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PARAMETER ESTIMATION AND CHANGE-POINT DETECTION FOR PROCESS AR(P)/ARCH(Q) WITH UNKNOWN PARAMETERS

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The problem of parameter estimation and change point detection of process AR(p)/ARCH(q) is considered. Sequential estimators with bounded standard deviation are proposed and their asymptotic properties are studied. The obtained estimators are used in a sequential change-point detection algorithm; due to usage of the estimators the false alarm and delay probabilities are bounded from above. The results of simulation are presented.
Keywords: AR/ARCH; guaranteed parameter estimation; change-point detection.

The problem of change point detection arises often in different applications connected with time series analysis, financial mathematics, image processing etc. Two types of algorithms are used to detect the change point: a posteriori methods, when the estimation of the change point is conducted in a sample of a fixed size, and sequential methods, when the decision on change point can be taken after obtaining a next observation. Sequential methods include a special stopping rule that determines a stopping time. At this instant a decision on change point can be made. There are two types of errors typical for sequential change point detection procedures: false alarm, when one makes a decision that change is occurred before a change point (type 1 error), and delay, when one makes a decision that change is not occurred after a change point (type 2 error). The properties of the sequential procedures are connected with these errors and include probabilities of the errors, mean delay time and mean time between false alarms.

Last decades, autoregressive type processes and autoregressive conditional heteroscedasticity processes are widely used in various applications, such as forecasting of financial indexes, geographic information systems, medical data analysis, etc. For example in paper [1], autoregressive models are used for description of financial data. In the references therein, one can find examples of applications in other fields, including business, economics, finance and quality control. Processes with non-constant parameters also can be used for such purposes. In [2] a piecewise constant model is set off against usual GARCH model for volatility modelling. A two-sample test for a change in variability is proposed, which works well even in case of skewed distributions. Paper [3] describes a usage of mixtures of structured autoregressive models for the analysis of electroencephalogram. On-line posterior estimation of the model parameters and related quantities is achieved using a sequential Monte Carlo algorithm.

One of recent papers [4] is devoted to change point detection in casual time series such as AR(∞), ARCH(∞), etc. The procedure is based on a discrepancy between the historical parameter estimator and the updated parameter estimator, where both these estimators are quasi-likelihood estimators. To construct these estimators historical observations supposed to be available. It is proven that if the change occurs then it is asymptotically detected with the probability one. Asymptotic behavior of the test statistic can be described using the standard Brownian motion. The power of the test is estimated by simulation. In paper [5], change-point detection is applied to analysis of financial data. A fractionally integrated process is considered and changes in the fractional integration parameter supposed to be detected. The authors use AR(p) model, for some large enough p, to approximate the process under consideration. The application of the tests to World inflation rates detected the presence of changes in persistence for most countries. In [6], some historic data set which is stationary and does not contain a change is used to construct an estimator for the initial set of parameters. Then new incoming observations are monitored for a change. It is shown that the algorithm can be applied to mean change model and to non-linear first-order autoregressive time series.

Theoretical properties of the described procedures are studied asymptotically when the number of observations before a change point tends to infinity. For small samples, usually simulation study is conducted. In this paper, we develop an alternative approach in the frame of guaranteed sequential methods. Due to a special stopping rule, we construct statistics with variances bounded from above by a known constant. Consequently, we can estimate the probabilities of false alarm and delay non-asymptotically, but we also investigate asymptotic properties of the statistics.

1. Model AR/ARCH

We consider scalar autoregressive process AR(p)/ARCH(q) specified by the equation

$$x_k = \lambda_1 x_{k-1} + \ldots + \lambda_p x_{k-p} + \sqrt{\alpha_0 + \alpha_1 x_{k-1}^2 + \ldots + \alpha_q x_{k-q}^2}\,\xi_k \tag{1}$$

Here $\{\xi_k\}_{k\geq 1}$ – is a sequence of independent identically distributed random variables with zero mean and unit variance. The density distribution function $f_\xi(x)$ of $\{\xi_k\}_{k\geq 1}$ is strictly positive for any value of x. Parameters $\Lambda = [\lambda_1,\ldots,\lambda_p]$ and $A = [\alpha_0,\ldots,\alpha_q]$ are supposed to be unknown.

