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THE CHROMATICITY OF THE JOIN OF TREE AND NULL GRAPH

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The chromaticity of the graph G, which is join of the tree T_p and the null graph O_q, is studied. We prove that G is chromatically unique if and only if $1 \leqslant p \leqslant 3$, $1 \leqslant q \leqslant 2$; a graph H and $T_p + O_{p-1}$ are χ-equivalent if and only if $H = T'_p + O_{p-1}$, where T'_p is a tree of order p; H and $T_p + O_p$ are χ-equivalent if and only if $H \in \{T'_p + O_p, T''_{p+1} + O_{p-1}\}$, where T'_p is a tree of order p, T''_{p+1} is a tree of order $p+1$. We also prove that if $p \leqslant q$, then $\chi'(G) = ch'(G) = \Delta(G)$; if $\Delta(G) = |V(G)| - 1$, then $\chi'(G) = ch'(G) = \Delta(G)$ if and only if $G \neq K_3$.

Keywords: *chromatic number, chromatically equivalent, chromatically unique graph, chromatic index, list-chromatic index.*

1. Introduction

All graphs considered in the paper are finite undirected graphs without loops or multiple edges. If G is a graph, then $V(G)$, $E(G)$ (or V and E in short) and $\overline{G}$ denote its vertex set, edge set and its complementary graph, respectively. The set of all neighbours of a subset $S \subseteq V(G)$ is denoted by $N_G(S)$ (or $N(S)$ in short). If $S = \{v\}$, then $N(S)$ is denoted by $N(v)$. For a vertex $v \in V(G)$, the degree of v is denoted by $\deg_G(v)$ (or $\deg(v)$), it equals $|N_G(v)|$. The subgraph of G induced by $W \subseteq V(G)$ is denoted by $G[W]$. Let R be a subset of edges in G, $|R| = r$; denote by $G - R$ the graph obtained by deleting all edges in R from G.

The null graphs and complete graphs of order n are denoted by O_n and K_n, respectively. The K_3 is called a *triangle*. Let $t_1(G)$, $t_2(G)$, and $t_3(G)$ be the numbers of triangles, of induced subgraphs C_4, and of complete subgraphs K_4 in G, respectively. Unless otherwise indicated, our graph-theoretic terminology follows [1].

An *acyclic* graph, one not containing any cycles, is called *forest*. A connected forest is called a *tree*, a tree of order n is denoted by T_n.

A graph $G = (V, E)$ is called *r-partite graph* if V admits a partition into r classes $V = V_1 \cup V_2 \cup \ldots \cup V_r$ such that the subgraphs of G induced by V_i, $i = 1, \ldots, r$, are empty. If $r = 2$, then G is called *bipartite graph*, if $r = 3$, then G is called *tripartite graph*. An r-partite graph in which every two vertices from different partition classes are adjacent is called *complete r-partite graph* and is denoted by $K_{|V_1|,|V_2|,\ldots,|V_r|}$. The complete r-partite graph $K_{|V_1|,|V_2|,\ldots,|V_r|}$ with $|V_1| = |V_2| = \ldots = |V_r| = s$ is denoted by K_s^r.

Let $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ be two graphs such that $V_1 \cap V_2 = \varnothing$. Their *union* $G = G_1 \cup G_2$ has, as expected, $V(G) = V_1 \cup V_2$ and $E(G) = E_1 \cup E_2$. Their *join* is denoted $G_1 + G_2$ and consists of $G_1 \cup G_2$ and all edges joining V_1 with V_2.

Let $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ be two graphs. We call G_1 and G_2 *isomorphic*, and write $G_1 \cong G_2$, if there exists a bijection $f : V_1 \to V_2$ with $uv \in E_1$ if and only if $f(u)f(v) \in E_2$ for all $u, v \in V_1$.

Let $G = (V, E)$ be a graph and λ is a positive integer.

A λ-*coloring* of G is a bijection $f : V(G) \to \{1, 2, \dots, \lambda\}$ such that $f(u) \neq f(v)$ for any adjacent vertices $u, v \in V(G)$. The smallest positive integer λ such that G has a λ-coloring is called the *chromatic number* of G and is denoted by $\chi(G)$. We say that a graph G is n-*chromatic* if $n = \chi(G)$.

