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ALGORITHMIC THEORY OF SOLVABLE GROUPS[1]

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The purpose of this survey is to give some picture of what is known about algorithmic and decision problems in the theory of solvable groups. We will provide a number of references to various results, which are presented without proof. Naturally, the choice of the material reported on reflects the author's interests and many worthy contributions to the field will unfortunately go without mentioning. In addition to achievements in solving classical algorithmic problems, the survey presents results on other issues. Attention is paid to various aspects of modern theory related to the complexity of algorithms, their practical implementation, random choice, asymptotic properties. Results are given on various issues related to mathematical logic and model theory. In particular, a special section of the survey is devoted to elementary and universal theories of solvable groups. Special attention is paid to algorithmic questions regarding rational subsets of groups. Results on algorithmic problems related to homomorphisms, automorphisms, and endomorphisms of groups are presented in sufficient detail.

Keywords: *solvable groups, algorithmic and decision problems, algorithms.*

1. Introduction

Awareness of the algebraic nature of many important concepts of topology and function theory in the 1880s led to the formation of a combinatorial group theory. Groups, already represented in the works of F. Klein, H. Poincare and other mathematicians, gained the right to independence after W. Dick discovered a universal way to define them using generators and defining relations [50]. H. Poincare [198, 199] established the first contacts between combinatorial topology and group theory. He introduced the fundamental groups of manifolds into consideration, while at the same time finitely defined groups of finite simplicial complexes were distinguished as effective objects. E. S. Fedorov [58] discovered a remarkable application of groups to the geometry of crystals. F. Klein proposed in his inaugural lecture in 1872 at the University of Erlangen (Germany) the famous *Erlangen program*, classifying geometries by their basic symmetry groups [106]. This program is an influential synthesis of much of the mathematics of the time.

It turned out that many important topology problems are algorithmic in nature. At the very beginning of the twentieth century, the basic algorithmic problems were formulated for a class of finitely defined groups. The word problem was posed by M. Dehn [42]: Is there an algorithm that, from two arbitrary group words from the generating elements of the group, determines whether they define the same element of the group? H. Tietze [250] developed the Tietze transformations for group presentations, and was the first to pose the group isomorphism problem: Is there an algorithm that finds out, from two arbitrary finite group

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assignments by generating elements and defining relations, whether these assignments define isomorphic groups?

From the very beginning, combinatorial group theory was closely intertwined with computability theory. We are currently seeing many new successful interactions between group theory and computer science. Complexity theory and automata theory began to play more important role in the group theory, especially in the algorithmic directions. Questions of random choice and asymptotic properties of groups have acquired significant importance. On the other hand, various mathematical fields such as algebraic cryptography and data compression have led to new questions in group theory.

More than thirty-five years ago, the author, together with V. N. Remeslennikov, published the survey [210], devoted to the algorithmic and model-theoretic questions in group theory. The survey is widespread, its objective was to give a fairly complete description of group-theoretic results of an algorithmic nature in their historical development, as well as to present the methods of model-theoretical research in group theory. These two lines of research are closely interrelated and have a common focus, as they seek to answer one general question: what properties and characteristics of groups can be effectively identified? Some aspects of this research are reflected in [152, 186].

V. N. Remeslennikov

The content of the papers [186, 210] and the monograph [152] is largely due to the significantly increased interest in research in combinatorial group theory at that time. This area was formed in the 60–70s of the twentieth century. Two monographs with the same title “Combinatorial group theory” written by W. Magnus, A. Karrass and D. Solitar [132], and by R. Lyndon and P. Shupp [125] played a significant role in its formation. The title of [132] has a subtitle “Representation of groups in terms of generators and relations”. These monographs laid the foundations for combinatorial group theory as one of the most actively developing areas of group theory and mathematics in general in the following decades to our time. Both monographs were translated into Russian and subsequently reprinted several times. W. Magnus et al.’s book focuses on representing groups in terms of generators and defining relations. The authors consider free constructions: free groups and products, free amalgamated products, Higman — Neumann — Neumann (HNN) extensions. The term “combinatorial” itself arose from the frequent and significant use of combinatorial methods. The book touched on algorithmic problems, from the classic Dehn problems to problems that only arose at that time. The value of the book [132] for the further development of the combinatorial group theory is very great. It is a tutorial, a problem source, and a research sample.

The book [125] is clearly an important contribution to the mathematical literature. It contains proof of Whitehead’s theorems and related theorems by J. McCool, proof of the Karrass — Solitar theorem on subgroups of free products with one amalgamated subgroup by Nielsen methods and its obvious promise applications. It also contains discussion of cohomology, graph-theoretical connections, discussion of HNN extensions, elegant treatment of one-relator groups, proof of the Higman embedding theorem, connections with logic, the use of van Kampen diagrams and the consideration of small cancellation theory and its applications are very good advances.

The history of combinatorial group theory is described by W. Magnus and B. Chandler in [34]. Results on combinatorial algebra are presented in the monograph by L. A. Bokut and G. P. Kukin [32]. O. Kharlampovich and M. Sapir presented in [103] a survey of results on algorithmic problems in varieties of algebraic systems.