2. Sequential parameter estimator for AR(*p*)/ARCH(*q*)

For parameter estimation of the process (1) we use the approach proposed in [7] for classification of autoregressive processes with unknown noise variance bounded from above. At the first stage, we construct a special factor to compensate the influence of the noise variance. Then, we estimate autoregressive parameters by using this factor.

Since the noise variance of the observed process is unbounded from above, we transform the model by introducing the following notation

$$m_{k-1} = \max\{1, |x_{k-1}|, \ldots, |x_{k-s}|\},$$

where $s = \max\{p,q\}$. Dividing equation (1) by m_{k-1}, we obtain

$$y_k = Z_k \Lambda + \sqrt{X_k A}\,\xi_k , \tag{2}$$

where

$$y_k = \frac{x_k}{m_{k-1}},\ Z_k = \left[\frac{x_{k-1}}{m_{k-1}},\ldots,\frac{x_{k-p}}{m_{k-1}}\right],\ X_k = \left[\frac{1}{m_{k-1}^2},\frac{x_{k-1}^2}{m_{k-1}^2},\ldots,\frac{x_{k-q}^2}{m_{k-1}^2}\right].$$

It is obvious, that the noise variance of the process (3) is bounded from above by the unknown value $\alpha_0 + \ldots + \alpha_q$. We can construct the compensating factor by first n observations in the following form

$$\Gamma_n = B_n \sum_{k=s+1}^{s+n} \frac{x_k^2}{\min\{1, x_{k-1}^2, \ldots, x_{k-q}^2\}},\quad B_n = E\left(\sum_{k=1}^{n}\xi_k^2\right)^{-1} \tag{3}$$

where n observations are taken at the interval where all the values $|x_k|$ are sufficiently large. In [8], we use a similar approach to compensate the noise variance of AR(p) process with unknown noise variance; it was proven that the compensating factor satisfies condition analogous to

$$E\frac{1}{\Gamma_n} \leq \frac{1}{(\alpha_0 + \alpha_1 + \ldots + \alpha_q)} \tag{4}$$

This proof can be generalized for our case with minimum changes so we omit it.

We construct the estimator of the parameter vector Λ in the form

$$\widehat{\Lambda}(H)=C^{-1}(\tau)\left(\sum_{k=n+s+1}^{\tau} v_k y_k Z_k^T\right),\ C(t)=\sum_{k=n+s+1}^{\tau} v_k Z_k^T Z_k\ , \tag{5}$$

where τ is the random stopping time defined as follows

$$\tau=\tau(H)=\min\{t\geq n+s-1:\quad \nu_{\min}(t)\geq H\}, \tag{6}$$

$\nu_{\min}(t)$ is the minimum eigenvalue of the matrix $C(t)$, H is a certain positive parameter. Then we define the weights v_k. Let m be the minimum value of t for which the matrix $C(n+t)$ is not degenerate. The weights on the interval $[n+s+1,n+m]$ are defined as

$$v_k=\begin{cases}\left(\Gamma_n Z_k Z_k^T\right)^{-1/2}, & \text{if } Z_k \text{ is linearly independent with } \{Z_{n+s+1},\dots,Z_{k-1}\};\\ 0, & \text{elsewhere.}\end{cases} \tag{7}$$

The weights on the interval $[n+m,\tau-1]$ are defined from the equations

$$\frac{\nu_{\min}(t)}{\Gamma_n}=\sum_{k=n+s-1}^{t} v_k^2 Z_k Z_k^T. \tag{8}$$

The last weight v_τ is found from condition

$$\frac{\nu_{\min}(\tau)}{\Gamma_n}\geq\sum_{k=n+s+1}^{\tau} v_k^2 Z_k Z_k^T,\quad \nu_{\min}(\tau)=H. \tag{9}$$

Theorem 1. The stopping time $\tau(H)$ is finite with probability one and the mean-square accuracy of the estimator $\widehat{\Lambda}(H)$ is bounded from above

$$E\left\|\widehat{\Lambda}(H)-\Lambda\right\|^2\leq\frac{H+p-1}{H^2}. \tag{10}$$