Let $V(G) = \{v_1, v_2, \dots, v_n\}$, two λ-colorings f and g are considered different if and only if $f(v_k) \neq g(v_k)$ for some $k \in \{1, 2, \dots, n\}$. Let $P(G, \lambda)$ (or simply $P(G)$ if there is no danger of confusion) denote the number of distinct λ-colorings of G. It is well-known that for any graph G, $P(G, \lambda)$ is a polynomial in λ, called the *chromatic polynomial* of G. The notion of chromatic polynomials was first introduced by Birkhoff [2] in 1912 as a quantitative approach to tackle the four-color problem. Two graphs G and H are called *chromatically equivalent* (or, in short, χ-*equivalent*), and we write $G \sim H$, if $P(G, \lambda) = P(H, \lambda)$. A graph G is called *chromatically unique* (χ-*unique*) if $G' \cong G$ (i.e., G' is isomorphic to G) for any graph G' such that $G' \sim G$. For examples, all cycles are χ-unique [3]. The notion of χ-unique graphs was first introduced and studied by Chao and Whitehead [4] in 1978. The readers can see the surveys [3, 5, 6] for more information on χ-unique graphs.

An edge coloring of a graph G can be defined similarly. Namely, an *edge* λ-*coloring* of a graph G is a mapping $f : E(G) \to \{1, 2, \dots \lambda\}$ such that two adjacent edges have distinct images. The *chromatic index* of G, denoted by $\chi'(G)$, is the smallest positive integer λ such that G has an edge λ-coloring. In 1964, Vizing [7] proved that $\chi'(G)$ is equal to either $\Delta(G)$ or $\Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of G. A graph G is said to be *Class one* (resp., *Class two*) if $\chi'(G) = \Delta(G)$ (resp., $\Delta(G) + 1$). For examples, all cycles C_n with n even are Class one; all cycles C_n with n odd are Class two. Let $(L(e))_{e \in E(G)}$ be a family of sets. We call an edge coloring f of G with $f(e) \in L(e)$ for all $e \in E(G)$ a *list edge coloring* from the lists $L(e)$. The least integer k such that G has an edge coloring from any family of lists of size k is the *list-chromatic index* of G and is denoted by $ch'(G)$. The idea of list colorings of graphs is due independently to V. G. Vizing [8] and to P. Erdös, A. L. Rubin, and H. Taylor [9].

In [10] we have characterized chromatically uniqueness of the graph $K_2^r + O_k$, in [11] we have characterized chromatically uniqueness of the graph $G = K_2^n + K_r$, and in [6] we have determined chromatic index and characterized chromatically uniqueness split graphs.

In this paper, we study the chromaticity of G, which is join of the tree T_p and the null graph O_q. We prove that G is chromatically unique if and only if $1 \leqslant p \leqslant 3$, $1 \leqslant q \leqslant 2$; H and $T_p + O_{p-1}$ are χ-equivalent if and only if $H = T'_p + O_{p-1}$, where T'_p is a tree of order p; H and $T_p + O_p$ are χ-equivalent if and only if $H \in \{T'_p + O_p, T''_{p+1} + O_{p-1}\}$, where T'_p is a tree of order p, T''_{p+1} is a tree of order $p + 1$. We also prove that if $p \leqslant q$, then $\chi'(G) = ch'(G) = \Delta(G)$; if $\Delta(G) = |V(G)| - 1$, then $\chi'(G) = ch'(G) = \Delta(G)$ if and only if $G \neq K_3$.

2. Vertex colorings

For a graph G and a positive integer k, a partition $\{A_1, A_2, \dots, A_k\}$ of $V(G)$ is called *a* k-*independent partition in* G if each A_i is a non-empty independent set of G. Let $\alpha(G, k)$ denote the number of k-independent partitions in G. Hence, $P(G, \lambda) = \sum\limits_{1 \leqslant k \leqslant n} \alpha(G, k)(\lambda)_k$, where $(\lambda)_k = \lambda(\lambda - 1) \dots (\lambda - k + 1)$.

The polynomial $\sigma(G, x) = \sum\limits_{1 \leqslant k \leqslant n} \alpha(G, k)x^k$ is called *the* σ-*polynomial of* G.

The polynomial $h(G, x) = \sum\limits_{1 \leqslant k \leqslant n} \alpha(\overline{G}, k)x^k$ is called *the adjoint polynomial of* G.

