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ON THE STABILIZER OF A COLUMN IN A MATRIX GROUP OVER A POLYNOMIAL RING[1]

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An original non-standard approach to describing the structure of a column stabilizer in a group of $n \times n$ matrices over a polynomial ring or a Laurent polynomial ring of n variables is presented. The stabilizer is described as an extension of a subgroup of a rather simple structure using the $(n-1) \times (n-1)$ matrix group of congruence type over the corresponding ring of $n-1$ variables. In this paper, we consider cases where $n \leqslant 3$. For $n = 2$, the stabilizer is defined as a one-parameter subgroup, and the proof is carried out by direct calculation. The case $n = 3$ is nontrivial; the approach mentioned above is applied to it. Corollaries are given to the results obtained. In particular, we prove that for the stabilizer in the question, it is not generated by its a finite subset together with the so-called tame stabilizer of the given column. We are going to study the cases when $n \geqslant 4$ in a forthcoming paper. Note that a number of key subgroups of groups of automorphisms of groups are defined as column stabilizers in matrix groups. For example, this describes the subgroup $\mathrm{IAut}(M_r)$ of automorphisms that are identical modulo a commutant of a free metabelian group M_r of rank r. This approach demonstrates the parallelism of theories of groups of automorphisms of groups and matrix groups that exists for a number of well-known groups. This allows us to use the results on matrix groups to describe automorphism groups. In this work, the classical theorems of Suslin, Cohn, as well as Bachmuth and Mochizuki are used.

Keywords: *matrix group over a ring, elementary matrices, stabilizer of a column, ring of polynomials, ring of Laurent polynomials, residue, free metabelian group, automorphism group.*

1. Introduction

In the group theory, matrix methods have been used by a number of authors to produce new interesting results on endomorphisms and automorphisms of groups. J. S. Birman [1] has given a matrix characterization of automorphisms of a free group F_r of rank r with basis $f_1, \ldots, f_r$ among arbitrary endomorphisms (the "inverse function theorem") as follows. For an endomorphism ϕ define the matrix $J_\phi = (d_j \phi(f_i))$, $1 \leqslant i, j \leqslant r$ (the "Jacobian matrix" of ϕ), where d_j denotes partial Fox derivation (with respect to f_j) in the free group ring $\mathbb{Z}[F_r]$ (see [2, 3]). Then ϕ is an automorphism if and only if the matrix J_ϕ is invertible.

S. Bachmuth [4] has obtained an inverse function theorem of the same kind on replacing the Jacobian matrix J_ϕ by its image $\bar{J}_\phi$ over the abelianized group ring $\mathbb{Z}[F_r/F_r']$. Thus he established a matrix characterization of automorphisms of a free metabelian group M_r. U. U. Umirbaev [5] has generalized Birman's result to primitive systems of free groups, V. A. Roman'kov [6, 7] and E. I. Timoshenko [8] have characterized primitive systems of free

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metabelian groups. By definition, *primitive system* is a system of elements of a relatively free group that can be a part of some basis of this group.

For any commutative associative ring K with identity, an $r \times r$ elementary matrix (transvection) $T_{ij}(a)$ over K is a matrix of the form $E + aE_{ij}$ where $i \neq j$, $a \in K$, E_{ij} is the $r \times r$ matrix whose (ij) component is 1 and all other components are zero. As usual, E denotes the identity matrix. Let $\mathrm{SL}(r, K)$ be the group of all the $r \times r$ matrices of determinant 1 whose entries are elements of K, and let $\mathrm{E}(r, K)$ be the subgroup of $\mathrm{SL}(r, K)$ generated by the elementary matrices. By $\Lambda_{nk}^K = K[a_1, \dots, a_k, a_{k+1}^{\pm 1}, \dots, a_n^{\pm 1}]$ we denote a mixed polynomial ring over K. In particular, $\Lambda_{nn}^K = K[a_1, \dots, a_n]$ is the polynomial ring and $\Lambda_{n0}^K = K[a_1^{\pm 1}, \dots, a_n^{\pm 1}]$ is the Laurent polynomial ring in n variables over K.

