МАТЕМАТИКА

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COMMUTATIVE FEEBLY INVO-CLEAN GROUP RINGS

A commutative ring R is called *feebly invo-clean* if any its element is of the form $v+e-f$, where v is an involution and e,f are idempotents. For every commutative unital ring R and every abelian group G we find a necessary and sufficient condition only in terms of R, G and their sections when the group ring $R[G]$ is feebly invo-clean. Our result improves two recent own achievements about commutative invo-clean and weakly invo-clean group rings, published in Univ. J. Math. & Math. Sci. (2018) and Ural Math. J. (2019), respectively.

Keywords: *invo-clean rings*, *weakly invo-clean rings*, *feebly invo-clean rings*, *group rings*.

1. Introduction and Conventions

Throughout the current paper, we will assume that all groups G are multiplicative abelian and all rings R with Jacobson radical $J(R)$ are associative, containing the identity element 1 which differs from the zero element 0. The standard terminology and notation are mainly in agreement with [9 and 10], whereas the specific notion and notation shall be explained explicitly below. As usual, both objects R and G form the group ring $R[G]$ of G over R.

The next concepts appeared in [1, 2, and 3], respectively.

Definition 1.1. A ring R is said to be *invo-clean* if, for each $r\in R$, there exist an involution v and an idempotent e such that $r=v+e$. If $r=v+e$ or $r=v-e$, the ring is called *weakly invo-clean*.

The next necessary and sufficient condition for a commutative ring R to be invo-clean was established in [1, 2], namely: *A ring R is invo-clean if, and only if, $R\cong R_1\times R_2$, where R_1 is a nil-clean ring with $z^2=2z$ for all $z\in J(R_1)$, and R_2 is a ring of characteristic 3 whose elements satisfy the equation $x^3=x$*. Moreover, it was proved in [6] that *a ring R is weakly invo-clean $\Leftrightarrow$ either R is invo-clean or R can be decomposed as $R=K\times\mathbb{Z}_5$, where $K=\{0\}$ or K is invo-clean*.

The above two notions could be expanded as follows:

Definition 1.2. A ring R is said to be *feebly invo-clean* if, for each $r\in R$, there exist an involution v and idempotents e,f such that $r=v+e-f$.

We will give up in the sequel an useful criterion for a commutative ring to be feebly invo-clean in order to be successfully applied to commutative group rings (compare with Proposition 2.2).

It was asked in [6] to find a suitable criterion only in terms of the commutative unital ring R and the abelian group G when the group ring $R[G]$ is feebly invo-clean. So, the goal of this short article is to address that question in the affirmative. Some related results in this area can also be found in [4 and 7].

2. The Characterization Result

We begin here with the following key formula from [8] which will be freely used below without concrete citation: Suppose that R is a commutative ring and G is an abelian group. Then

$$J(R[G]) = J(R)[G] + \left\langle r(g-1) | g \in G_p, pr \in J(R) \right\rangle,$$

where G_p designates the p-primary component of G.

The next two technicalities are crucial for our further considerations.

Lemma 2.1. *Let K be a commutative ring of characteristic 5. Then K is feebly invo-clean $\Leftrightarrow$ $x^5 = x$ holds for any $x \in K$.*

Proof. The "left-to-right" implication is almost trivial as writing $x = v + e - f$ with $v^2 = 1$, $e^2 = e$ and $f^2 = f$, we have that $x^5 = (v+e-f)^5 = v^5 + e^5 - f^5 = v + e - f = x$, as asserted.

As for the "right-to-left" implication, we process like this: Given an arbitrary non-identity element x in K. Then the subring, S, generated by 1 and x will have the same property, namely its characteristic is again 5 and $y^5 = y$ for all $y \in S$. So, with no harm of generality, we may replace K by this subring S, and thus it needs to prove the wanted representation property in S only. To that purpose, we claim that S is isomorphic to a quotient of the factor-ring $\mathbb{Z}_5[X]/(X^5 - X) \cong \mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_5$ of the polynomial ring $\mathbb{Z}_5[X]$ over $\mathbb{Z}_5$. In fact, we just consider the map $\mathbb{Z}_5[X] \to S$, defined by mapping $X \to x$, which is elementary checked to be a surjective homomorphism with kernel which contains the ideal generated by $X^5 - X$, and henceforth the classical Homomorphism Theorem works to get the desired claim. Working now in the direct product of five copies of the five-element field $\mathbb{Z}_5 = \{0,1,2,3,4 | 5 = 0\}$, a plain technical argument gives our wanted initial assertion that S and hence K are both feebly invo-clean. This is subsumed by the presentations $0 = 1 + 0 - 1$, $1 = 1 + 0 - 0$, $2 = 1 + 1 - 0$, $3 = 4 + 0 - 1$ and $4 = 4 + 0 - 0$, where $4^2 = 1$, $1^2 = 1$ and $0^2 = 0$. □