Proof. According to [9], the stopping time $\tau(H)$ is finite with probability one if

$$\sum_{k=n+s+1}^{\infty} v_k^2 Z_k Z_k^T=\infty \text{ a.s.}$$

The equation for v_k (14) can be rewritten in the form

$$\frac{1}{\Gamma_n}\min_{x:\|x\|=1}\left(x,\left(C(k-1)+v_k Z_k^T Z_k\right)x\right)=\frac{\nu_{\min}(k-1)}{\Gamma_n}+v_k^2 Z_k Z_k^T.$$

It implies that for any vector $x:\|x\|=1$

$$\left(x,C(k-1)x\right)+v_k\left(x,Z_k^T Z_k x\right)=\left(x,C(k-1)x\right)+v_k\left(Z_k x\right)^2\leq\nu_{\min}(k-1)+v_k^2\Gamma_n Z_k Z_k^T;$$

hence,

$$-v_k^2\Gamma_n Z_k Z_k^T+v_k\left(Z_k x\right)^2+\left(\left(x,C(k-1)x\right)-\nu_{\min}(k-1)\right)\leq 0.$$

For a certain vector $b_k:\|b_k\|=1$ the inequality turns into the equality and the weight v_k is a root of the quadratic equation. As $\left(b_k,C(k-1)b_k\right)-\nu_{\min}(k-1)\geq 0$, then the equation has two roots: non-positive and non-negative. It gives us the equation for the weight

$$v_k=\min_{x:\|x\|=1}\frac{\left(Z_k b_k\right)^2+\sqrt{\left(Z_k b_k\right)^4+4\Gamma_n Z_k Z_k^T\left(\left(b_k,C(k-1)b_k\right)-\nu_{\min}(k-1)\right)}}{2\Gamma_n Z_k Z_k^T}.$$

Consequently, v_k tends to zero if and only if $Z_k b_k$ tends to zero and at the same time b_k tends to the eigenvector corresponding to the minimum eigenvalue of the matrix $C(k–1)$ as k tends to infinity. As the first component of the vector Z_k depends on ξ_k (1) which can take any value then v_k does not tend to zero with non-zero probability and the instant τ is finite with the probability one.

For the mean-square accuracy of $\widehat{\Lambda}(H)$ (5), by using (2), Cauchy–Schwarz–Bunyakovskii inequality, inequality $\|C(t)\| \geq \nu_{\min}(t)$ and (4), we obtain

$$E\left\|\Lambda^*(H)-\Lambda\right\|^2 = E\left\|C^{-1}(\tau)\right\|^2 \left\|\sum_{k=n+s-1}^{\tau} v_k \sqrt{X_k A}\xi_{k+1} Z_k^T\right\|^2 \leq \frac{\alpha_1+\ldots+\alpha_q}{H^2} E\left\|\sum_{k=n+s-1}^{\tau} v_k Z_k^T \xi_{k+1}\right\|^2 .$$

For the second multiplier,

$$E\left\|\sum_{k=n+s+1}^{\tau} v_k \xi_{k+1} Z_k^T\right\|^2 = E\sum_{k=n+s+1}^{\tau} v_k^2 Z_k Z_k^T \xi_{k+1}^2 + 2E\sum_{k=n+s+1}^{\tau}\sum_{l=n+s+2}^{k-1} v_l v_k Z_l Z_k^T \xi_{l+1}\xi_{k+1}. \quad (11)$$

Consider a truncated stopping instant $\tau(N)=\min\{\tau, N\}$. Consider the sum differing from the first summand only in the upper limit. Let $F_k=\sigma(\xi_1,\ldots,\xi_k)$ be the σ-algebra generated by $\{\xi_1,\ldots,\xi_k\}$, then τ defined by (6) is a Markovian instant with respect to $\{F_k\}$. Hence,

$$E\sum_{k=n+s+1}^{\tau(N)} v_k^2 Z_k Z_k^T \xi_{k+1}^2 = E\sum_{k=n+s+1}^{N} E\left[v_k^2 Z_k Z_k^T \chi_{k\leq\tau}\xi_{k+1}^2 \middle| F_k\right] = E\sum_{k=n+s+1}^{N} v_k^2 Z_k Z_k^T \chi_{k\leq\tau} E\left[\xi_{k+1}^2 \middle| F_k\right] = E\sum_{k=n+s+1}^{\tau(N)} v_k^2 Z_k Z_k^T .$$