The origins of the theory of solvable (some authors use the term “soluble”) groups go back to the first half of the nineteenth century, when Évariste Galois determined a necessary and sufficient condition for a polynomial to be solvable by radicals, thereby solving a problem standing for 350 years. His work laid the foundations for Galois theory and group theory, two major branches of abstract algebra. He realized that the algebraic solution to a polynomial equation is related to the structure of a group of permutations associated with the roots of the polynomial, the Galois group of the polynomial. He found that an equation could be solved in radicals if one can find a series of subgroups of its Galois group, each one normal in its successor with abelian quotient, or its Galois group is solvable.

Évariste Galois

This proved to be a fertile approach, which later mathematicians adapted to many other fields of mathematics besides the theory of equations to which Galois originally applied it. The achievements of Galois theory stimulated intensive study of permutation groups, and indeed in the early stages of its development, group theory was preoccupied almost exclusively with finite groups.

However, under the influence of geometry, topology, and the theory of differential equations, there arose a pressing need to consider infinite groups of transformations. The theory of infinite groups began to develop in the 20s of the twentieth century. Free groups first arose in the study of hyperbolic geometry, as examples of Fuchsian groups (discrete groups acting by isometries on the hyperbolic plane). The algebraic study of free groups was initiated by Jakob Nielsen in 1920s, who gave them their name and established many of their basic properties [179, 180]. Otto Schreier published an algebraic proof of the Nielsen — Schreier theorem in [243]. Max Dehn realized the connection of groups with topology, and obtained the first proof of the Nielsen — Schreier theorem [42]. Kurt Reidemeister included a comprehensive treatment of free groups in [204] and in his book [205], the first monograph on combinatorial group theory and topology. Parametric groups made their appearance in the works of S. Lie [114].

In the 1930s of the twentieth ccentury, Wilhelm Magnus invented the connection between the lower central series of free groups and free Lie algebras (see [133]).

Wilhelm Magnus

From the Preface of [133]: “Magnus has had such a profound influence on combinatorial group theory because many of his ideas, startingly and strikingly simple, have provided not only deep insights into a very difficult subject but also powerful methods for dealing with these difficulties. His ideas have also found application in topology, K-theory, the theory of Lie and associative algebras, computational complexity, and also in logic. The expert in group theory, however, will be astonished to find that this reprinting of Magnus' papers contains a very large amount of very important work on diffraction problems and related topics in analysis. Indeed Magnus is one of the very few mathematicians who has done significant work in two completely different fields. There is a large number of mathematicians who know Magnus for his work in analysis but are totally unaware of his work in group theory. His books, his teaching, his many doctoral students, his effect on the thinking of his colleagues both in private conversation and in seminars have also helped to establish him as a mathematician of the first rank and enriched the mathematical community.” — G. Baumslag and B. Chandler.

Intensive research on solvable groups began in the 30s of the twentieth century. This research was initiated by P. Hall, who just completed his great sequence of papers on finite solvable groups [83]. His PhD-student K. A. Hirsch published a sequence of five papers [85 –

89], where he introduced and investigated polycyclic groups. From the very beginning it became clear that the theory of infinite solvable groups needs in some original methods of studying. It turned out later that the methods come from ring theory, matrix group theory and homological algebra. Thus, the theory of infinite solvable groups has a broader connection in algebra.

In 1950s A. I. Mal'cev established the basic theory of solvable matrix groups [141]. He also invented the notion of a rank and investigated solvable groups of finite rank [139]. He showed the undecidability of the elementary theory of finite groups, of free nilpotent groups, of free soluble groups and many others. This Mal'cev's works determined the perspective direction of research for many years.

A. I. Mal'cev is a great mathematician known for his fundamental achievements in algebra and mathematical logic. He is one of the founders of the general theory of algebraic systems and model theory and the founder of the Siberian School of Algebra and Logic.

At the same time, P. Hall made a significant contribution to the development of the theory of soluble groups. Namely, he published a series of papers [78 – 82] on finitely generated solvable and nilpotent groups. In these papers, he proved a number of results that are important in theory and determine further research in this area.

Since that time the solvable group theory became one of the central topics in group theory.

Solvable groups are interesting not only in and of themselves. They are an effective tool for investigating more general objects of group theory. Suffice it to recall the Sylow subgroups, solvable radicals, Borel subgroups and so on. Below we give two examples of the results obtained by passing to solvable factor groups.

The following result was proposed by M. Dehn and proved by his student, W. Magnus, in his doctoral thesis (see [129]). It is well-known as the *freedom theorem* of Magnus or

The *Freiheitssatz*: Let

$$G = \langle x_1, \ldots, x_n : r \rangle$$

be a group presentation given by n generators x_i and a single cyclically reduced relator r. If x_1 appears in r, then the subgroup of G generated by $x_2, \ldots, x_n$ is a free group, freely generated by $x_2, \ldots, x_n$.