Proposition 2.2. *A commutative ring R is feebly invo-clean $\Leftrightarrow$ $R = P \times K$ for two rings P, K, where $P = \{0\}$ or P is invo-clean, and $K = \{0\}$ or K possesses characteristic 5 such that $x^5 = x$, $\forall x \in K$.*

Proof. "$\Rightarrow$". It follows from the corresponding characterization method used in [3, Theorem 2.6].

"$\Leftarrow$". Firstly, it needs to show that K is feebly invo-clean. This, however, follows directly from Lemma 2.1. Furthermore, one suffices to observe again with [3, Theorem 2.6] at hand that the direct product of such a ring K with an invo-clean ring remains a feebly invo-clean ring, thus getting resultantly that R is feebly invo-clean, as expected. □

We are now ready to proceed by proving the following preliminary statement (see [5] as well).

Proposition 2.3. *Suppose R is a non-zero commutative ring and G is an abelian group. Then $R[G]$ is invo-clean if, and only if, R is invo-clean having the decomposition $R = R_1 \times R_2$ such that precisely one of the next three items holds*:

(0) $G = \{1\}$

or

(1) $|G| > 2$, $G^2 = \{1\}$, $R_1 = \{0\}$ *or* R_1 *is a ring of* $char(R_1) = 2$, *and* $R_2 = \{0\}$, *or* R_2 *is a ring of* $char(R_2) = 3$

or

(2) $|G| = 2$, $2r_1^2 = 2r_1$ *for all* $r_1 \in R_1$ (*in addition* $4 = 0$ *in* R_1), *and* $R_2 = \{0\}$ *or* R_2 *is a ring of* $char(R_2) = 3$.

Proof. If G is the trivial i.e., the identity group, there is nothing to do, so we shall assume hereafter that G is non-identity.

"**Necessity.**" Since there is an epimorphism $R[G] \to R$, and an epimorphic image of an invo-clean ring is obviously an invo-clean ring (see, e.g., [1]), it follows at once that R is again an invo-clean ring. According to the criterion for invo-cleanness alluded to above, one writes that $R = R_1 \times R_2$, where R_1 is a nil-clean ring with $a^2 = 2a$ for all $a \in J(R_1)$ and R_2 is a ring whose elements satisfy the equation $x^3 = x$. Therefore, it must be that $R[G] \cong R_1[G] \times R_2[G]$, where it is not too hard to verify by [1] that both $R_1[G]$ and $R_2[G]$ are invo-clean rings.

First, we shall deal with the second direct factor $R_2[G]$ being invo-clean. Since $\mathrm{char}(R_2) = 3$, it follows immediately that $\mathrm{char}(R_2[G]) = 3$ too. Thus an application of an assemble of facts from [1, 2] allows us to deduce that all elements in $R_2[G]$ also satisfy the equation $y^3 = y$. So, given $g \in G \subseteq R[G]$, it follows that $g^3 = g$, that is, $g^2 = 1$.