As $\tau(N)\to\tau$ while $N\to\infty$, and taking into account (7)–(9) we obtain

$$E\sum_{k=n+s+1}^{\tau(N)} v_k^2 Z_k Z_k^T \xi_{k+1}^2 \to E\sum_{k=n+s+1}^{\tau} v_k^2 Z_k Z_k^T = E\sum_{k=n+s+1}^{\tau} v_k^2 Z_k Z_k^T + E\sum_{k=n+s+1}^{\tau} v_k^2 Z_k Z_k^T \leq E\left(\frac{H+p-1}{\Gamma_n}\right).$$

Similarly, we can show that the second summand in (11) is equal to zero. The obtained results together with (4) imply (10).

3. Asymptotic properties of the estimator

We establish properties of estimator (5) for sufficiently large values of H. In paper [10], we have proven a martingale central limit theorem for the vector case, using the stochastic exponent method.

Theorem 2. Let $\left\{F_k^n\right\}_{k\geq 0}$ be a non-decreasing sequence of σ-algebras, $F_0^n=\{\varnothing,\Omega\}$, $F_k^n \subseteq F_{k+1}^n$ for all $0 \leq k < n$. Let $Y^n=\sum_{k=1}^{\tau_n}\eta_k^n$, where $\tau_n \leq n$ is a Markovian instant with respect to $\left\{F_k^n\right\}$. Suppose for all $n \geq 1$ the sequence $\eta^n=\left(\eta_k^n, F_k^n\right)$ is a martingale-difference sequence, $E\left\|\eta_k^n\right\|^2<\infty$, and

$$(A)\ \max_{0\leq k\leq n}\left\|\eta_k^n\right\|^2 \xrightarrow{P} 0, \quad (B)\ \sum_{k=0}^{\tau_n} E\left[\eta_k^n\left(\eta_k^n\right)^T \middle| F_{k-1}^N\right] \xrightarrow{P} \Sigma ,$$

where Σ is a symmetric positive definite matrix with $E\Sigma < \infty$. Then $Y^n \xrightarrow{d} Y$ where Y is a random vector with the characteristic function

$$E\exp\left\{i\lambda^T Y\right\} = E\exp\left\{-\frac{1}{2}\lambda^T\Sigma\lambda\right\}.$$

The theorem allows establishing asymptotic properties of estimator (7). Lemma 1 proven in [11] allows us to obtain more precise results than in [8] and [10].

Lemma 1. Let $\xi_1, \ldots, \xi_n$ be independent identically distributed standard Gaussian variables. Then, for any $\lambda_1, \ldots, \lambda_n$, $\lambda_i \geq 0$, $\lambda_1 + \ldots + \lambda_n = 1$ and for sufficiently large C

$$P\left\{\lambda_1\xi_1^2+\ldots+\lambda_n\xi_n^2 > C\right\} \leq P\left\{\xi_1^2 > C\right\}.$$

Theorem 3. If for noise variables ξ_k in (1), $E\xi_k^4<\infty$, and process (1) is ergodic, then the mean-squared deviation of estimator (7) is bounded from above

$$P\left\{\left\|\hat{\Lambda}(H)-\Lambda\right\|^2 > x\right\} \le 2\left(1-\Phi\left(\frac{xH^2}{H-p+1}\right)\right), \quad (12)$$

where $\Phi(\cdot)$ is the standard Gauss distribution function.