Magnus' method of proof of the Freiheitssatz relies on free amalgamation products of groups. This method initiated the use of these products in the study of infinite discrete groups.

N. S. Romanovskii used a different approach in his proving the *generalized freedom theorem* for groups with several relations (solution of the Lyndon problem) [217]: Let the group

$$G = \langle x_1, \ldots, x_n : r_1, \ldots, r_m \rangle$$

have deficiency $d = n - m > 0$. Then there exist a subset of d of the given generators which freely generates a subgroup of G isomorphic to F_d. A similar assertion was also proved for groups defined by generators and relations in varieties of solvable and nilpotent groups and pro-p groups. He essentially used solvable groups as a tool for these proofs.

In [228], the author proved that the automorphism group of the free pro-p group $\tilde{F}_r(p)$ of rank $r \geqslant 2$ is topologically infinitely generated. A similar assertion was also proved for free profinite groups $\tilde{F}_r$ and for free metabelian pro-p groups $M_r(p)$. His methods of proofs are also related to solvable groups.

The modern theory of solvable groups is presented in the monograph [113] by J. C. Lennox and D. J. S. Robinson. See also the author's monograph [239]. The theory of nilpotent groups is presented in the lectures [9] by G. Baumslag and [82] by P. Hall.

The main object of research in the monograph by E. I. Timoshenko [256] — free groups in varieties of solvable groups and their universal theories. In addition, groups of automorphisms and semigroups of endomorphisms of solvable groups are described. A significant part of the results belongs to the author of the monograph.

Algebraic geometry over groups arose in the mid-90s of the twentieth century in the works of B. I. Plotkin [194, 195] on the one hand and in the works of G. Baumslag, V. N. Remeslennikov, A. G. Myasnikov and O. G. Kharlampovich [17, 168, 101, 102] on the other. The current state of algebraic geometry over groups and more generally over algebraic systems is presented in [196, 197], and in [41].

2. Algorithmic problems

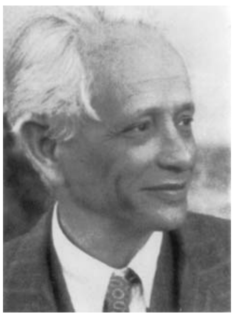
In the very beginning of the twentieth century M. Dehn and H. Titze proposed the following three algorithmic problems:

- The word problem (Dehn [42]): Given a group presentation $G = \langle X : R \rangle$ and words $w(X), u(X)$ in the alphabet X determine if $w(X) =_G u(X)$.
- The conjugacy problem (Dehn [42]): Given a group presentation $G = \langle X : R \rangle$ and words $w(X), u(X)$ in the alphabet X determine if there exists some $g(X)$ such that $g(X)^{-1}w(X)g(X) =_G u(X)$.
- The isomorphism problem (Tietze [250]): Given two group presentations $G = \langle X : R \rangle$ and $H = \langle Y : S \rangle$ determine if they define isomorphic groups.

Subsequently, the following problem began to be added to this list of problems:

- The subgroup membership problem: Given a group presentation $G = \langle X : R \rangle$ and a finite set of words $g(X), w_1(X), \ldots, w_k(X)$ in the alphabet X find out whether or not $g(X) \in \mathrm{gp}(w_1(X), \ldots, w_k(X))$.

The subgroup membership problem is often called the *generalized word problem* or simply *the membership problem* in the literature of combinatorial group theory.



Max Dehn

Until the 1950s of the twentieth century only positive results could be obtained since totally new methods were needed even to state the problem of finding a group with unsoluble word problem with the formal precision. In particular, W. Magnus published in [130] a complete proof of the solution of the word problem for the class of one-relator groups.

The proof of the algorithmic undecidability of the word problem in the class of all finitely defined groups, obtained by Petr Sergeevich Novikov in 1952 is one of the best results in algorithmic group theory and mathematics in general.

Theorem 1 (P. S. Novikov [187, 188]). There exists a finitely presented group G such that the word problem for G is undecidable.

A wonderful example of P. S. Novikov was of fundamental importance for further research on algorithmic issues in group theory. Obviously conjugacy and membership problems are also unsolvable in the class of finitely presented groups.

W. W. Boone gave in [33] an independent proof of Novikov's result. See [267] for some other results on the word problem in groups.

Let us especially note the importance of the work of S. I. Adian [1], in which a number of algorithmic problems are solved. In particular, he showed the undecidability of the isomorphism problem in the class of finitely defined groups. S. I. Adian in some his proofs based on the idea of the following Markov property.

Markov Property: An abstract property $\mathbb{P}$ of finitely presented groups is a *Markov property* if there are two finitely presented groups G^+ and G^- such that

— G^+ has property $\mathbb{P}$;
— G^- cannot be embedded as a subgroup in any finitely presentable group with property $\mathbb{P}$.

Theorem 2 (S. I. Adian [1]). If $\mathbb{P}$ is a Markov property of finitely presented groups, then $\mathbb{P}$ is not recursively recognisable.