Let K_p^+ be the vertex gluing of K_p and K_2.

For convenience, denote $\sigma(G,x)$ by $\sigma(G)$, $h(G,x)$ by $h(G)$, and $G \cong H$ by $G = H$. The following lemmas will be used to prove our main results.

Lemma 1 [3]. If $G = K_n$ is the complete graph on n vertices, then $\chi(G) = n$ and G is χ-unique.

Lemma 2. If $G = K_{n_1,n_2,\dots,n_r}$ is the complete r-partite graph, then $\chi(G) = r$.

Lemma 3 [12]. Let G and H be two χ-equivalent graphs. Then

(i) $|V(G)| = |V(H)|$;
(ii) $|E(G)| = |E(H)|$;
(iii) $\chi(G) = \chi(H)$;
(iv) G is connected if and only if H is connected;
(v) G is 2-connected if and only if H is 2-connected;
(vi) $t_1(G) = t_1(H)$;
(vii) $t_2(G) - 2t_3(G) = t_2(H) - 2t_3(H)$;
(viii) $\alpha(G,k) = \alpha(H,k)$ for each $k = 1, 2, \dots$

Lemma 4 [12].

(i) All trees of the same order are χ-equivalent. Further, the graph G of order n is a tree if and only if $P(G,\lambda) = \lambda(\lambda-1)^{n-1}$;
(ii) A tree T_n is χ-unique if and only if $1 \leqslant n \leqslant 3$;
(iii) If $G = T_n$ is a tree of order n, then $\chi(G) = 2$.

Lemma 5 [11]. The graph $G = K_2^m + K_n$ is χ-unique.

Lemma 6 [13]. Let G and H be two disjoint graphs. Then

(i) $\sigma(G+H,x) = \sigma(G,x)\sigma(H,x)$;
(ii) $h(G \cup H,x) = h(G,x)h(H,x)$.

Lemma 7 [14]. Let G and H be two graphs. Then

(i) $P(G,\lambda) = P(H,\lambda)$ if and only if $\sigma(G,x) = \sigma(H,x)$;
(ii) $P(G,\lambda) = P(H,\lambda)$ if and only if $h(\overline{G},x) = h(\overline{H},x)$.

Lemma 8. If $p \geqslant 2$, then $\chi(T_p + O_q) = 3$.

Proof. If $p \geqslant 2$, then the complete graph K_3 is a subgraph of $G = T_p + O_q$. So $\chi(G) \geqslant 3$. Let $V(G) = V_1 \cup V_2$ is a partition of $V(G)$ such that $G[V_1] = T_p$, $G[V_2] = O_q$. The graph $G[V_1]$ is a tree, by (iii) of Lemma 4, $G[V_1]$ has a coloring f_1 using two colors $1, 2$. Set mapping

$$f : V(G) \to \{1, 2, 3\}$$

such that $f(v) = f_1(v)$ if $v \in V_1$, $f(v) = 3$ if $v \in V_2$. Then f is a 3-coloring of G, i.e., $\chi(G) \leqslant 3$. Thus, $\chi(G) = 3$. ■

Theorem 1. $G = T_p + O_q$ is χ-unique if and only if $1 \leqslant p \leqslant 3$, $1 \leqslant q \leqslant 2$.

Proof. First we prove the necessity. Suppose that $G = T_p + O_q$ is χ-unique. Suppose the contrary, that $p \geqslant 4$. Set $G^1 = (K^1 \cup I^1, E^1)$ with

$$K^1 = \{v_1, v_2, \dots, v_p\}, \quad I^1 = \{u_1, u_2, \dots, u_q\},$$
$$E^1 = \{v_1v_2, v_1v_3, \dots, v_1v_p\} \cup \{v_iu_j : i = 1, 2, \dots p, j = 1, 2, \dots q\}.$$

Set $G^2 = (K^2 \cup I^2, E^2)$ with

$$K^2 = \{v_1, v_2, \dots, v_p\}, \quad I^2 = \{u_1, u_2, \dots, u_q\},$$
$$E^2 = \{v_1v_2, v_2v_3, \dots, v_{p-1}v_p\} \cup \{v_iu_j : i = 1, 2, \dots p, j = 1, 2, \dots q\}.$$

By (i) of Lemma 4, (i) of Lemma 6, and (i) of Lemma 7, it follows that

$$P(G^1,\lambda) = P(G^2,\lambda) = P(G,\lambda).$$

It is not difficult to see that

$$\Delta(G^1) = \max\{\deg(u) : u \in V(G^1)\} = \deg(v_1) = p+q-1$$

and

$$\Delta(G^2) = \max\{\deg(u) : u \in V(G^2)\} = \max\{p, q+2\}.$$

If $q \geqslant 2$, then $\max\{p, q+2\} < p+q-1$, it follows that $\Delta(G^2) < \Delta(G^1)$. So $G^1 \not\cong G^2$ and G is not χ-unique, a contradiction.

