N})^{\mathrm{T}} = (\Phi_N(x,\delta), S_N(x))^{\mathrm{T}}; \quad d_N = N; \quad t = (t_1, t_2)^{\mathrm{T}} = (\Phi(x,\delta), S(x))^{\mathrm{T}};$$

$$H(t) = \frac{1}{\delta}\left(1 - \frac{t_1}{t_2}\right) = \frac{1}{\delta}\left(1 - \frac{\Phi(x,\delta)}{S(x)}\right) = \bar{a}_x(\delta); \qquad H(t_N) = \frac{1}{\delta}\left(1 - \frac{\Phi_N(x,\delta)}{S_N(x)}\right) = \bar{a}_x^N(\delta);$$

$$\nabla H(t) = \left(H_1(t), H_2(t)\right)^{\mathrm{T}} = \left(-\frac{1}{\delta S(x)}, \frac{\Phi(x,\delta)}{\delta S^2(x)}\right)^{\mathrm{T}} \neq 0.$$

The sequence $\{H(t_N)\}$ satisfies the condition 1) of Theorem 1 with $C_0 = \dfrac{2}{\delta}$ and $\gamma = 0$. Indeed,

$$|H(t_N)| = \frac{1}{\delta}\left|1 - \frac{\Phi_N(x,\delta)}{S_N(x)}\right| \leq \frac{1}{\delta}\left(1 + \frac{\Phi_N(x,\delta)}{S_N(x)}\right) \leq \frac{1}{\delta}\left(1 + \frac{e^{\delta x}\sum_{i=1}^{N}\exp(-\delta X_i)\mathrm{I}(X_i > x)}{\sum_{i=1}^{N}\mathrm{I}(X_i > x)}\right) \leq$$

$$\leq \frac{1}{\delta}\left(1 + \frac{e^{\delta x}e^{-\delta x}\sum_{i=1}^{N}\mathrm{I}(X_i > x)}{\sum_{i=1}^{N}\mathrm{I}(X_i > x)}\right) = \frac{2}{\delta}.$$

Further, the function $H(t)$ satisfies the condition 1) in view of $t_2 = S(x) > 0$, the condition 2) due to Lemma 3.1 [6], provided that $E\{\mathrm{I}^i(X > x)\} = S(x) \leq 1$, $E\{e^{i\delta x}e^{-i\delta X}\mathrm{I}^i(X > x)\} \leq e^{i\delta x}e^{-i\delta x}S(x) = S(x) \leq 1$ for all $i \in \Re$.

We know that $S_N(x)$ is an unbiased and consistent estimator of $S(x)$. Show that $\Phi_N(x,\delta)$ is an unbiased estimator of the functional $\Phi(x,\delta)$:

$$E\Phi_N(x,\delta) = \frac{e^{\delta x}}{N}E\left\{\sum_{i=1}^{N}\exp(-\delta X_i)\mathrm{I}(X_i > x)\right\} = \Phi(x,\delta).$$

Now, calculate the variance of $\Phi_N(x,\delta)$:

$$D\Phi_N(x,\delta) = D\left\{\frac{e^{\delta x}}{N}\sum_{i=1}^{N}\mathrm{I}(X_i > x)e^{-\delta X_i}\right\} = \frac{e^{2\delta x}}{N^2}\sum_{i=1}^{N}D\left\{\mathrm{I}(X_i > x)e^{-\delta X_i}\right\} = \frac{1}{N}\left(\Phi(x,2\delta) - \Phi^2(x,\delta)\right).$$

The ratio of two unbiased estimators can have a bias. Considering that all the conditions of Theorem 1 are fulfilled and $E(t_N - t) = 0$, in accordance with (4) we get the order of the bias of $\bar{a}_x^N(\delta)$:

$$\left|E(\bar{a}_x^N(\delta) - \bar{a}_x(\delta)) - E\left[\nabla H(t)(t_N - t)\right]\right| = \left|E(\bar{a}_x^N(\delta) - \bar{a}_x(\delta))\right| = O\left(N^{-1}\right).$$