Therefore, the following properties of finitely defined groups are not recognized recursively, namely: to be trivial (finite, abelian, nilpotent, solvable, free, torsion-free, or residually finite) group, having a solvable word problem, and so on.

M. O. Rabin [201] proved similar results, which are now called the *Adian – Rabin theorem*.

S. I. Adian and V. G. Durnev [2] presented a detailed survey of results concerning the main decision problems of group theory and semigroup theory. They discuss results on the word problem, isomorphism problem, recognition problems, and other algorithmic questions related to them. The classical theorems of A. A. Markov and E. L. Post, P. S. Novikov, S. I. Adian and M. O. Rabin, G. Higman, W. Magnus, and R. C. Lyndon are given with complete proofs.

Further in the paper, we do not present here other results of algorithmic theory pertaining to classes of groups other than solvable.

For simplicity, we will simplify expressions, speaking not about group representations, but about groups, not about words in the generators of a given representation, but about group elements, etc.

After the obtained negative results on the solvability of algorithmic problems in the class of all finitely defined groups, the interest of researchers was turned to various classes of groups. The methods of assigning groups have expanded. The algorithmic problems themselves became more diverse. Algorithmic problems of the following two types began to be considered:

— *Decision problems*: Given a property $\mathbb{P}$ and an object $\mathcal{O}$, find out whether or not the object $\mathcal{O}$ has the property $\mathbb{P}$.
— *Search problems*: Given the property $\mathbb{P}$ and the information "$\mathcal{O}$ satisfies $\mathbb{P}$", find out at least one specific implementation of $\mathbb{P}$ to $\mathcal{O}$.

For example, if we know that elements w and u are conjugate in the group G, the search problem is to find a conjugating element $g \in G$ such that $g^{-1}wg = u$.

The issues of solvability of search problems are especially important for applications and algorithms used in practice. For theory and practical applications, the complexity of the algorithms is essential. At present, the issues of the complexity of algorithms, in particular — computational complexity, have become of paramount importance. There is a huge amount of research in this area. Some results related to the complexity of algorithms for solvable groups will be touched upon in this review.

Friendly definitions:

— Sol = solvable groups;

— ThAlg(Sol) — Algorithmic theory of Sol = Information about Sol-groups, their elements, subgroups, subsets, structure, etc., that can, in principle at least, be obtained by machine computation, namely Turing machines, automata, computers, and so on.

Relative presentation in the variety $\mathfrak{L}$:

$$G = \langle x_1, \dots, x_n : \{r_\lambda : \lambda \in \Lambda\}; \mathfrak{L}\rangle.$$

That is

$$G = F(X, \mathfrak{L})/ncl\{r_\lambda : \lambda \in \Lambda\},$$

where $F(X, \mathfrak{L})$ is a free group in the variety $\mathfrak{L}$ with basis $X = \{x_1, \dots, x_n\}$, and Λ is finite or, more generally, recursive enumerable set.

Presentation by generators:

$$G = \mathrm{gp}(g_1, \dots, g_n) \leqslant \bar{G},$$

where $\bar{G}$ is some bigger group, for example, $\bar{G} = GL_n(K)$, the general linear (matrix) group over K, or $\bar{G} = \pi_1(S)$, the fundamental group of a topological space S.

Presentation by action:

$$G = Aut(H),$$

where H is some other group (more generally, some structure), or

$$G = \pi_1(S),$$

where S is some topological space.

Classical algorithmic problems:

For a group G:

— *The word problem* (WP): $w = 1$?
— *The conjugacy problem* (CP): $\exists g : g^{-1}wg = u$?
— *The membership problem* (MP): $w \in H \leqslant G$?

For a class of groups $\mathcal{C}$:

— *The isomorphism problem* (IP): $G \simeq H$?

We also highlight the following two problems concerning automorphisms and homomorphisms, which can also be considered classical because of their high importance. The first problem is formulated for an arbitrary group G:

— *The automorphic conjugacy problem* (J. H. C. Whitehead [266]): Is there an algorithm that finds out from two arbitrary group words from the generating elements of the group, do they determine automorphically conjugate elements of the group? In other words, is there an automorphism of a group that takes one of the given elements to another?

The automorphism problem for a free group F_r of rank r was algorithmically solved by J. H. C. Whitehead himself in a classic 1936 paper [266] and his solution came to be known as *Whitehead's algorithm*. This proof was topological.

Subsequently, E. S. Rapaport [202] and later, based on her work, P. J. Higgins and R. C. Lyndon in [84] gave a purely combinatorial and algebraic re-interpretation of Whitehead's algorithm. The exposition of Whitehead's algorithm in the book of R. Lyndon and P. Schupp [125] is based on this combinatorial approach.

In 1946, Emil Post [200] invented the following

The Post correspondence problem (PCP). Given an alphabet Σ, an instance of PCP is a finite set of pairs of strings (g_i, h_i), where $1 \leqslant i \leqslant s$, over Σ. A solution to this instance is a sequence of selections $i_1, i_2, \dots$ (repetition is possible) such that

$$g_{i_1} g_{i_2} \dots g_{i_n} = h_{i_1} h_{i_2} \dots h_{i_n}.$$

Is an effective procedure answering for any instance on the question: Does a solution exist for this instance?