If $q = 1$, then $\Delta(G^2) = \Delta(G^1) = p$. It is not difficult to see that

$$|\{u \in V(G^1) : \deg_{G^1}(u) = p\}| = 2 \quad \text{and} \quad |\{u \in V(G^2) : \deg_{G^2}(u) = p\}| = 1.$$

It follows that $G^1 \not\cong G^2$ and G is not χ-unique, a contradiction. Thus, $1 \leqslant p \leqslant 3$.

Suppose that $q \geqslant 3$. For $p = 3$, we set $G^3 = (K^3 \cup I^3, E^3)$ with

$$K^3 = \{v_1, v_2, v_3\}, \quad I^3 = \{u_1, u_2, \ldots, u_q\},$$
$$E^3 = \{v_1v_2, v_2v_3\} \cup \{v_iu_j : i = 1,2,3, j = 1,2,\ldots q\},$$

and set $G^4 = (K^4 \cup I^4, E^4)$ with

$$K^4 = \{v_1, v_2, v_3\}, \quad I^4 = \{u_1, u_2, \ldots, u_q\},$$
$$E^4 = \{v_2u_1, u_1u_2, u_2u_3, \ldots, u_{q-1}u_q\} \cup \{v_1v_2, v_2v_3\} \cup \{v_1u_j, v_3u_j : j = 1,2,\ldots,q\}.$$

It is not difficult to see that $G = G^3 = T_3 + O_q$ and $G^4 = T_{q+1} + O_2$. By (i) of Lemma 4, (i) of Lemma 6, and (i) of Lemma 7, we have

$$\begin{aligned}
\sigma(G^3,x) &= \sigma(T_3 + O_q, x) = \\
&= \sigma(T_3,x)\sigma(O_q,x) = \\
&= \sigma(O_1 + O_2, x)\sigma(O_q,x) = \\
&= \sigma(O_1,x)\sigma(O_2,x)\sigma(O_q,x) = \\
&= \sigma(O_1,x)\sigma(O_q,x)\sigma(O_2,x) = \\
&= \sigma(O_1 + O_q, x)\sigma(O_2,x) = \\
&= \sigma(T_{q+1},x)\sigma(O_2,x) = \\
&= \sigma(T_{q+1} + O_2, x) = \\
&= \sigma(G^4,x).
\end{aligned}$$

It follows that $P(G^3,\lambda) = P(G^4,\lambda)$. Otherwise,

$$\Delta(G^3) = \max\{\deg(u) : u \in V(G^3)\} = \deg(v_2) = q+2,$$
$$\Delta(G^4) = \max\{\deg(u) : u \in V(G^2)\} = \deg(v_1) = \deg(v_3) = q+1.$$

So $G^3 \not\cong G^4$ and G is not χ-unique, a contradiction.

For $p = 2$, we set $G^5 = (K^5 \cup I^5, E^5)$ with

$$K^5 = \{v_1, v_2\}, \quad I^5 = \{u_1, u_2, \ldots, u_q\},$$
$$E^5 = \{v_1v_2\} \cup \{v_iu_j : i = 1,2, j = 1,2,\ldots q\},$$

and set $G^6 = (K^6 \cup I^6, E^6)$ with

$$K^6 = \{v_1, v_2\}, \quad I^6 = \{u_1, u_2, \ldots, u_q\},$$
$$E^6 = \{v_2u_1, u_1u_2, u_2u_3, \ldots, u_{q-1}u_q\} \cup \{v_1v_2\} \cup \{v_1u_j : j = 1, 2, \ldots q\}.$$

It is clear that $P(G^5, \lambda) = P(G^6, \lambda)$ and

$$|\{u \in V(G^5) : \deg_{G^5}(u) = q+1\}| = |\{v_1, v_2\}| = 2,$$
$$|\{u \in V(G^6) : \deg_{G^6}(u) = q+1\}| = |\{v_1\}| = 1.$$

So $G^5 \ncong G^6$ and G is not χ-unique, a contradiction.