Find the components of the covariance matrix $\sigma(\bar{a}_x(\delta)) = \begin{bmatrix}\sigma_{11}\sigma_{12}\\ \sigma_{21}\sigma_{22}\end{bmatrix}$ for the statistics $\Phi_N(x,\delta), S_N(x)$:

$$\sigma_{11} = ND\{\Phi_N(x,\delta)\} = \Phi(x,2\delta) - \Phi^2(x,\delta); \quad \sigma_{22} = ND\{S_N(x)\} = S(X)(1 - S(x)); \quad \sigma_{12} = \sigma_{21} =$$
$$= N\,\mathrm{cov}(S_N(x), \Phi_N(x,\delta)) = N\left(E\{S_N(x)\Phi_N(x,\delta)\} - E\{S_N(x)\}E\{\Phi_N(x,\delta)\}\right) = (1 - S(x))\Phi(x,\delta).$$

Using the previous results on the bias and the covariance matrix, we obtain

$$u^2(\bar{a}_x^N(\delta)) = E\left[\nabla H(t)(t_N - t)\right]^2 + O\left(N^{-3/2}\right) = \frac{\tilde{C}(\bar{a}_x(\delta))}{N} + O\left(N^{-3/2}\right),$$

$$\tilde{C}(\bar{a}_x(\delta)) = H_1^2(t)\sigma_{11} + H_2^2(t)\sigma_{22} + 2H_1(t)H_2(t)\sigma_{12} = \frac{\Phi(x,2\delta) - \Phi^2(x,\delta)/S(x)}{\delta^2 S^2(x)}. \tag{5}$$

The proof is completed.

3. Asymptotic Normality of $\bar{a}_x^N(\delta)$

To find the limit distribution of (3), we need the following two Theorems.

Theorem 3 (The usual central limit theorem) [7, Appendix 5]. If $\xi_1, \xi_2, ..., \xi_N, ...$ is a sequence of independent and identically distributed *s*-dimensional vectors, $E\{\xi_k\} = 0, \quad \sigma(x) = E\{\xi_k^{\mathrm{T}} \xi_k\}, \ t_N = \frac{1}{N}\sum_{k=1}^{N} \xi_k$, then, as $N \to \infty$, $\sqrt{N} t_N \Rightarrow \mathrm{N}_s(0, \sigma(x))$.

Theorem 4 (asymptotic normality of $H(t_N)$) [2]. Let

1) $\sqrt{N} \cdot t_N \Rightarrow \mathrm{N}_s\{\mu, \sigma(x)\}$;

2) $H(z)$ be differentiable at the point μ, $\nabla H(\mu) \neq 0$.

Then

$$\sqrt{N}(H(t_N) - H(\mu)) \Rightarrow \mathrm{N}_1\left(\sum_{j=1}^{s} H_j(\mu)\mu_j, \ \sum_{p=1}^{s}\sum_{j=1}^{s} H_j(\mu)\sigma_{jp} H_p(\mu)\right).$$

Theorem 5. Under the conditions of Theorem 2

$$\sqrt{N}(\bar{a}_x^N(\delta) - \bar{a}_x(\delta)) \Rightarrow \mathrm{N}_1\left(0, \frac{\Phi(x,2\delta) - \Phi^2(x,\delta)/S(x)}{\delta^2 S^2(x)}\right).$$

Proof. In the notation of Theorem 3, we have $s = 2, \ \ \sigma(x) = \sigma(\bar{a}_x(\delta))$. Thus,

$$\sqrt{N}\left\{(\Phi_N(x,\delta), S_N(x)) - (\Phi(x,\delta), S(x))\right\} \Rightarrow \mathrm{N}_2\left((0,0), \sigma\left(\bar{a}_x(\delta)\right)\right).$$

The function $H(z)$ is differentiable at the point $t = \left(\Phi(x,\delta), S(x)\right)$ and $\nabla H(t) \neq 0.$ Consequently, all the conditions of Theorem 4 hold, and using (5), we have $\sqrt{N}(\bar{a}_x^N(\delta) - \bar{a}_x(\delta)) \Rightarrow \mathrm{N}_1\left(0, \tilde{C}(\bar{a}_x(\delta))\right).$

Theorem 5 is proved.