It was also proved in [200] that PCP in the classical setting is unsolvable.

This gives rise to a more general definition often found in the literature, according to which any two homomorphisms α, β with a common domain F and a common codomain G form an instance of the Post correspondence problem, which now asks whether there exists a nonempty word $w \in F$ such that $\alpha(w) = \beta(w)$.

Obviously, PCP can be posed for a free algebraic system F.

PCP(F): For a pair of endomorphisms $\alpha, \beta \in \mathrm{End}(F)$, is there a (nontrivial) element (word) $w \in F$ such that $\alpha(w) = \beta(w)$?

Moreover, PCP can be formulated for any algebraic system A as follows. Let $F(A)$ be a free algebraic system in the variety $\mathrm{Var}(A)$ generated by A, and $\alpha, \beta : F(A) \to A$ be a pair of homomorphisms.

PCP(A): Is there a (nontrivial) element (word) $w \in F(A)$ such that $\alpha(w) = \beta(w)$?

Thus, we can formulate PCP for any group G.

— Let $F(G)$ be a free group in the variety $\mathrm{Var}(G)$ generated by G, and $\alpha, \beta : F(G) \to G$ be a pair of homomorphisms. Is there a nontrivial element $w \in F(G)$ such that $\alpha(w) = \beta(w)$?

In this paper we give a special Section 7 devoted to the Post correspondence problem and its generalizations.

A word $u(x) = u(x_1, \dots, x_r)$ in certain variables $x = (x_1, \dots, x_r)$ is called an *identity* in a group G if under substitution of any sequence $g = (g_1, \dots, g_r)$ of elements of G into $u(x)$ in place of x we obtain the equality $u(g) = 1$. In other words, G satisfies the identity $u(x) \equiv 1$. A quasi-identity is an implication of the form $u_1(x) = 1 \wedge \dots \wedge u_n(x) = 1 \to u(x) = 1$.

The I-theory (Q-theory) of a class $\mathcal{C}$ of groups is the totality of all identities (quasi-identities) that are true on all the groups in $\mathcal{C}$.

A. I. Mal'cev posed in [108] (Question 2.40 (a)) the following *identity (quasi-identity) problem*: Does there exist a finitely axiomatizable variety of groups whose I-theory (Q-theory) is non-decidable?

Further decision problems:

For a group G:

— *The twisted conjugacy problem* (TCP): For endomorphism $\varphi \in \mathrm{End}(G)$) and elements $g, f \in G$ to decide whether there exists an element $x \in G$ such that $\varphi(x)g = fx$.
— *The bi-twisted conjugacy problem* (BTCP): For endomorphisms $\varphi, \psi \in \mathrm{End}(G)$) and elements $g, f \in G$ to decide whether there exists an element $x \in G$ such that $\varphi(x)g = f\psi(x)$.
— *The generation and presentation problem* (GPP):
Find generators or presentation of a subgroup, centralizer, an automorphism group, etc.
— *The equation problem* (EqP):

$$\exists x_1 \dots \exists x_n \big(w(x_1, \dots, x_n) = 1\big)?$$

— *The endomorphism* (*automorphism*) *problem* (EndoP or AutoP):

$$\exists \varphi \in \mathrm{End}(G)\ (\mathrm{Aut}(G))\big(\varphi(g) = f\big)?$$

For a class $\mathcal{C}$ of groups:

— *The epimorphism problem* (EpiP): $\exists \varphi \in \mathrm{Hom}(G, H)\big(\varphi(G) = H\big)$?

Recall that a group G is called *residually finite* if for each nontrivial element $g \in G$ there exists a finite group K and a homomorphism $\varphi : G \to K$ such that $\varphi(g) \neq 1$. A. I. Mal'cev proved that every finitely presented residually finite group G has the decidable WP [142].

3. Finitely generated nilpotent and polycyclic groups

In his series of papers [78 – 72] P. Hall established a remarkable connection between the theory of polycyclic groups and commutative algebra.

Philip Hall

He noted that, since the class of finitely presented groups is closed under extensions, polycyclic groups are finitely presented. These groups satisfies *max*, the maximal condition for subgroups, and they admit many other nice properties.

A. I. Mal'cev [142] showed that residual finiteness of some recursive enumerable property $\mathbb{P}$ of a group G implies decidability of $\mathbb{P}$ in G. Subsequently, many proofs of the solvability of algorithmic problems were based on the corresponding finite residuality.

Classical decision problems. Positive solutions:

— M. F. Newman [178]: the conjugacy problem is solvable for any finitely generated nilpotent group.

— S. Blackburn [31]: every finitely generated nilpotent group G is *conjugacy separable*, i.e., residually finite with respect to the conjugacy property. In other words, for every pair $g, f \in G$ of elements that are not conjugate in G there is a homomorphism $\mu : G \to K$ onto finite group K for which $\mu(g), \mu(f)$ are not conjugate in K.