If $p = 1$, then G is a tree T_n with $n = q+1 \geqslant 4$. By (ii) of Lemma 4, G is not χ-unique, a contradiction.

Now we prove the sufficiency. If $p = 1$ and $q = 1$, then $G = K_2$, if $p = 2$ and $q = 1$, then $G = K_3$. By Lemma 1, G is χ-unique.

If $p = 1$ and $q = 2$, then $G = T_3$. By (ii) of Lemma 4, G is χ-unique.

If $p = 2$ and $q = 2$ or $p = 3$ and $q = 1$, then $G = K_2^1 + K_2$, if $p = 3$ and $q = 2$, then $G = K_2^2 + K_1$. By Lemma 5, G is χ-unique. ■

Theorem 2.

(i) H and $T_p + O_{p-1}$ are χ-equivalent if and only if $H = T'_p + O_{p-1}$, where T'_p is a tree of order p;

(ii) H and $T_p + O_p$ are χ-equivalent if and only if $H \in \{T'_p + O_p, T''_{p+1} + O_{p-1}\}$, where T'_p is a tree of order p, T''_{p+1} is a tree of order $p+1$.

Proof. If $p = 2$, then, by Theorem 1, $T_p + O_{p-1}$ and $T_p + O_p$ are χ-unique. It follows that the theorem is obviously true. Hence we may assume that $p \geqslant 3$.

Suppose that H and $G = T_p + O_q$ are χ-equivalent, where $p-1 \leqslant q \leqslant p$. By (iii) of Lemma 3 and Lemma 8, $\chi(H) = 3$. So H is a tripartite graph. We may assume that $D = K_{a,b,c}$ and $R = \{e_1, e_2, \ldots, e_r\} \subseteq E(D)$ such that $H = D - R$ and $a \leqslant b \leqslant c$. It is clear that

$$r = |E(D) - |E(H)| = |E(D)| - |E(G)| = ab + ac + bc - pq - p + 1$$

and

$$a + b + c = |V(D)| = |V(H)| = |V(G)| = p + q.$$

Denote by $t_1(e_i)$ the number of triangles containing the edge e_i in D for every $i = 1, 2, \ldots, r$. It is not difficult to see that $t_1(e_i) \leqslant c$ for every $i = 1, 2, \ldots, r$. Then

$$t_1(H) \geqslant t_1(D) - rc,$$

and the equality holds only if $t_1(e_i) = c$ for every $i = 1, 2, \ldots, r$.

By (vi) of Lemma 3, $t_1(G) = t_1(H)$, it follows that

$$t_1(D) - t_1(G) = t_1(D) - t_1(H) \leqslant rc.$$

Since $t_1(D) = abc, t_1(G) = (p-1)q$, we have

$$\begin{aligned} f(c) &= t_1(D) - t_1(G) - rc = \\ &= abc - (p-1)q - (ab + ac + bc - pq - p + 1)c = \\ &= abc - (p-1)q - [ab + (p + q - c)c - pq - p + 1]c = \\ &= (c-1)(c-p+1)(c-q) \leqslant 0. \end{aligned}$$

By $p \geqslant 3$ and $p-1 \leqslant q \leqslant p$, it follows that $c \geqslant (p+q)/3 > 1$. Let $\{V_1, V_2, V_3\}$ be the 3-independent partition in H such that $|V_1| = a$, $|V_2| = b$, $|V_3| = c$. It is not difficult to see that if $f(c) = 0$, then $t_1(e_i) = c$ for every $i = 1, 2, \ldots, r$, so edge $e_i \in R$ has one end vertex in V_1 and another end vertex in V_2. It follows that $H = H[V_1 \cup V_2] + O_q$

(i) If $q = p-1$, then $G = T_p + O_{p-1}$. In this case, $f(c) = (c-1)(c-p+1)^2 \geqslant 0$, $f(c) = 0$ if and only if $c = p-1$. So $t_1(e_i) = c = p-1$ for every $i = 1, 2, \ldots, r$. By (i) of Lemma 6 and (i) of Lemma 7, we have

$$\begin{aligned}\sigma(H,x) &= \sigma(H[V_1 \cup V_2] + O_{p-1}, x) = \\ &= \sigma(H[V_1 \cup V_2], x)\sigma(O_{p-1}, x) = \\ &= \sigma(G,x) = \\ &= \sigma(T_p + O_{p-1}, x) = \\ &= \sigma(T_p, x)\sigma(O_{p-1}, x).\end{aligned}$$