4. Construction of Estimators Using Expected Lifetime

Suppose we know the expected lifetime

$$EX = a. \tag{6}$$

The estimator by making use of such information according to [8–17] can be taken in the following form:

$$\bar{a}_x^N(\delta, \lambda) = \frac{1}{\delta}\left(1 - \frac{\Phi_N(x,\delta)}{S_N(x)} - \lambda(\bar{x} - a)\right), \tag{7}$$

where $\bar{x} = \frac{1}{N}\sum_{i=1}^{N} X_i$ is an estimator of *a*, parameter λ we will find minimizing the principal term of the asymptotic MSE of $\bar{a}_x^N(\delta, \lambda)$ (7). The estimator (7) combines the available empirical information containing in (3) and prior information (6).

For the estimator $\bar{a}_x^N(\delta, \lambda)$ in the notation of Theorem 1, we have: $s = 3$,

$$t_N = (t_{1N}, t_{2N}, t_{3N})^{\mathrm{T}} = (\Phi_N(x,\delta), S_N(x), \bar{x})^{\mathrm{T}}; \ d_N = N; \ t = (t_1, t_2, t_3)^{\mathrm{T}} = (\Phi(x,\delta), S(x), a)^{\mathrm{T}};$$

$$H(t) = H(t_1, t_2, t_3) = \frac{1}{\delta}\left(1 - \frac{t_1}{t_2} - \lambda(t_3 - a)\right) = \frac{1}{\delta}\left(1 - \frac{\Phi(x,\delta)}{S(x)} - \lambda(a - a)\right) = \bar{a}_x(\delta);$$

$$H(t_N) = \frac{1}{\delta}\left(1 - \frac{\Phi_N(x,\delta)}{S_N(x)} - \lambda(\bar{x} - a)\right) = \bar{a}_x^N(\delta, \lambda);$$

$$\nabla H(t) = \left(H_1(t), H_2(t), H_3(t)\right)^{\mathrm{T}} = \left(-\frac{1}{\delta S(x)}, \frac{\Phi(x,\delta)}{\delta S^2(x)}, -\frac{\lambda}{\delta}\right)^{\mathrm{T}} \neq 0.$$

5. Bias and MSE of $\bar{a}_x^N(\delta,\lambda)$

Arguing as in Section 1, it is easy to show that the sequence $\{H(t_N)\}$ satisfies the condition 1) of Theorem 1 with $C_0 = \frac{2+|\lambda|(\omega+a)}{\delta}$, $\omega<\infty$ is the limiting age, and $\gamma=0$; also, the statistic t_N satisfies the condition 2) due to Lemma 3.1 [6], provided that $EX^i \le \omega^i < \infty$ for all $i \in \Re$. From here, for the bias of (7) we obtain the following result:

$$\left|E(\bar{a}_x^N(\delta,\lambda)-\bar{a}_x(\delta)) - E[\nabla H(t)(t_N-t)]\right| = \left|E(\bar{a}_x^N(\delta,\lambda)-\bar{a}_x(\delta))\right| = \left|b(\bar{a}_x^N(\delta,\lambda))\right| = O\left(N^{-1}\right).$$

Now, find the covariance matrix $\sigma(\bar{a}_x(\delta,\lambda)) = \begin{bmatrix} \sigma_{11}\sigma_{12}\sigma_{13} \\ \sigma_{21}\sigma_{22}\sigma_{23} \\ \sigma_{31}\sigma_{32}\sigma_{33} \end{bmatrix}$ for statistics $\Phi_N(x,\delta),\ S_N(x),\ \bar{x}$:

$\sigma_{33} = ND\{\bar{x}\} = D(X);\quad \sigma_{13}=\sigma_{31} = N\operatorname{cov}(\bar{x},\Phi_N(x,\delta)) = C_1(x,\delta) - a\Phi(x,\delta);$

$\sigma_{23}=\sigma_{32} = N\operatorname{cov}(S_N(x),\bar{x}) = C_2(x) - aS(x),$ where $C_1(x,\delta) = e^{\delta x}\int_x^\infty e^{-\delta u}u\,dF(u)$, $C_2(x) = \int_x^\infty u\,dF(u)$, and σ_{11}, σ_{12}, σ_{21}, σ_{22} are defined in Section 1. Using (5), the above results for the bias and the covariance matrix $\sigma(\bar{a}_x(\delta,\lambda))$, we obtain:

$$u^2\left(\bar{a}_x^N(\delta,\lambda)\right) = E[\nabla H(t)(t_N-t)]^2 + O\left(N^{-3/2}\right) = \frac{\tilde{C}(\bar{a}_x(\delta,\lambda))}{N} + O\left(N^{-3/2}\right),$$