— V. N. Remeslennikov [206] and E. Formanek [59]: every polycyclic group is conjugacy separable. Therefore, the conjugacy problem for any polycyclic group is decidable.

Let $Fin(G)$ denote the set of isomorphism classes of finite quotients of the group G. Two groups G and H are said to *have the same finite quotients* if $Fin(G) = Fin(H)$. Obviously, for a finitely generated abelian group A we have $Fin(A) = \{A\}$. G. A. Noskov proved that $Fin(M) = \{M\}$ for any free metabelian group M [184].

P. F. Pickel constructed infinitely many nonisomorphic finitely presented metabelian groups with the same finite quotients, using modules over a suitably chosen ring [193]. These groups also give an example of infinitely many nonisomorphic split extensions of a fixed finitely presented metabelian group by a fixed finite abelian group, all having the same finite quotients. G. Baumslag proved that there exists non-isomorphic meta-cyclic groups G and H for which $Fin(G) = Fin(H)$ [10].

F. Grunewald and P. Zalesskii introduced in [72] a notion of a *genus* $g(\mathcal{C}, G)$ for a class of groups $\mathcal{C}$ and $G \in \mathcal{C}$. It consists of isomorphism classes of groups from $\mathcal{C}$ having the same profinite completion as G. They showed finiteness results for $g(\mathcal{C}, G)$ for several important families of groups including finitely generated virtually free groups. They also developed formulas for the number of elements in $g(\mathcal{C}, G)$ in various cases. By these they found interesting examples where $g(\mathcal{C}, G)$ contains only one element.

Let G be a finitely generated group. By $\tilde{G}$ we denote the profinite completion of G, G and $\tilde{G}$ have the same finite quotients. The key result to formalize the precise connection between the collection of finite quotients of G and those of $\tilde{G}$ is the following. Suppose that G and H are finitely generated abstract groups. Then $\tilde{G}$ and $\tilde{H}$ are isomorphic if and only if $Fin(G) = Fin(H)$. This is basically proved in [47]. A. W. Reid [203] introduced the mild difference in the statement by emploing the great result by N. Nikolov and D. Segal [182] to replace topological isomorphism with isomorphism.

Now we list the known positive results on the Isomorphism problem for the classes $\mathcal{N}$ and $\mathcal{P}$ of finitely generated nilpotent and polycyclic groups, respectively:

— P. F. Pickel [192] proved that the genus of every finitely generated nilpotent group N is finite. Consequencly, the isomorphism problem to a fixed finitely generated nilpotent group N is decidable.
— F. Grunewald, P. F. Pickel, and D. Segal [69] established that every $g(\mathcal{PF})$-class of polycyclic-by-finite groups is the union of finitely many isomorphism classes.
— F. Grunewald and D. Segal [70, 71] constructed some rather general algorithms, which can (in theory) be applied in diverse situations. In particular, they gave an algorithm that solves the isomorphism problem for finitely generated nilpotent groups.
— R. A. Sarkisjan [241, 242] independently solved the isomorphism problem for finitely generated nilpotent groups under certain conditions, the validity of which was not known at that time. Later it turned out that the condition is met.

Theorem 3 (D. Segal [244]). There is an algorithm which does the following: given a finitely presented virtually polycyclic group G, given elements $a_1, \ldots, a_n, b_1, \ldots, b_n$ of G, and given finitely generated subgroups $A_1, \ldots, A_m, B_1, \ldots, B_m$ of G, it decides whether there exists an automorphism α of G such that $\alpha(a_i) = b_i$ (and $\alpha(A_i) = B_i$) for $i = 1, \ldots, n$, and $j = 1, \ldots, m$.

As a consequence of this statement, we obtain that the isomorphism problem for the class of virtually polycyclic groups is decidable. Indeed, the solvability of the classical isomorphism problem for virtually polycyclic groups is an immediate consequence of Theorem 3: For any pair of groups A and B we can write down a presentation for $G = A \times B$, and observe that $A \simeq B$ if and only if there exists an automorphism α of G with $\alpha(A) = B$.

The Further decision problems. Positive and negative solutions:

— V. A. Roman'kov [231] proved that TCP is solvable for any polycyclic group. He also proved in [225] that EqP and EndoP are not solvable for free nilpotent groups of class $\geqslant 9$.
— V. N. Remeslennikov [208] established that EpiP is not solvable for the variety $\mathfrak{N}_2$ of nilpotent groups of class $\geqslant 2$.

Theorem 4 (G. Baumslag, F. B. Cannonito, D. J. S. Robinson, and D. Segal [12]).
Let $G = \langle x_1, \ldots, x_n | r_1, \ldots, r_m \rangle$ be a presentation of a polycyclic group. Then there is a uniform algorithm which, when given a finite subset U of G, produces a finite presentation of $\mathrm{gp}(U)$. Hence we can efficiently find a polycyclic presentation of G, the Hirsch number $h(G)$, the Fitting ($\mathrm{Fitt}(G)$) and Frattini ($\mathrm{Fratt}(G)$) subgroups, the center $\mathrm{C}(G)$, decide if G is torsion-free, and so on.