Next, we shall treat the invo-cleanness of the group ring $R_1[G]$. Since $\mathrm{char}(R_1)$ is a power of 2 (see [1]), it follows the same for $R_1[G]$. Consequently, utilizing once again an assortment of results from [1, 2], we infer that $R_1[G]$ should be nil-clean, so that $z^2 = 2z$ for all $z \in J(R_1[G])$. That is why, invoking the criterion from [7], we have that G is a 2-group. We claim that even $G^2 = 1$. In fact, for an arbitrary $g \in G$, we derive with the aid of the aforementioned formula from [8] that $1 - g \in J(R_1[G])$, because $2 \in J(R_1)$. Hence $(1-g)^2 = 2(1-g)$ which forces that $1 - 2g + g^2 = 2 - 2g$ and that $g^2 = 1$, as desired. We now assert that $\mathrm{char}(R_1) = 2$ whenever $|G| > 2$. To that purpose, there are two nonidentity elements $g \neq h$ in G with $g^2 = h^2 = 1$. Furthermore, again appealing to the formula from [8], the element $1 - g + 1 - h = 2 - g - h$ lies in $J(R_1[G])$, because $2 \in J(R_1)$. Thus $(2-g-h)^2 = 2(2-g-h)$ which yields that $2 - 2g - 2h + 2gh = 0$. Since $gh \neq 1$ as for otherwise $g = h^{-1} = h$, a contradiction, this record is in canonical form. This assures that $2 = 0$, as wanted.

However, in the case when $|G|=2$, i.e. when $G=\{1,g|g^2=1\}=\langle g\rangle$, we can conclude that $2r^2=2r$ for any $r\in R_1$. Indeed, in view of the already cited formula from [8], the element $r(1-g)$ will always lie in $J(R_1[G])$, because $2\in J(R_1)$. We therefore may write $[r(1-g)]^2=2r(1-g)$ which ensures that $2r^2-2r^2g=2r-2rg$ is canonically written on both sides. But this means that $2r^2=2r$, as pursued. Substituting $r=2$, one obtains that $4=0$. Notice also that $2r^2=2r$ for all $r\in R_1$ and $a^2=2a$ for all $a\in J(R_1)$ will imply that $a^2=0$.

"**Sufficiency.**" Foremost, assume that (1) is true. Since R_1 has characteristic 2, whence it is nil-clean, and G is a 2-group, an appeal to [7] allows us to get that $R_1[G]$ is nil-clean as well. Since $z^2=2z=0$ for every $z\in J(R_1)$, it is routinely checked that $\delta^2=2\delta=0$ for each $\delta\in J(R_1[G])$, exploiting the formula from [8] for $J(R_1[G])$ and the fact that $R_1[G]$ is a modular group algebra of characteristic 2. That is why, by a consultation with [1], one concludes that $R_1[G]$ is invo-clean, as expected. Further, by a new usage of [1], we derive that $R_2[G]$ is an invo-clean ring of characteristic 3. To see that, given $x\in R_2[G]$, we write $x=\sum_{g\in G} r_g g$ with $r_g\in R_2$ satisfying $r_g^3=r_g$. Since $G^2=1$ will easily imply that $g^3=g$, one obtains that $x^3=(\sum_{g\in G} r_g g)^3=\sum_{g\in G} r_g^3 g^3=\sum_{g\in G} r_g g=x$, as needed. We finally conclude with the help of [1] that $R[G]\cong R_1[G]\times R_2[G]$ is invo-clean, as expected.

Let us now point (2) be fulfilled. Since $G^2=1$, similarly to (1), R_2 being invo-clean of characteristic 3 implies that $R_2[G]$ is invo-clean, too. In order to prove that $R_1[G]$ is invo-clean, we observe that R_1 is nil-clean with $2\in J(R_1)$. According to [7], the group ring $R_1[G]$ is also nil-clean. What remains to show is that for any element δ of $J(R_1[G])$ the equality $\delta^2=2\delta$ is valid. Since in conjunction with the explicit formula quoted above for the Jacobson radical, an arbitrary element in $J(R_1[G])$ has the form $j+j'g+r(1-g)$, where $j,j'\in J(R_1)$ and $r\in R_1$, we have that $[j+j'g+r(1-g)]^2\in(J(R_1)^2+2J(R_1))[G]+r^2(1-g)^2$. However, using the given conditions, $z^2=2z=2z^2$ and thus $z^2=2z=0$ for any $z\in J(R_1)$. Consequently, one checks that $[j+j'g+r(1-g)]^2=r^2(1-g)^2=2r^2(1-g)=2r(1-g)=2[j+j'g+r(1-g)]$, because $2r^2=2r$, as required. Therefore, $R_1[G]$ is invo-clean with [1] at hand. Finally, again [1] gives that $R[G]\cong R_1[G]\times R_2[G]$ is invo-clean, as promised. □

It is worthwhile noticing that concrete examples of an invo-clean ring of characteristic 4, such that its elements are solutions of the equation $2r^2=2r$, are the rings $\mathbb{Z}_4$ and $\mathbb{Z}_4\times\mathbb{Z}_4$.