$$\tilde{C}(\bar{a}_x(\delta,\lambda)) = \sum_{p=1}^{3}\sum_{j=1}^{3} H_j(t)\sigma_{jp}H_p(t) = H_1^2(t)\sigma_{11} + H_2^2(t)\sigma_{22} + H_3^2(t)\sigma_{33} + 2H_1(t)H_2(t)\sigma_{12} +$$

$$+2H_1(t)H_3(t)\sigma_{13} + 2H_2(t)H_3(t)\sigma_{23} = \tilde{C}(\bar{a}_x(\delta)) + \frac{\lambda^2\sigma_{33}}{\delta^2} - \frac{2\lambda H_1\sigma_{13}}{\delta} - \frac{2\lambda H_2\sigma_{23}}{\delta} =$$

$$= \tilde{C}(\bar{a}_x(\delta)) + \lambda^2 Q_1 - 2\lambda Q_2, \tag{8}$$

where $Q_1 = \frac{\sigma_{33}}{\delta^2} > 0$, $Q_2 = \frac{H_1\sigma_{13}+H_2\sigma_{23}}{\delta}$. Then the minimum of $\tilde{C}(\bar{a}_x(\delta,\lambda))$ with respect to λ is achived at $\lambda_0 = \frac{Q_2}{Q_1}$. Such λ_0 minimizes the principal term of MSE $u^2(\bar{a}_x^N(\delta,\lambda))$, and this minimum is as follows:

$$\frac{\tilde{C}\left(\bar{a}_x(\delta,\lambda_0)\right)}{N} = \frac{1}{N}\left(\tilde{C}\left((\bar{a}_x(\delta)\right) - \frac{Q_2^2}{Q_1}\right) < \frac{\tilde{C}\left((\bar{a}_x(\delta)\right)}{N}. \tag{9}$$

6. Bias, MSE, and Asymptotic Normality of $\bar{a}_x^N(\delta,\lambda_0)$

In accordance with (9), the estimator

$$\bar{a}_x^N(\delta,\lambda_0) = \frac{1}{\delta}\left(1 - \frac{\Phi_N(x,\delta)}{S_N(x)} - \lambda_0(\bar{x}-a)\right) \tag{10}$$

will be called the optimal (in the mean square sense) estimator. The non-negative quantity $\frac{Q_2^2}{NQ_1}$ in (9) determines the decrease of the principal term of MSE for the optimal estimator by using auxiliary information (6).

Theorem 6. If $S(x)>0$ and $S(t)$ is continuous at x, then

1) for the bias of (10), the following relation holds:

$$\left|b\left(\bar{a}_x^N(\delta,\lambda_0)\right)\right| = O\left(N^{-1}\right);$$

2) the MSE of (10) is given by the formula

$$u^2\left(\bar{a}_x^N(\delta,\lambda_0)\right)=E\left(\bar{a}_x^N(\delta,\lambda_0)-\bar{a}_x(\delta)\right)^2=\frac{\tilde{C}\left(\bar{a}_x(\delta,\lambda_0)\right)}{N}+O\left(N^{-3/2}\right),$$

where $\tilde{C}\left(\bar{a}_x(\delta,\lambda_0)\right)$ is defined by the formula (9).

Theorem 7. Under the conditions of Theorem 2

$$\sqrt{N}\left(\bar{a}_x^N(\delta,\lambda_0)-\bar{a}_x(\delta)\right)\Rightarrow \mathrm{N}_1\left(0,\tilde{C}\left(\bar{a}_x(\delta,\lambda_0)\right)\right).$$

Proof. The statements of Theorems 6 and 7 follow from Theorems 1 and 4 with the usage of the arguments of Sections 3–5.