Then the famous Suslin's Stability theorem [9] implies that for any $r \geqslant 3$ and any ring $\Lambda_{nk}^{\mathbb{F}}$, where $\mathbb{F}$ is an arbitrary field, $\mathrm{SL}(r, \Lambda_{nk}^{\mathbb{F}}) = \mathrm{E}(r, \Lambda_{nk}^{\mathbb{F}})$.

By $\mathrm{GE}(r, K)$ we denote the subgroup of $\mathrm{GL}(r, K)$ generated by $\mathrm{E}(r, K)$ and all diagonal matrices. It follows that for any $r \geqslant 3$ and any ring $\Lambda_{nk}^{\mathbb{F}}$, $\mathrm{GL}(r, \Lambda_{nk}^{\mathbb{F}}) = \mathrm{GE}(r, \Lambda_{nk}^{\mathbb{F}})$.

In contrast, $\mathrm{GL}(2, \Lambda_{nk}^{\mathbb{F}})$ has a number of specific properties. In [10], P. M. Cohn proved that

$$\begin{pmatrix} 1 + a_1 a_2 & -a_1^2 \\ a_2^2 & 1 - a_1 a_2 \end{pmatrix} \in \mathrm{GL}(2, \Lambda_{22}^{\mathbb{F}}) \setminus \mathrm{GE}(2, \Lambda_{22}^{\mathbb{F}}).$$

In [11], S. Bachmuth and H. Y. Mochizuki proved that if $n \geqslant 2$ then

$$\mathrm{GL}(2, \Lambda_{\mathrm{n}0}^{\mathbb{Z}}) \neq \mathrm{GE}(2, \Lambda_{\mathrm{n}0}^{\mathbb{Z}}).$$

In passing, we note that the coincidence of $\mathrm{GL}(2, \Lambda_{10}^{\mathbb{Z}})$ with $\mathrm{GE}(2, \Lambda_{10}^{\mathbb{Z}})$ is still an open problem.

Let M_r be the free metabelian group of rank r with basis $\{x_1, \dots, x_r\}$, and $A_r = M_r/M_r'$ be the abelianization of M_r, the free abelian group with the corresponding basis $\{a_1, \dots, a_r\}$. The group ring $\mathbb{Z}[A_r]$ can be considered as a Laurent polynomial ring Λ_{r0}.

For any group G, $\mathrm{IAut}(G)$ denotes the subgroup of the automorphism group $\mathrm{Aut}(G)$ consisting of all automorphisms that induce the identity map on the abelianization $G_{ab} = G/G'$. In the similar way the subsemigroup $\mathrm{IEnd}(G)$ of the endomorphism semigroup $\mathrm{End}(G)$ is defined too.

In [4], S. Bachmuth introduced the following embedding:

$$\beta : \mathrm{IAut(M_r)} \to \mathrm{GL(r, \Lambda_{r0}^{\mathbb{Z}})}, \ \beta : \phi \mapsto \bar{\mathrm{J}}_\phi, \ \phi \in \mathrm{IAut(M_r)}.$$

This embedding is called *Bachmuth's embedding.*

The image $\beta(\mathrm{IAut}(M_r))$ in $\mathrm{GL}_r(\Lambda_{r0}^{\mathbb{Z}})$ consists of all matrices A such that

$$A\bar{a}_r = \bar{a}_r \quad \text{for} \quad \bar{a}_r = \begin{pmatrix} a_1 - 1 \\ a_2 - 1 \\ \vdots \\ a_r - 1 \end{pmatrix}.$$

In other words, $\mathrm{IAut}(M_r) = \mathrm{Stab}_{\mathrm{GL(r,\Lambda_{r0}^{\mathbb{Z}})}}(\bar{a}_r)$ (the stabilizer of $\bar{a}_r$ in $\mathrm{GL}(r, \Lambda_{r0}^{\mathbb{Z}})$). Thus, this is an example showing that key subgroups can act as column stabilizers in matrix groups. In [12], V. Shpilrain obtained a matrix characterization of IA-endomorphisms with non-trivial fixed points ("eigenvectors") which, although is similar to the corresponding well-known characterization in linear algebra, also reveals a subtle difference. All these and some other results show a remarkable parallelism between the theory of automorphisms and endomorphisms of a free (or a free metabelian) group and the theory of linear operators in a vector space.