Proof. For estimator (7).

$$\left\|\hat{\Lambda}(H)-\Lambda\right\|^2 = \left\|C^{-1}(\tau)\right\|^2 \left\|\sum_{k=n+1}^{\tau} v_k\sqrt{X_k A}\xi_{k+1}Z_k^T\right\|^2 \le \frac{\alpha_0+\ldots+\alpha_q}{H}\left\|\sum_{k=n+1}^{\tau}\frac{v_k}{\sqrt{H}}Z_k^T\xi_{k+1}\right\|^2. \quad (13)$$

Introduce a truncated stopping instant $\tau_N = \min\{\tau, N\}$ and the following notation:

$$\eta_k^N = \frac{v_k}{\sqrt{H}} Z_k^T \xi_{k+1}\chi_{k\le\tau}.$$

then $\left\|\eta_k^N\right\|^2 = \frac{1}{H}\left\|v_k Z_k^T\xi_{k+1}\chi_{k\le\tau}\right\|^2 \le \frac{p}{H}\xi_{k+1}^2$. Using the Chebyshev and Cauchy–Schwarz–Bunyakovskii inequalities and (8), one obtains

$$P\left\{\max_{1\le k\le N}\left\|\eta_k^N\right\|^2 > a\right\} \le P\left\{\sum_{k=n+s+1}^{N}\left\|\eta_k^N\right\|^2\chi_{\left\|\eta_k^N\right\|^2>a} > a\right\} \le \frac{1}{a}E\sum_{k=n+s+1}^{N}\left\|\eta_k^N\right\|^2\chi_{\left\|\eta_k^N\right\|^2>a}$$

$$= \frac{1}{aH}E\sum_{k=n+s+1}^{N}\chi_{k\le\tau}E\left[\left\|v_k Z_k\xi_{k+1}\right\|^2\chi_{\left\|\eta_k^N\right\|^2>a}\middle| F_k\right] \le \frac{1}{aH}E\sum_{k=n+s+1}^{\tau_N}\left\|v_k Z_k\right\|^2\sqrt{E\xi_{k+1}^4}\sqrt{P\left\{\xi_{k+1}^2 > \frac{aH}{v_k^2\left\|Z_k\right\|^2}\right\}}$$

$$\le \frac{E\xi_{k+1}^4}{(aH)^{3/2}}E\sum_{k=n+s+1}^{\tau_N}\left\|v_k Z_k\right\|^2\sqrt{v_k^2\left\|Z_k\right\|^2} \le E\frac{\xi_{k+1}^4}{a^{3/2}H^{1/2}\Gamma_n}\max_{1\le k\le N}\sqrt{v_k^2\left\|Z_k\right\|^2}.$$

As $\left\|Z_k\right\|^2 \le p$, condition (A) of Theorem 2 holds true for $H\to\infty$. To check condition (B), consider matrix

$$\Sigma_N = \sum_{k=n+s+1}^{\tau_N}E\left[\eta_{k+1}^N\left(\eta_{k+1}^N\right)^T\middle| F_k^N\right] = \sum_{k=n+s+1}^{\tau_N}\frac{v_k^2 Z_k^T Z_k}{H}E\left[\xi_{k+1}^2\middle| F_k^N\right] = \frac{1}{H}\sum_{k=n+s+1}^{\tau_N}v_k^2 Z_k^T Z_k.$$

Taking into account (7)–(9), one obtains that the trace of the matrix is bounded from above

$$\mathrm{tr}\Sigma_N = \frac{1}{H}\sum_{k=n+s+1}^{\tau_N}v_k^2\left\|Z_k\right\|^2 \le \frac{H+p-1}{\Gamma_n H}. \quad (14)$$

Then, $\Sigma_t(i,j)$ is a Cauchy sequence with respect to the convergence in probability. For any $t > m$

$$\left|\Sigma_t(i,j)-\Sigma_m(i,j)\right| \le \frac{1}{H\Gamma_n}\sum_{k=m}^{t}v_k^2\left|Z_k(i)Z_k(j)\right|\chi_{k\le\tau} \le \frac{1}{H\Gamma_n}\sum_{k=m}^{t}v_k^2\left\|Z_k\right\|^2\chi_{k\le\tau}.$$

Inequality (14) imply the convergence in probability of the last sum as t, $m\to\infty$. This and (13) provide condition (*B*). Letting N go to infinity, one obtains the vector Y from Theorem 2

$$Y = \lim_{N\to\infty}Y^N = \sum_{k=n+s+1}^{\tau}\frac{v_k}{\sqrt{H}}Z_k^T\xi_{k+1}$$