7. Adaptive Estimator

The statistic $\bar{a}_x^N(\delta,\lambda_0)$ can be used as an estimator for $\bar{a}_x(\delta)$ if we know λ_0; otherwise, it is required to construct an adaptive estimator. We need a more detailed formula for λ_0:

$$\lambda_0=\frac{1}{S(x)DX}\left[\frac{\Phi(x,\delta)}{S(x)}\left(C_2(x)-aS(x)\right)-C_1(x,\delta)+a\Phi(x,\delta)\right]. \tag{11}$$

Using (11), we consider the following adaptive estimator:

$$\bar{a}_x^N(\delta,\hat{\lambda}_0)=\frac{1}{\delta}\left(1-\frac{\Phi_N(x,\delta)}{S_N(x)}-\hat{\lambda}_0(\bar{x}-a)\right) \tag{12}$$

with

$$\hat{\lambda}_0=\frac{1}{s^2S_N(x)}\left[\frac{\Phi_N(x,\delta)}{S_N(x)}\left(\hat{C}_2(x)-aS_N(x)\right)-\hat{C}_1(x,\delta)+a\Phi_N(x,\delta)\right], \tag{13}$$

where $s^2=\frac{1}{N-1}\sum_{i=1}^{N}(X_i-\bar{x})^2$ is an unbiased estimator of the variance DX,

$$\hat{C}_2(x)=N^{-1}\sum_{i=1}^{N}X_i\mathrm{I}(X_i>x),\quad \hat{C}_1(x,\delta)=N^{-1}\sum_{i=1}^{N}e^{-\delta X_i}X_i\mathrm{I}(X_i>x).$$

Theorem 8. Under the conditions of Theorem 2,

$$\sqrt{N}\left(\bar{a}_x^N(\delta,\hat{\lambda}_0)-\bar{a}_x(\delta)\right)\Rightarrow \mathrm{N}_1\left(0,\tilde{C}\left(\bar{a}_x(\delta,\lambda_0)\right)\right).$$

Proof. The following equality holds:

$$\sqrt{N}\left(\bar{a}_x^N(\delta,\hat{\lambda}_0)-\bar{a}_x(\delta)\right)=\sqrt{N}\left(\bar{a}_x^N(\delta,\lambda_0)-\bar{a}_x(\delta)\right)+R_N,$$

where $R_N=\delta^{-1}\left(\lambda_0-\hat{\lambda}_0\right)\sqrt{N}(\bar{x}-a)$. All the estimators, used in (13), converge almost surely to their true values according to the strong law of large numbers (the Second Theorem of Kolmogorov [18]). Thus, from the First Continuity Theorem of Borovkov [7], estimator $\hat{\lambda}_0$ converges almost surely to λ_0. Based on the central limit theorem $\sqrt{N}(\bar{x}-a)\Rightarrow \mathrm{N}_1\left(0,DX\right)$, we retrieve $R_N\Rightarrow 0$. Now, the statement of Theorem 8 is proved by making use of Theorem 7.

Conclusion

The paper deals with the problem of estimating the present values of the continuous whole life annuity using auxiliary information about the expectation of life. It is shown that the usage of such auxiliary information can often provide the MSE smaller than that of standard estimators. We proved the results on asymptotic properties of the proposed estimators: unbiasedness, consistency and normality. Also, the main parts of the asymptotic MSEs of the estimators were found. An adaptive estimator is constructed; such estimator is equivalent (in the sense of asymptotic distribution) to the estimator with the optimal weight coefficient λ_0.

Note that the improved estimators of life annuities (3), (10) and (12) can be obtained by substituting of empirical survival functions by the smooth empirical survival functions (cf. [19–32]).

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Дмитриев Ю.Г., Кошкин Г.М. ОЦЕНИВАНИЕ ПОЖИЗНЕННОЙ РЕНТЫ С ИСПОЛЬЗОВАНИЕМ ИНФОРМАЦИИ О СРЕДНЕЙ ПРОДОЛЖИТЕЛЬНОСТИ ЖИЗНИ. *Вестник Томского государственного университета. Управление, вычислительная техника и информатика.* 2018. № 45. С. 22–29.

Рассматривается проблема оценивания актуарной непрерывной пожизненной ренты с использованием дополнительной информации о средней продолжительности жизни. По данным продолжительностей жизни индивидуумов строятся непараметрические оценки пожизненной ренты. Показано, что использование дополнительной информации приводит к среднеквадратической ошибке, меньшей, чем у стандартной оценки. Также предлагается адаптивная оценка. Показано, что адаптивная оценка эквивалентна в смысле асимптотического распределения оптимальной оценке. Доказана асимптотическая нормальность всех оценок.

Ключевые слова: непараметрическая оценка; пожизненная рента; дополнительная информация; среднеквадратическая ошибка, асимптотическая нормальность.

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