We thereby come to our main theorem which states the following:

Theorem 2.4. *Let G be an abelian group and let R be a commutative non-zero ring. Then the group ring $R[G]$ is feebly invo-clean if, and only if, at most one of the next points is valid*:

(1) $G=\{1\}$ *and* R *is feebly invo-clean.*

(2) $G\neq\{1\}$ *and* $R\cong P\times K$, *where* $P\cong R_1\times R_2$ *is an invo-clean ring and either* $K=\{0\}$ *or* K *is a ring of* $char(K)=5$ *which is a subdirect product of a family of copies of the field* $\mathbb{Z}_5$ *such that either*

(2.1) $P=\{0\}$ *and* $G^4=\{1\}$

or

(2.2) $|G|>2$, $G^2=\{1\}$, $P\neq\{0\}$ *with* $R_1=\{0\}$ *or* R_1 *is a ring of* $char(R_1)=2$ *and* $R_2=\{0\}$ *or* R_2 *is a ring of* $char(R_2)=3$

or

(2.3) $|G|=2$, $P\neq\{0\}$ *with* $2r_1^2=2r_1$ *for all* $r_1\in R_1$ (*in addition* $4=0$ *in* R_1) *and* $R_2=\{0\}$ *or* R_2 *is a ring of* $char(R_2)=3$.

Proof. If G is trivial, there is nothing to prove because of the validity of the isomorphism $R[G]\cong R$, so let us assume hereafter that G is non-trivial.

"**Necessity.**" As the feebly invo-cleanness of the group ring $R[G]$ implies the same property for R, utilizing Proposition 2.2 we come to the fact that $R[G]\cong P[G]\times K[G]$ will imply feebly invo-cleanness of both group rings $P[G]$ and $K[G]$ whence $P[G]$ is necessarily invo-clean whereas $K[G]$ is either zero or a subdirect product of a family of copies of the field $\mathbb{Z}_5$. After that, under the presence of $P[G]\neq\{0\}$, we just need apply Proposition 2.3 to deduce the described above things in points (2), (2.2) and (2.3). Letting now $P[G]=\{0\}$, we shall deal only with $K[G]$. To that goal, what we now assert is that the group ring $K[G]$ having the property $x^5=x$ for all $x\in K[G]$ with $\mathrm{char}(K[G])=5$ yields that K has the property $y^5=y$ for all $y\in K$ with $\mathrm{char}(K)=5$ and $G^4=\{1\}$. Indeed, since $K\subseteq K[G]$ and $G\subseteq K[G]$, this can be extracted elementarily thus substantiating our initial statement after all.

"**Sufficiency.**" Item (2) ensures that $R[G]\cong P[G]\times K[G]$ and so it is simple verified that the feebly invo-cleanness of both $P[G]$ and $K[G]$ will assure feebly invo-cleanness of $R[G]$ as well. That is why, we will be concentrated separately on these two group rings. Firstly, the stated above conditions are a guarantor with the aid of Proposition 2.3 that $P[G]$ is invo-clean. Secondly, it is pretty easily seen that as $y^5=y$ and $g^5=g$ for all $y\in K$ and $g\in G$, because K is a subdirect product of copies of the field $\mathbb{Z}_5$ possessing characteristic 5 and $G^4=\{1\}$, we may conclude that $x^5=x$ holds in $K[G]$ too, as required. This substantiates our former assertion after all. □

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Ключевые слова: инво-чистые кольца, слабо инво-чистые кольца, мало инво-чистые кольца, групповые кольца.

Коммутативное кольцо R называется мало инво-чистым, если каждый его элемент имеет вид $v + e - f$, где v – инволюция, а e, f – идемпотенты. Для каждого коммутативного унитального кольца R и каждой абелевой группы G найдены необходимые и достаточные условия, когда групповое кольцо $R[G]$ мало инво-чисто. Результаты статьи улучшают два последних достижения автора по коммутативным инво-чистым и слабо инво-чистым групповым кольцам, опубликованные в Универсальном журнале математики и математических наук (2018) и Уральском математическом журнале (2019) соответственно.

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