According to (13),

$$P\left\{\left\|\hat{\Lambda}(H)-\Lambda\right\|^2 > x\right\} \le P\left\{\left\|\sum_{k=n+s+1}^{\tau}\frac{v_k}{\sqrt{H}}Z_k^T\xi_{k+1}\right\|^2 > \frac{xH}{B}\right\} = P\left\{\left\|Y\right\|^2 > \frac{xH}{B}\right\}$$

Using the result of Theorem 2 and Fubini's theorem to change the order of integration one obtains

$$P\left\{\left\|Y\right\|^2 > \frac{xH}{B}\right\} = \int_{YY^T > xH/B}\int_{-\infty}^{\infty}\frac{\exp\left\{i\lambda^T Y\right\}}{2\pi}E\exp\left\{-\frac{1}{2}\lambda^T\Sigma\lambda\right\}d\lambda dY = E\frac{1}{\sqrt{(2\pi)^p\left|\Sigma\right|}}\int_{YY^T > xH/B}\exp\left\{-\frac{1}{2}Y^T\Sigma^{-1}Y\right\}dY.$$

The matrix Σ is symmetric and positive definite; hence, an orthogonal transformation T, resulting in the matrix Σ to diagonal form Σ', exists. Using the change of variables $S = Y\Sigma^{-1/2}T^T$ one obtains

$$P\left\{\|Y\|^2 > \frac{xH}{B}\right\} = \frac{1}{\sqrt{(2\pi)^p}} E \int\limits_{\sum_{i=1}^{p}\nu_i s_i^2 > xH/B} \exp\left\{-\frac{1}{2}S^T S\right\} dS = EP\left\{\sum_{i=1}^{p}\nu_i s_i^2 > xH/B\right\},$$

where ν_i are the eigenvalues of the matrix Σ', and s_i are the independent components of the Gaussian vector S. As the sum of ν_i is equal to Σ, (14) and Lemma 1 imply (12).

4. Change-point detection

Describe now the change point detection problem for process (1). Suppose that after a certain instant θ, parameter vectors (Λ, A) change their values from (Λ_0, A_0) to (Λ_1, A_1), and $\|\Lambda_0 - \Lambda_1\|^2 \geq \Delta$. We construct a series of sequential estimation plans $\left(\tau_i, \hat{\Lambda}_i\right)$, where $\{\tau_i\}$ is the increasing sequence of the stopping instances ($\tau_0 = -1$), and $\hat{\Lambda}_i$ is the guaranteed parameter estimator (5) on the interval $[\tau_{i-1} + 1, \tau_i]$. Then we choose an integer $l > 1$ and associate the statistic J_i with the i-th interval for all $i > l$

$$J_i = \left(\hat{\Lambda}_i - \hat{\Lambda}_{i-l}\right)^T \left(\hat{\Lambda}_i - \hat{\Lambda}_{i-l}\right). \qquad (15)$$

This statistic is the squared deviation of the estimators with numbers i and $i - l$. Due to using estimators (5) with properties (10) and (12) the proposed statistics change their expectation after a change point.

Theorem 4. The expectation of statistic J_i (15) satisfies the following inequality:

$$E\left[J_i \middle| \tau_i < \theta\right] \leq \frac{4(H+p-1)}{H^2}, \quad E\left[J_i \middle| \tau_{i-l} < \theta \leq \tau_{i-1}\right] \geq \Delta - 4\frac{\sqrt{\Delta(H+p-1)}}{H}. \qquad (16)$$

Proof is based on property (10); it is very similar to one described in [7].

Hence, the Theorem allows us to construct the following change-point detection algorithm. The J_i values are compared with a certain threshold δ, where

$$\frac{4(H+p-1)}{H^2} < \delta < \Delta - 4\frac{\sqrt{\Delta(H+p-1)}}{H}.$$

The change point is considered to be detected when the value of the statistic exceeds δ.