The main goal of this paper and a forthcoming paper is to present an original non-standard approach to the description of column stabilizers in matrix groups over rings. We consider matrix groups over mixed polynomial rings Λ_{nk}^K, where K is an arbitrary commutative domain with identity element. For simplicity, we formulate some of statements only in the following important cases: $K = \mathbb{Z}$ or $\mathbb{F}$, where $\mathbb{F}$ is an arbitrary field.

In this paper we consider only 2×2 matrices over Λ_{2k}^K and 3×3 matrices over Λ_{3k}^K. The case $n = 3$ is the first non-trivial in the subject. In the forthcoming paper we will extend our method to the description of column stabilizers for the cases $n \geqslant 4$. We restrict ourselves to considering columns with components of the following form: for $i \leqslant k$ this is a_i, for $i \geqslant k+1$ it is $a_i - 1$. In particular we consider either columns of variables for rings of polynomials, or columns with components of the form "a variable minus 1" for rings of Laurent polynomials. The proofs are independent of the particular choice of mixed polynomial ring Λ_{nk}.

For the case $n = 2$, we give an exhaustive description of the stabilizer of a column as a one-parameter subgroup. For $n = 3$, we describe a stabilizer of a column as extension of a clear subgroup by a concrete group of 2×2 matrices over ring on 2 variables. The idea of such description was originated in [13, 14]. Such a description was successfully used in [14] to prove that every automorphism of $M_r, r \geqslant 4$, is induced by an automorphism of F_r, i.e., is tame. This description was also used in [7] to prove that M_3 contains primitive elements that are not images of the primitive elements of F_3.

At the last Section 3, we derive a number of consequences of the obtained results about stabilizers of columns in case $n = 3$.

Remark 1. The column stabilizer in a $n \times n$ matrix group over a field $\mathbb{F}$ can be described as follows. Having included the stabilized column as the last element of a basis of the corresponding linear space, we get each of the stabilizer matrices in the half-expanded form when the last row is of the form $(0, \ldots, 0, 1)$. The considering stabilizer consists of all matrices of the such form. It has as a homomorphic image the corresponding group of $(n-1) \times (n-1)$ matrices over $\mathbb{F}$ with the kernel isomorphic to the direct sum of $n-1$ terms $\mathbb{F} \oplus \ldots \oplus \mathbb{F}$. A similar description for a matrix group over a ring is possible if at least one component of the stabilized column is invertible. Our approach is useful for other cases.

2. Preliminaries

Let K be an arbitrary commutative associative domain with identity. For any $n \in \mathbb{N}$, let Λ_n^K denotes a mixed polynomial ring Λ_{nk}^K. Let Δ_n^K stays for $\mathrm{id}(a_1, \ldots, a_k, a_{k+1} - 1, \ldots, a_n - 1)$ of Λ_{nk}^K (the augmentation ideal of Λ_n^K). Denote $c_i = a_i$ for $i = 1, \ldots, k$ and $c_i = a_i - 1$ for $i = k+1, \ldots, n$. Further in the paper, we will omit k and K for brevity and simply write Λ_n.

Each element $g \in \Lambda_l$, $l \leqslant n$, has for every $t \geqslant 1$ the unique expression of the form

$$g = \sum_{i=0}^{t} g_i c_l^i,$$

where $g_i \in \Lambda_{l-1}$ for $i = 0, \ldots, t-1$, and $g_t \in \Lambda_l$. Since every ring Λ_l embeds into a field of fractions, we can consider a Λ_l-submodule $\Lambda_l^{(-)} = \Lambda_l + c_l^{-1}\Lambda_l$, and each element of $\Lambda_l^{(-)}$ has for each $t \geqslant 0$ the unique expression of the form

$$g = \sum_{i=-1}^{t} g_i c_l^i, \tag{1}$$

where $g_i \in \Lambda_{l-1}$ for $i = -1, \ldots, t-1$, and $g_t \in \Lambda_l$.