For nilpotent groups, an algorithm to solve the conjugacy problems for subgroups is described in [115].

G. Baumslag, C. F. Miller III, and G. Ostheimer [16] described an algorithm for deciding whether or not a given finitely generated torsion-free nilpotent group is decomposable as the direct product of nontrivial subgroups.

Let O be a binomial ring, i.e., an integral domain containing the ring of integers $\mathbb{Z}$ and containing with every element λ all binomial coefficients

$$\binom{\lambda}{n} = \frac{\lambda(\lambda-1)\cdots(\lambda-n+1)}{n!}, \quad n \in \mathbb{N}.$$

P. Hall [82] itroduced the class of nilpotent O-power groups. M. I. Kargapolov et al. [95] solved in a uniform way various algorithmic problems for O-power groups: word, conjugacy, and membership problems, determination of the O-periodic part, determination of intersection of two O-subgroups, and description of the O-subgroups in terms of generators and defining relations. Note, that in the case $O = \mathbb{Z}$ we have the usual nilpotent groups.

Gilbert Baumslag, an outstanding mathematician and great enthusiast of solvable groups and algorithms.

See other results the O-power groups and its generalizations in [4, 111, 134, 135], etc.

4. Metabelian groups

P. Hall [78] proved that every finitely generated metabelian group G satisfies max_n (the maximal property for normal subgroups). Therefore, G is finitely defined in the variety $\mathfrak{A}^2$ of all metabelian groups.

The basis of any finitely generated metabelian group G is its commutant G', which can be considered as a module over a finitely generated commutative group ring $\mathbb{Z}[G/G']$. Since this ring is Noetherian, G' as a module is finitely generated. Therefore, there exists a finite description of the commutant G', despite the fact that it is not always finitely generated as a subgroup. The following theorem is of fundamental importance.

Theorem 5 (G. Baumslag, F. B. Cannonito, and D. J. S. Robinson [11]). There is an algorithm that, given a finitely generated metabelian group G by generating elements and defining relations, finds a finite representation of $\mathbb{Z}[G/G']$-module G'.

Corollary 1. This statement has a number of consequences. There is an algorithm, that:

1) finds the center of $\mathrm{C}(G)$ and its finite representation, an algorithm that finds a finite set of elements whose normal closure in the group coincides with the Fitting subgroup $\mathrm{Fitt}(G)$;
2) determines the presence of nontrivial elements of finite order, which determines the order for a given element, determines all possible finite orders of elements of a group;
3) ascertaining the conjugacy of two sets of group elements (using one of Noskov's lemmas);
4) finding the Frattini subgroup $\mathrm{Fratt}(G)$.

On the whole, this allows us to speak of a satisfactory basic algorithmic theory of finitely generated metabelian groups.

W. Magnus invented his famous Magnus embedding, which became a very efficient instrument in the theory of solvable groups.

The Classical decision problems. Positive solutions:

— WP: P. Hall [81] proved that every finitely generated abelian-by-nilpotent group is residually finite. In particular, finitely generated metabelian groups are always residually finite. Since every finitely generated metabelian group G is finitely presented in $\mathfrak{A}^2$, therefore, the word problem is decidable in G.

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— E. I. Timoshenko presented in [253] a direct algorithm that solves the word problem in an arbitrary finitely generated metabelian group.
— CP: G. A. Noskov [183] proved that the conjugacy problem is decidable in an arbitrary finitely generated metabelian group.
— MP: N. S. Romanovskii [216] proved that the membership problem is decidable in an arbitrary finitely generated metabelian group. In [218], he proved that the membership problem is decidable in an arbitrary abelian-by-nilpotent group.
— M. I. Kargapolov and E. I. Timoshenko [96] proved that in general case a finitely generated metabelian group is not conjugate separable.

5. Solvable groups of arbitrary length

The Classical decision problems. Positive solutions:

— O. Kharlampovich [98]: The WP is decidable in any subvariety of $\mathfrak{N}_2\mathfrak{A}$. (Consequently R. Bieri and R. Strebel [27] proved that every finitely presented group $G \in \mathfrak{N}_2\mathfrak{A}$ is residually finite.)
— C. K. Gupta and N. S. Romanovskii [222]: Any polynilpotent group with a single primitive defining relation has a decidable word problem.

The Classical decision problems. Negative solutions:

V. N. Remeslennikov [207] constructed an example of a group finitely defined in the variety $\mathfrak{A}^5$ with an unsolvable word problem. In addition, a finitely defined in $\mathfrak{A}^4$ group G and a finitely generated subgroup $H \leqslant G$ were given, such that the membership problem with respect to H is unsolvable.