The probabilities of false alarm and delay in the change-point detection in any observation cycle are important characteristics of any change point detection procedure. Due to the application of the guaranteed parameter estimators in the statistics, we can obtain the upper bounds for these probabilities.

Theorem 5. The probability of false alarm P_0 and the probability of delay P_1 in any observation cycle $[\tau_{i-1} + 1, \tau_i]$ are bounded from above

$$P_0 \leq 4(H+p-1)\big/\delta H^2, \quad P_1 \leq 4(H+p-1)\Big/\left(\sqrt{\Delta}-\sqrt{\delta}\right)^2 H^2. \qquad (17)$$

Proof is based on property (15), Cauchy–Schwarz–Bunyakovskii inequality and the following equalities

$$P_0 = P\left\{J_i > \delta \middle| \tau_i < \theta\right\}, \quad P_1 = P\left\{J_i < \delta \middle| \tau_{i-l} < \theta \leq \tau_{i-1}\right\}.$$

Asymptotic properties of the estimators let us establish the following asymptotic upper bounds of the error probabilities.

Theorem 6. For ergodic process (1) in the conditions of Theorem 3, for sufficiently large H

$$P_0 \leq 4\left(1-\Phi\left(\sqrt{\delta}H\Big/2\sqrt{(H+p-1)}\right)\right), \quad P_1 \leq 4\left(1-\Phi\left(\left(\sqrt{\Delta}-\sqrt{\delta}\right)H\Big/2\sqrt{(H+p-1)}\right)\right). \qquad (18)$$

where $\Phi(x)$ is the standard normal distribution function.

5. Simulation results and their discussion

We conducted numerical simulation of the proposed estimation and change point detection algorithms for AR(p)/ARCH(q) process. For every set of the parameters, 100 replications of the experiment were performed. First, we considered the parameter estimation problem for the AR(2)/ARCH(2) process rewritten in a special form (2). The noise variation of the process is bounded from above by the value $\alpha_0 + \alpha_1 + \alpha_2 = 0,6 + 0,1 + 0,3 = 1$. The number n was chosen as the integral part of $H^{1/2}$. Table 1 presents the results. Here H is the parameter of the procedure, $\hat{\lambda}_1$ and $\hat{\lambda}_2$ are the estimators of the parameters $\lambda_1 = 0,5$ and $\lambda_2 = 0,1$, T is the mean interval of the estimation, Γ_n is the compensating factor (3), σ^2 is the sample standard deviation, D is the upper bound for the standard deviation of the estimator given by inequality (10).

Table 1

Parameter estimation for the AR(2)/ARCH(2) process

H	$\hat{\lambda}_1$	$\hat{\lambda}_2$	T	σ^2	Γ_n	D
50	0,4996	0,0976	748	0,0067	1,99	0,0204
75	0,4996	0,1008	1 073	0,0044	1,89	0,0135
100	0,4989	0,0986	1 355	0,0036	1,79	0,0101
125	0,4994	0,1016	1 661	0,003	1,74	0,0081
150	0,5011	0,0996	1 961	0,0026	1,71	0,0067

One can see that the mean number of the observations increases linearly by H. This property is important for sequential estimators [12]. The sample mean square error of the estimation is about three times less than the theoretical one. It is connected with rather complicated structure of the AR/ARCH process. It has unbounded noise variation so we rewrite the equation in a special form by dividing the equation by the value not less than 1. As a result, the minimum eigenvalue of matrix C in (5) grows rather slowly and that implies the increase of the estimation interval.

Further we conducted simulations of the proposed change-point detection algorithm. Simulations were conducted for the AR(2)/ARCH(2) process specified by the equation in the special form (2) with parameters given in Table 2.

Table 2

Parameters the AR(2)/ARCH(2) process

H	λ_1	λ_2	α_0	α_1	α_2
Before the change point	0,5	0,1	0,6	0,1	0,3
After the change point	0,1	0,3	0,6	0,3	0,1

In this process the noise variance is bounded from above by 1 both before and after the change point.