Denote

$$\bar{c}_n = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}.$$

All along the paper we assume $n = 3$ (with one small exception for $n = 2$ at the beginning of the next Section 2). Let $G = \mathrm{Stab}(\bar{c}_3)$ be the subgroup of $\mathrm{GL}(3, \Lambda_3)$ consisting of all matrices g such that

$$g\bar{c}_3 = \bar{c}_3,$$

in other words, G is the stabilizer of the column $\bar{c}_3$ in $\mathrm{GL}(3, \Lambda_3)$. We will show how to construct an explicit matrix group $H \leqslant \mathrm{GL}(2, \Lambda_2)$ and a homomorphism ρ of G onto H for which $\mathrm{Ker}(\rho)$ is easily understood as a subgroup of $\Lambda_3 \oplus \Lambda_3$. In other words, we will describe G as an extension of $\mathrm{Ker}(\rho)$ by $\mathrm{Im}(\rho)$ with explicitly desribed factors. We will give a number of applications of these results.

3. On the stabilizer of a column in GL$(3, \Lambda_3)$

Before considering the case of 3×3 matrices, we show how the stabilizer of the vector $\bar{c}_2$ is arranged in the group of 2×2 matrices over Λ_2.

Proposition 1. In $\mathrm{GL}(2, \Lambda_2)$,

$$\mathrm{Stab}(\bar{c}_2) = \left\{ \begin{pmatrix} 1 + ac_1c_2 & -ac_1^2 \\ ac_2^2 & 1 - ac_1c_2 \end{pmatrix} \right\}, \tag{2}$$

where $a \in \Lambda_2$.

Proof. Obviously, every matrix A in $\mathrm{M}(2, \Lambda_2)$ such that $A\bar{c}_2 = \bar{c}_2$ has the form

$$\begin{pmatrix} 1 + ac_2 & -ac_1 \\ bc_2 & 1 - bc_1 \end{pmatrix}. \tag{3}$$

A matrix of the form (3) is invertible if and only if its determinant is 1. By direct computation we obtain that this happens if and only if this matrix has the form (2). ∎

From (1) follows that each element $g \in \Lambda_3^{(-)}$ can be uniquely expressed in the form

$$g = \sum_{i=-1}^{2} g_i c_3^i, \tag{4}$$

where $g_{-1}, g_0, g_1 \in \Lambda_2$ and $g_2 \in \Lambda_3$. Let G be the stabilizer of the column $\bar{c}_3$ in the group $\mathrm{GL}(3, \Lambda_3)$. Denote

$$C = \begin{pmatrix} 1 & 0 & c_1 \\ 0 & 1 & c_2 \\ 0 & 0 & c_3 \end{pmatrix}.$$

For $E + A \in G, A = (a_{ij})$, we have the following equality

$$C^{-1}(E + A)C = \begin{pmatrix} 1 + a_{11} - a_{31}c_1c_3^{-1} & a_{12} - a_{32}c_1c_3^{-1} & 0 \\ a_{21} - a_{31}c_2c_3^{-1} & 1 + a_{22} - a_{32}c_2c_3^{-1} & 0 \\ a_{31}c_3^{-1} & a_{32}c_3^{-1} & 1 \end{pmatrix}.$$

Denote

$$R(A)=\begin{pmatrix} 1+a_{11}-a_{31}c_1c_3^{-1} & a_{12}-a_{32}c_1c_3^{-1} \\ a_{21}-a_{31}c_2c_3^{-1} & 1+a_{22}-a_{32}c_2c_3^{-1} \end{pmatrix}.$$

Each element of $R(A)$ belongs to $\Lambda_3^{(-)}$. Then we have a homomorphism θ of G onto a group of matrices over $\Lambda_3^{(-)}$ defined by the map

$$\theta : E + A \mapsto R(A).$$

Using (4), we obtain a unique decomposition of the form

$$R(A) = E + c_3^2 A_2 + c_3 A_1 + A_0 + c_3^{-1} A_{-1}, \tag{5}$$

where $A_1, A_0, A_{-1} \in \mathrm{M}_2(\Lambda_2)$, $A_2 \in \mathrm{M}_2(\Lambda_3)$. It follows, that for each pair of matrices $E+A$, $E+B \in G$, one has $A_{-1}B_{-1} = 0$. We put

$$X=\begin{pmatrix} c_1c_2 & -c_1^2 \\ c_2^2 & -c_1c_2 \end{pmatrix}.$$

Note that $X^2 = 0$.