It follows that $\sigma(H[V_1 \cup V_2], x) = \sigma(T_p, x)$. So $P(H[V_1 \cup V_2], \lambda) = P(T_p, \lambda)$. By (i) of Lemma 4, $H[V_1 \cup V_2] = T'_p$, where T'_p is a tree of order p. Thus, $H = T'_p + O_{p-1}$.

It is not difficult to see that if $H = T'_p + O_{p-1}$, then $P(H, \lambda) = P(T_p + O_{p-1}, \lambda)$.

(ii) If $q = p$, then $G = T_p + O_p$. So $f(c) \leqslant 0$ if and only if $p-1 \leqslant c \leqslant p$, $f(z) = 0$ if and only if $c = p-1$ or $c = p$. Now we consider separately two cases.

Case 1: $c = p$. We have

$$\begin{aligned}\sigma(H,x) &= \sigma(H[V_1 \cup V_2] + O_p, x) = \\ &= \sigma(H[V_1 \cup V_2], x)\sigma(O_p, x) = \\ &= \sigma(G,x) = \\ &= \sigma(T_p + O_p, x) = \\ &= \sigma(T_p, x)\sigma(O_p, x).\end{aligned}$$

It follows that $\sigma(H[V_1 \cup V_2], x) = \sigma(T_p, x)$. So $P(H[V_1 \cup V_2], \lambda) = P(T_p, \lambda)$. By (i) of Lemma 4, $H[V_1 \cup V_2] = T'_p$, where T'_p is a tree of order p. Thus, $H = T'_p + O_p$.

Case 2: $c = p-1$. We have

$$\begin{aligned}\sigma(H,x) &= \sigma(H[V_1 \cup V_2] + O_{p-1}, x) = \\ &= \sigma(H[V_1 \cup V_2], x)\sigma(O_{p-1}, x) = \\ &= \sigma(G,x) = \\ &= \sigma(T_p + O_p, x) = \\ &= \sigma(T_p, x)\sigma(O_p, x) = \\ &= \sigma(O_1 + O_{p-1}, x)\sigma(O_p, x) = \\ &= \sigma(O_1, x)\sigma(O_{p-1}, x)\sigma(O_p, x) = \\ &= \sigma(O_1, x)\sigma(O_p, x)\sigma(O_{p-1}, x) = \\ &= \sigma(O_1 + O_p, x)\sigma(O_{p-1}, x) = \\ &= \sigma(T_{p+1}, x)\sigma(O_{p-1}, x).\end{aligned}$$

It follows that $\sigma(H[V_1 \cup V_2], x) = \sigma(T_{p+1}, x)$. So $P(H[V_1 \cup V_2], \lambda) = P(T_{p+1}, \lambda)$. By (i) of Lemma 4, $H[V_1 \cup V_2] = T''_{p+1}$, where T''_{p+1} is a tree of order $p+1$. Thus, $H = T''_{p+1} + O_{p-1}$. It is not difficult to see that if $H \in \{T'_p + O_p, T''_{p+1} + O_{p-1}\}$, then $P(H, \lambda) = P(T_p + O_p, \lambda)$. ■

3. Edge colorings

We need the following lemmas 9–13 to prove our results.

Lemma 9 [15]. Every bipartite graph G satisfies $\chi'(G) = \Delta(G)$.

Lemma 10 [15]. $ch'(G) \geqslant \chi'(G)$ for all graphs G.

Lemma 11 [15]. Every bipartite graph G satisfies $ch'(G) = \chi'(G)$.

Lemma 12 [16]. If G is a graph of order $2n+1$ and $\Delta(G) = 2n$, then G is Class one if and only if $|E(\overline{G})| \geqslant n$.

Lemma 13 [12]. If $G = T_n$ is a tree of order n, then $|E(G)| = n-1$.

Theorem 3. If $p \leqslant q$, then graph $G = T_p + O_p$ satisfies

(i) $\chi'(G) = \Delta(G)$;

(ii) $ch'(G) = \Delta(G)$.