The change point $\theta = 10\ 000$ and $\Delta = 0,2$. Table 3 presents the results of the simulation. Here H and δ are the parameters of the procedure, T_1 is the mean delay in the change-point detection, $\hat{p}_0$ and $\hat{p}_1$ are the sample probabilities of the false alarm and of the delay, respectively, P_0 and P_1 are the asymptotic upper bounds for the probabilities expressed by formulas (18). False alarms were registered only in one case.

Table 3

Change-point detection for the AR(2)/ARCH(2) process

H	δ	T_1	$\hat{p}_0$	$\hat{p}_1$	P_0	P_1
150	0,03	1 595	0,0	0,0	0,581	0,189
175	0,03	2 093	0,0	0,0	0,506	0,141
200	0,03	3 290	0,0	0,0	0,439	0,107
150	0,05	1 819	0,0	0,0	0,345	0,345
175	0,05	2 366	0,0	0,0	0,281	0,281
200	0,05	4 048	0,0	0,0	0,229	0,229
150	0,07	1 891	0,0	0,0	0,213	0,53
175	0,07	2 413	0,0	0,054	0,162	0,457
200	0,03	4 347	0,0	0,0	0,121	0,395

In the example above, the difference between the parameters before and after change point is not significant ($\Delta = 0,2$) so we considered the second example of process with parameters given in Table 4.

Table 4

Parameters the AR(2)/ARCH(2) process

H	λ_1	λ_2	α_0	α_1	α_2
Before the change point	0,5	0,1	0,6	0,1	0,3
After the change point	–0,1	0,8	0,6	0,3	0,1

Here $\Delta = 0,85$, so it is possible to choose the parameter H less than in the first case. The results of the simulation are presented in Table 5.

Table 5

Change-point detection for the AR(2)/ARCH(2) process

H	δ	T_1	$\hat{p}_0$	$\hat{p}_1$	P_0	P_1
50	0,1	601	0,002	0,118	0,537	0,0679
100	0,1	1 237	0,0	0,11	0,231	0,0052
150	0,1	1 663	0,0	0,102	0,107	0,0004
50	0,2125	804	0,0	0,135	0,213	0,213
100	0,2125	1 320	0,0	0,07	0,044	0,044
150	0,2125	1 841	0,0	0,0	0,001	0,001
50	0,3	897	0,0	0,11	0,11	0,38
100	0,3	1 400	0,0	0,0	0,013	0,125
150	0,3	2 035	0,0	0,07	0,002	0,045

One can see that when the difference between the parameters is sufficiently large then the sample error probabilities are many fewer than their theoretical upper bounds. Moreover, generally no false alarms and skipping the change point were registered.

Conclusion

The change point detection algorithm for the AR(p)/ARCH(q) process with unknown parameters before and after the change point has been constructed. The algorithm is based on the weighted least square method. The guaranteed sequential estimators of unknown parameters are used. The choice of weights and stopping rule guarantees the prescribed accuracy of the estimation and hence the prescribed error probabilities in every observation interval. The results of numerical simulation prove the possibility to use the suggested algorithm used for change point detection of recurrent processes with unknown noise variance. However, the algorithms should be improved through more accurate compensation of the noise variance.

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Воробейчиков С.Э., Буркатовская Ю.Б. ОЦЕНКА ПАРАМЕТРА И ОБНАРУЖЕНИЯ РАЗЛАДОК ПРОЦЕССА AR(p)/ARCH(q) С НЕИЗВЕСТНЫМИ ПАРАМЕТРАМИ. *Вестник Томского государственного университета. Управление, вычислительная техника и информатика.* 2019. № 46. С. 40–48

Рассматриваются задачи оценивания параметров и обнаружения разладок процесса AR(p)/ARCH(q) с неизвестными параметрами. Строится последовательная оценка по взвешенному методу наименьших квадратов. Использование специального момента остановки и весов позволяет ограничить среднеквадратическое отклонение оценки заранее заданной величиной. Предложенные оценки применяются в алгоритме обнаружения изменения параметров и позволяют ограничить сверху вероятности ложной тревоги и запаздывания в обнаружении разладки. Исследованы неасимптотические и асимптотические свойства алгоритмов.

Ключевые слова: AR/ARCH; гарантированное оценивание параметров; обнаружение разладок.

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