The following lemma is proved by direct calculation.

Lemma 1. For each $A \in \mathrm{M}(2,\Lambda_2)$, if $AX = XA = 0$, then $A = \alpha X$ for some $\alpha \in \Lambda_2$.

Theorem 1. In the above notation, there exist elements $\alpha, \beta, \gamma, \delta \in \Lambda_2$ such that for each $E + A \in G$,

$$A_{-1} = \alpha X,\ A_0 X = \beta X,\ X A_0 = \gamma X,\ X A_1 X = \delta X.$$

Proof. Let $T = -c_2E_{31} + c_1E_{32}$. Then $E + T \in G$ and $R(T) = E + c_3^{-1}X, T_{-1} = X$. Then $A_{-1}X = XA_{-1} = 0$. By Lemma 1, there is $\alpha \in \Lambda_2$ for which $A_{-1} = \alpha X$.

Next, $\theta(E + A)\theta(E + T) = R(A)R(T)$ has the (-1)-component $X + A_0X + A_{-1} = (1+\alpha)X+A_0X$ and so $A_0X = \beta X$, $\beta \in \Lambda_2$. Similarly, $R(T)R(A)$ has the (-1)-component $X + XA_0 + A_{-1} = (1 + \alpha)X + XA_0$, hence $A_0 = \gamma X$, $\gamma \in \Lambda_2$.

Note, that

$$XE_{11}X = c_1c_2X,\ XE_{12}X = c_2^2X,\ XE_{21}X = -c_1^2X,\ XE_{22}X = -c_1c_2X.$$

It follows that for every $B \in \mathrm{M}(2,\Lambda_2)$, $B = (b_{ij})$,

$$XBX = \delta X,$$

where $\delta = c_1c_2(b_{11} - b_{22}) + c_2^2 b_{21} - c_1^2 b_{12} \in \Lambda_2$. ■

Thus, we can associate the elements $\alpha, \beta, \gamma, \delta \in \Lambda_2$ with the matrix $E + A \in G$. These elements are called *residues* of $E + A$ with respect to c_3 Now we give explicit formulas for the residues in terms of the elements of the matrices A_i, $i = 1, 0, -1$. These formulas are obtained by direct computations:

$$\begin{gathered} \alpha = -(a_{31})_0c_2^{-1} = (a_{32})_0c_1^{-1},\ \beta = -(a_{31})_1c_1 - (a_{32})_1c_2,\ \gamma = (a_{11})_0 - (a_{21})_0c_1c_2^{-1}, \\ \delta = -(a_{21})_1c_1^2 + (a_{12})_1c_2^2 + (a_{11})_1 - (a_{22})_1)c_1c_2. \end{gathered} \tag{6}$$

Theorem 2. The map

$$\rho : G \to \mathrm{GL}_2(\Lambda_2), \mathrm{A} \mapsto \begin{pmatrix} 1+\beta & \alpha \\ \delta & 1+\gamma \end{pmatrix}$$

ia a homomorphism.

Proof. Let $E+A' \in G$ and let $R(A') = E+c_3^2A'_2+c_3A'_1+A'_0+c_3^{-1}A'_{-1}$ be decomposition of the form (5). Let $\alpha', \beta', \gamma', \delta'$ be the residues of $E+A'$ with respect to c_3.

Then $(E+A)(E+A') = E+\tilde{A}$, where $\tilde{A} = A+A'+AA'$, and

$$R(\tilde{A}) = E + c_3^2\tilde{A}_2 + c_3\tilde{A}_1 + \tilde{A}_0 + c_3^{-1}\tilde{A}_{-1}$$

be a decomposition of the form (5).