Proof. Let $V(G) = V_1 \cup V_2$ is a partition of $V(G)$ such that $G[V_1] = T_p$, $G[V_2] = O_q$, $V_1 = \{v_1, v_2, \ldots, v_p\}$, $V_2 = \{u_1, u_2, \ldots, u_q\}$. Set $G_1 = G[V_1]$ and $G_2 = G - E(G[V_1])$. It is not difficult to see that G_1 and G_2 are bipartite graphs, $\Delta(G_2) = q = \deg_{G_2}(v)$ for every vertex $v \in V_1$ and $\Delta(G) = \Delta(G_1) + \Delta(G_2) = \Delta(G_1) + q = \deg(v)$ for some vertex $v \in V_1$.

(i) By (iii) of Lemma 4, $\chi(G_1) = 2$, so G_1 is a bipartite graph. By Lemma 9, G_1 has an edge coloring f_1 using $\Delta(G_1)$ colors $1, 2, \ldots, \Delta(G_1)$. Again by Lemma 9, G_2 has an edge coloring f_2 using $\Delta(G_2) = q$ colors $\Delta(G_1)+1, \ldots, \Delta(G_1)+q$. Since $\Delta(G) = \deg(v)$ for some vertex $v \in V_1$, it is clear that the mapping

$$f : E(G) \to \{1, 2, \ldots, \Delta(G_1), \Delta(G_1)+1, \ldots, \Delta(G_1)+q\}$$

such that $f(e) = f_1(e)$ if $e \in E(G_1)$ and $f(e) = f_2(e)$ if $e \in E(G_2)$ is an edge coloring of G. Since $\Delta(G) = \Delta(G_1) + q$, it follows that $\chi'(G) = \Delta(G)$.

(ii) By Lemma 10 and (i), we have $ch'(G) \geqslant \Delta(G) = \Delta(G_1) + q$. Now we prove that $ch'(G) \leqslant \Delta(G)$. Let $L(e)$ be the lists of colors of $e \in E(G)$ such that $|L(e)| = \Delta(G)$.

Let $L_1(e) \subseteq L(e)$ such that $|L_1(e)| = \Delta(G_1)$ for every $e \in E(G_1)$. Since G_1 is a bipartite graph, by Lemma 9 and Lemma 11, there exists g_1 being a list edge coloring of G_1 with the lists of colors $L_1(e)$ for every $e \in E(G_1)$.

For every $i = 1, 2, \ldots, p$, the subgraph induced by the edges of G_1 incident with v_i is denoted by $G_1(v_i)$. It is clear that $|g_1(G_1(v_i))| \leqslant \Delta(G_1)$. For every $i = 1, 2, \ldots, p$, $j = 1, 2, \ldots, q$, set $L'(v_iu_j) = L(v_iu_j) \backslash g_1(G_1(v_i))$. It follows that $|L'(v_iu_j)| \geqslant \Delta(G) - \Delta(G_1) = q$. Let $L_2(v_iu_j) \subseteq L'(v_iu_j)$ such that $|L_2(v_iu_j)| = \Delta(G_2) = q$ for every $i = 1, 2, \ldots, p$, $j = 1, 2, \ldots, q$. By Lemma 11, there exists g_2 being a list edge coloring of G_2 with the lists of colors $L_2(v_iu_j)$ for every $i = 1, 2, \ldots, p$, $j = 1, 2, \ldots, q$. Let g be the edge coloring of G such that $g(e) = g_1(e)$ if $e \in E(G_1)$ and $g(e) = g_2(e)$ if $e \in E(G_2)$. Then g is a list edge coloring of G with the lists of colors $L(e)$ for every $e \in E(G)$, i.e., $ch'(G) \leqslant \Delta(G)$. Thus, $ch'(G) = \Delta(G)$. ■

Theorem 4. Let $G = T_p + O_p$ be a graph with $\Delta(G) = p + q - 1$. Then

$$\chi'(G) = ch'(G) = \Delta(G)$$

if and only if $G \neq K_3$.