Here $\tilde{A}_{-1} = A_{-1}+A'_{-1}+A_0A'_{-1}+A_{-1}A'_0 = (\alpha+\alpha'+\alpha'\beta+\alpha\gamma')X$. Hence the corresponding residue is

$$\tilde{\alpha} = \alpha + \alpha' + \alpha'\beta + \alpha\gamma'.$$

Further, $\tilde{A}_0 = E + A_0 + A'_0 + A_0A'_0 + A_1A'_{-1} + A_{-1}A'_1$. Hence

$$\tilde{A}_0X = (1+\beta+\beta'+\beta\beta'+\alpha\delta')X : X\tilde{A}_0 = 1+\alpha+\alpha'+\gamma\gamma'+\delta\alpha'. \tag{7}$$

Hence $\tilde{\beta} = 1+\beta+\beta'+\beta\beta'+\alpha\delta'$ and $\tilde{\gamma} = 1+\gamma+\gamma'+\gamma\gamma'+\delta\alpha'$. Then

$$\tilde{A}_1 = A_1 + A'_1 + A_1A'_0 + A_0A'_1 + A_2A'_{-1} + A_{-1}A'_2.$$

Hence $\tilde{\delta} = 1+\delta+\delta'+\delta\beta'+\gamma\delta'$. Consequently,

$$\begin{pmatrix} 1+\tilde{\beta} & \tilde{\alpha} \\ \tilde{\delta} & 1+\tilde{\gamma} \end{pmatrix}\begin{pmatrix} 1+\beta & \alpha \\ \delta & 1+\gamma \end{pmatrix} = \begin{pmatrix} 1+\beta' & \alpha' \\ \delta' & 1+\gamma' \end{pmatrix},$$

is equivalent to $\rho(AA') = \rho(A)\rho(A')$. ■

Now we are to compute $\mathrm{Im}(\rho) = \rho(G)$. Let $\mathrm{GL}(2,\Lambda_2,\Delta_2)$ denote the congruence subgroup of $\mathrm{GL}(2,\Lambda_2)$ with respect to the augmentation ideal Δ_2 of Λ_2. We denote by $\mathrm{GL}(2,\Lambda_2,\Delta_2,\Delta_2^2)$ the subgroup of $\mathrm{GL}(2,\Lambda_2)$ consisting of the matrices corresponding to the following inclusion scheme:

$$\begin{pmatrix} 1+\Delta_2 & \Lambda_2 \\ \Delta_2^2 & 1+\Delta_2 \end{pmatrix}. \tag{8}$$

Theorem 3. $\mathrm{Im}(\rho) = \mathrm{GL}(2,\Lambda_2,\Delta_2,\Delta_2^2)$.

Proof. Let

$$B = \begin{pmatrix} 1+\beta & \alpha \\ \delta & 1+\gamma \end{pmatrix}$$

be an invertible matrix corresponding to the inclusion scheme (8). Then we have the following decompositions:

$$\beta = \beta_1c_1 + \beta_2c_2, \gamma = \gamma_1c_1 + \gamma_2c_2, \delta = \delta_{11}c_1^2 + \delta_{12}c_1c_2 + \delta'_{12}c_1c_2 + \delta_{22}c_2^2,$$

where $\beta_1, \ldots, \delta_{22} \in \Lambda_2$.

First we define a matrix that stabilizes the column $\bar{c}_3$ such that $\rho(C) = B$ subject to its invertibility:

$$C = \begin{pmatrix} 1+\gamma_2 c_2 + \delta'_{12} c_3 & -\gamma_2 c_1 + \delta_{22} c_3 & -\delta'_{12} c_1 - \delta_{22} c_2 \\ -\gamma_1 c_2 - \delta_{11} c_3 & 1+\gamma_1 c_1 - \delta_{12} c_3 & \delta_{11} c_1 + \delta_{12} c_2 \\ -\alpha c_2 - \beta_1 c_3 & \alpha c_1 - \beta_2 c_3 & 1+\beta_1 c_1 + \beta_2 c_2 \end{pmatrix}. \tag{9}$$

Obviously, $C\bar{c} = \bar{c}$. By direct computation we obtain that $\det(C) = \det(B) + r$,

$$r = \delta'_{12} c_3 + \gamma_1 \delta'_{12} c_1 c_3 - \delta_{12} c_3 - \delta'_{12} \delta_{12} c_3^2 - \gamma_2 \delta_{12} c_2 c_3 +$$
$$+\beta_2 \delta'_{12} c_2 c_3 - \beta_1 \delta_{12} c_1 c_3 - \beta_1 \delta_{22} c_2 c_3 + \gamma_1 \delta_{22} c_2 c_3 - \gamma_2 \delta_{11} c_1 c_3 + \delta_{11} \delta_{22} c_3^2 + \beta_2 \delta_{11} c_1 c_3.$$

Suppose, that $B \in \mathrm{GL}(2, \Lambda_1)$, then $\beta_2, \gamma_2, \delta'_{12}, \delta_{12}, \delta_{22} = 0$, hence $r = 0$, and $C \in G$. Similarly we obtain, that $C \in G$ if B does not depend of c_1. Since $B = B_1 B'$ where B_1 does not depend of c_2 and B' lies in the congruence subgroup with respect to c_2, i.e.,

$$B' \in \begin{pmatrix} 1+\Lambda_2 c_2 & \Lambda_2 c_2 \\ \Delta_2 c_2 & 1+\Lambda_2 c_2 \end{pmatrix}.$$

Both matrices, B_1 and B' are invertible, and $B_1 \in \mathrm{Im}(G)$. There are decompositions (6) in which $\beta_1, \gamma_1, \delta_{11} = 0$. Note that transvection $t = t_{21}((-\delta_{12} - \delta'_{12}) c_1 c_2$ lies in B and has a preimage in G. Then

$$B'' = B't \in \begin{pmatrix} 1+\Lambda_2 c_2 & \Lambda_2 c_2 \\ \Lambda_2 c_2^2 & 1+\Lambda_2 c_2 \end{pmatrix}.$$

The elements of B'' are decompositions (6) such that $\beta_1, \gamma_1, \delta'_{12}, \delta_{12}, \delta_{11} = 0$. The corresponding matrix C'' defined in the form (9) is invertible because its determinant is equal to $\det(B'')$. Hence $B'' \in \mathrm{Im}(\rho)$, and $B \in \mathrm{Im}(\rho)$. ■

Then G is an extension of $\mathrm{Ker}(\rho)$, that is described by formulas (6), by $\mathrm{Im}(\rho)$, that is consisting of all invertible matrices corresponding to (3). By the way, we note, that $\mathrm{Ker}(\rho)$ contains the subgroup H of all matrices in G of the form

$$\begin{pmatrix} 1+c_3^2\Lambda_3 & c_3^2\Lambda_3 & c_3\Lambda_3 \\ c_3^2\Lambda_3 & 1+c_3^2\Lambda_3 & c_3\Lambda_3 \\ c_3^2\Lambda_3 & c_3^2\Lambda_3 & 1+c_3\Lambda_3 \end{pmatrix}.$$

The quotient $\mathrm{Im}(\rho)/H$ is easily understood.

4. On the tame stabilizer of a column in GL$(3, \Lambda_3)$

In general, the stabilizer G of $\bar{c}_n$ in $\mathrm{GL}(n, K)$ for any commutative associative ring K with identity contains each matrix of the form

$$T_{i,j,k}(a) = E + ac_k E_{ij} - ac_j E_{ik}, \text{ for } i \neq j,k;\ j<k;\ a \in K. \tag{10}$$

Also, given the Proposition 1, G contains each matrix of the form

$$S_{i,j}(a) = E + ac_i c_j E_{ii} - ac_i^2 E_{ij} + ac_j^2 E_{ji} - ac_i c_j E_{jj}, \text{ for } i<j,\ a \in K \tag{11}$$

(see (2)).

We denote by G_t (*the tame stabilizer*) the subgroup of G generated by all matrices $T_{i,j,k}(a)$ and $S_{i,j}(a)$ defined by (10) and (11), respectively. A question arises: Does G_t

coincides with G? For $n = 2$, the answer “Yes” is obvious by Proposition 1. We will show below that the answer for $n = 3$ is “No”.

Now, let $G \leqslant \mathrm{GL}(3, \Lambda_3)$ be the stabilizer of $\bar{c}_3$ and let $G_t \leqslant G$ be the corresponding tame stabilizer. As above, Λ_3 denotes $\Lambda^{\mathbb{F}}_{30}$, $\Lambda^{\mathbb{Z}}_{30}$, or $\Lambda^{\mathbb{Z}}_{30}$.