Proof. Let $V(G) = V_1 \cup V_2$ is a partition of $V(G)$ such that $G[V_1] = T_p$, $G[V_2] = O_q$, $V_1 = \{v_1, v_2, \ldots, v_p\}$, $V_2 = \{u_1, u_2, \ldots, u_q\}$. Set $G_1 = G[V_1]$ and $G_2 = G - E(G[V_1])$. It is not difficult to see that G_1 and G_2 are bipartite graphs and $\Delta(G) = p + q - 1 = \deg(v)$ for some vertex $v \in V_1$. It follows that $\Delta(G_1) = p - 1$.

Suppose that $\chi'(G) = ch'(G) = \Delta(G)$. We have $chi'(K_3) = ch'(K_3) = 3$. So $G \neq K_3$.

Now suppose that $G \neq K_3$. If $p \leqslant q$, then by Theorem 3, $\chi'(G) = ch'(G) = \Delta(G)$. So we may assume that $p > q$. If $p = 2$, then $q = 1$, so $G = K_3$, a contradiction. It follows that $p \geqslant 3$. Without loss of generality we may assume that $\Delta(G_1) = \deg_{G_1}(v_1)$, so $\Delta(G) = \deg(v_1)$. Since $\Delta(G_1) = p-1$, it is not difficult to see that $E(G_1) = \{v_1v_2, v_1v_3, \ldots, v_1v_p\}$. We consider separately two cases.

Case 1: $p = q+1$.

If $q = 1$, then $p = 2$, so $G = K_3$, a contradiction. So we may assume that $q \geqslant 2$. By Lemma 13, it is not difficult to see that $|E(\overline{G})| = q^2 - q$. Since $q \geqslant 2$, it follows that $|E(\overline{G})| \geqslant q$. By Lemma 12, G is Class one.

By Lemma 10, $ch'(G) \geqslant \chi'(G) = \Delta(G)$. Let $L(e)$ be the lists of colors of $e \in E(G)$ such that $|L(e)| = \Delta(G)$. Set $G_3 = G - E(G[\{v_2, v_3, \ldots, v_p\} \cup V_2])$ and $G_4 = G[\{v_2, v_3, \ldots, v_p\} \cup V_2]$. It is clear that G_3 and G_4 are bipartite graphs with $\Delta(G_3) = \deg(v_1) = \Delta(G)$ and $\Delta(G_4) = q$. By Lemma 9 and Lemma 11, there exists g_3 being a list edge coloring of G_3 with the lists of colors $L(e)$ for every $e \in E(G_3)$. For every $i = 2, 3, \ldots, p$, $j = 1, 2, \ldots, q$, set $L'(v_iu_j) = L(v_iu_j) \setminus \{g_3(v_1v_i), g_3(v_1u_j)\}$. It follows that $|L'(v_iu_j)| \geqslant \Delta(G) - 2 = p+q-3 \geqslant q$ for every $i = 2, 3, \ldots, p$, $j = 1, 2, \ldots, q$. Let $L_4(v_iu_j) \subseteq L'(v_iu_j)$ such that $|L_4(v_iu_j)| = q$ for every $i = 2, 3, \ldots, p$, $j = 1, 2, \ldots, q$. By Lemma 11, there exists g_4 being a list edge coloring of G_4 with the lists of colors $L_2(v_iu_j)$ for every $i = 2, 3, \ldots, p$, $j = 1, 2, \ldots, q$. Let g be the edge coloring of G such that $g(e) = g_3(e)$ if $e \in E(G_3)$ and $g(e) = g_4(e)$ if $e \in E(G_4)$. Then g is a list edge coloring of G with the lists of colors $L(e)$ for every $e \in E(G)$, i.e., $ch'(G) \leqslant \Delta(G)$. Thus, $ch'(G) = \Delta(G)$.

Case 2: $p \geqslant q+2$.

It is clear that $G[v_1 \cup V_2] = T'_{q+1}$, where T'_{q+1} is a tree of order $q+1$. Therefore, $G = T'_{q+1} + O_{p-1}$. Since $q+1 < p-1$, by Theorem 3, $\chi'(G) = ch'(G) = \Delta(G)$. ■

Conclusion

The coloring problems are interesting topics in graph theory. Coloring graphs found application in many practical problems, for example, coding theory or security. Clearly, to estimate the chromatic as well as the chromatic uniqueness is very important. So far there have been many research results on this topic for different graph layers. However, the problem has not been generally solved, and further research is needed. This paper explores some of the coloring problems with graph G, which is join of the tree T_p and the null graph O_q, contributes to enriching the research results on the coloring problems.

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