We exlude Laurent polynomial rings $\Lambda^{\mathbb{F}}_{33}$ over a field. The following results show a connection between G_t and $\mathrm{GE}(2, \Lambda_2)$, that allows to show that G_t is small with respect to G.

Proposition 2. $\mathrm{Im}(G_t) \leqslant \mathrm{GL}(2, \Lambda_2)$.

Proof. If the matrix $A = (a_{ij}) \in G$ has the form $E + A'$, and all rows of the matrix A' are zero except for one row, then $\rho(A)$ lies in the subgroup $\mathrm{GE}(2, \Lambda_2)$. Indeed, formulas (6) show that in this case $\alpha = 0$ or $\delta = 0$. Then $\rho(A)$ is a triangular matrix. But every triangular matrix lies obviously in $\mathrm{GE}(2, \Lambda_2)$. This proves the statement for any matrix $A = T_{i,j,k}(a)$.

By formulas (6) for any matrix $S_{i,j}(a)$, one has $\alpha = 0$, and we conclude as above. ■

Theorem 4. Let G be the stabilizer of the column $\bar{c}_3$ in $\mathrm{GL}(3, \Lambda^{\mathbb{Z}}_{30})$. Then for every finite subset $L \subseteq G$

$$\mathrm{gp(L, G_t)} \neq \mathrm{G}.$$

Proof. By Bachmuth and Mochizuki result [11], if $n \geqslant 2$ then $\mathrm{GL}(2, \Lambda^{\mathbb{Z}}_{n0})$ can not be generated by any finite subset together with the subgroup $\mathrm{GE}(2, \Lambda^{\mathbb{Z}}_{n0})$. Hence,

$$\mathrm{gp}(\mathrm{GE}(2, \Lambda^{\mathbb{Z}}_{22}), \rho(L)) \neq \mathrm{GL}(2, \Lambda^{\mathbb{Z}}_{22}). \tag{12}$$

Then there is a matrix A that belongs to the difference between the two sides of (12). We will show that there is a similar matrix with elements corresponding to the scheme (8). To prove this assertion, we define the image $E + A_0$ of A that lies in $\mathrm{GL}(2, \mathbb{Z})$ under specialization homomorphism $\mathrm{GL}(2, \Lambda^{\mathbb{Z}}_{22}) \to \mathrm{GL}(2, \mathbb{Z})$ defined by the map $c_i \mapsto 1$, $i = 1, 2$.

In other words, A_0 is the 0 part of A under the decomposition form (5). Then $E + A_0 \in$ $\in \mathrm{GE}(2, \mathbb{Z})$. We multiply A by $(E + A_0)^{-1}$ and get new matrix $\tilde{A}$ with the same property. Suppose, that its (21) component $\tilde{a}_{21} = q_1c_1 + q_2c_2 + q_3$, where $q_1, q_2 \in \mathbb{Z}$, $q_3 \in \Delta^{\mathbb{Z}}_2$ does not lie in $\Delta^{\mathbb{Z}}_{22}$. This means that $q_1 \neq 0$ or $q_2 \neq 0$. Then we multiply $\tilde{A}$ by $t_{21}(-q_1c_1 - q_2c_2)$ and obtain matrix $\bar{A}$ that lies in the difference the two sides of (12) and corresponds to the scheme (8). Thus $\bar{A} \in \mathrm{Im}(\rho)$ but has no preimages in $\mathrm{gp}(\mathrm{GE}(2, \Lambda^{\mathbb{Z}}_{22}), \rho(\mathrm{L}))$. ■

Conclusion

The main results of this paper were obtained for matrix groups over polynomial rings under fairly rigorous assumptions regarding the stabilized vector. The similar results can be obtained for some other rings. The main advantage of the proposed method is the fact that the homomorphism ρ reduces the group of matrices over the module $\Lambda_3 + c_3^{-1}\Lambda_3$ to the group of matrices over Λ_2. This process can be considered as an elimination of the residue c_3^{-1}. Such reducing allows the using of the corresponding induction. This approach also demonstrates the parallelism of theories of groups of automorphisms of groups and matrix groups that exists for a number of well-known groups. See [15]. This allows us to use the results on matrix groups to describe automorphism groups. In the recent paper [16], some applications of the method are given.

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