Theorem 6 (O. Kharlampovich [97]). There is a finitely presented solvable group G of class 3 in which WP is undecidable. More exactly, G can be chosen in the centrally-nilpotent of class 2-by-abelian variety $\mathcal{Z}\mathfrak{N}_2\mathfrak{A}$ defined by identity $[[[x_1, x_2], [x_3, x_4]], [x_5, x_6]], y] \equiv 1$ [99]. Thus, WP is unsolvable in the variety $\mathfrak{N}_3\mathfrak{A}$.

O. Kharlampovich demonstrated how results of M. Minsky from recursion theory works in constructing counter examples in the solvable group theory.

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Subsequently, this was proved in a different way by G. Baumslag, D. Gildenhuys, and R. Strebel [13, 14]. They constructed a finitely presented solvable of class 3 group G and a recursive set of words $w_1, \ldots, w_n, \ldots$ in generators of G such that $w_i^p = 1$ with p a prime and $w_i \in C(G)$ for which there is no algorithm to decide if a given w_i equals the identity in G. This group can also be used to show that the IP is undecidable in the finitely presented solvable groups of class 3.

In [27], R. Bieri and R. Strebel constructed for every finitely generated $\mathbb{Z}Q$-module A, where Q is a finitely generated abelian group of torsion-free rank n, a subset of the unit sphere $\mathbb{S}^{n-1} \subseteq \mathbb{R}^n$. This subset is equivalent to the set of equivalence classes $[\nu]$ of valuations (homomorphisms) $\nu : Q \to \mathbb{R}^+$. One can attach to every finitely generated Q-module A the set

$$\Sigma_A = \{[\nu] : A \text{ is finitely generated over } Q_\nu\},$$

where $Q_\nu = \{g \in Q : \nu(g) \geqslant 0\}$. In [27], the relations between geometric properties of Σ_A and algebraic properties of A are investigated. In particular, this invariant determines which metabelian groups are finitely presented. For generalizations of concepts and results of the paper [27] see [28, 26].

In [15], an algorithm is presented which decides for a free metabeliab group (or, more generally, for the wreath product of two free abelian groups) whether the intersection of two finitely generated subgroups is finitely generated or trivial. The existence of an algorithm that solves this question for metabelian groups in general is unknown.

Free solvable groups of finite ranks

The Classical decision problems. Positive and negative solutions:

— M. I. Kargapolov and V. N. Remeslennikov [94] proved that the conjugacy problem is solvable for any free solvable group. V. N. Remeslennikov and V. G. Sokolov [211] established that any free solvable group is conjugacy separable.

— U. U. Umirbaev [261] constructed an example of a group G with undecidable word problem which is finitely presented in a variety of solvable groups $\mathfrak{S}_3$ of class $\geqslant 3$. This group G is defined by the relations from the last commutator subgroup of the corresponduing free solvable group. Early S. A. Agalakov [3] proved that there are a finitely generated not finitely separated subgroups in each non-abelian free solvable group of class $d \geqslant 3$.

The identity problem for the class of solvable groups was solved by Yu. G. Kleiman [104, 105].

Theorem 7 (Yu. G. Kleiman [104, 105]). There exists a finitely based variety of groups $\mathfrak{D} \subseteq \mathfrak{A}^7$ in whose free noncyclic groups the equality problem (hence also the identity problem) is unsolvable. Furthermore, it is possible to find a word $v(x)$ such that there exists no algorithm for determining whether or not an arbitrary identity $u(x) \equiv 1$ follows from $v(x) \equiv 1$.

Fox derivatives

For a given positive integer r and for the free group F_r with basis $\{f_1, \ldots, f_r\}$ the *Fox derivatives* are defined as follows.

For $j = 1, \ldots, r$, the (left) Fox derivative associated with f_j is the linear map $D_j : \mathbb{Z}[F_r] \to \mathbb{Z}[F_r]$ satisfying the conditions

$$D_j(f_j) = 1, D_j(f_i) = 0 \quad \text{for} \quad i \neq j,$$
$$D_j(uv) = D_j(u) + uD_j(v) \quad \text{for all} \quad u, v \in F_r.$$

Obviously, an element $u \in F_r$ is trivial if and only if $D_i(u) = 0$ for all $i = 1, \ldots, r$. Also note that for an arbitrary element g of F_n and every $j = 1, \ldots, n$, $D_j(g^{-1}) = -g^{-1}D_j(g)$. An introduction to the theory of the Fox derivatives and possible applications of them can be found in [239, 256].

The *trivialization* homomorphism $\varepsilon : \mathbb{Z}[F_r] \to \mathbb{Z}$ is defined on the generators of F_n by $f_i\varepsilon = 1$ for all $i = 1, \ldots, r$ and extended linearly to the group ring $\mathbb{Z}F_n$.

The Fox derivatives appear in another setting as well. Let ΔF_r denote the fundamental ideal of the group ring $\mathbb{Z}[F_r]$. It is a free left $\mathbb{Z}[F_r]$-module with a free basis consisting of

$\{f_1 - 1, \ldots, f_r - 1\}$. This it leads us to the following formula which is called the *main identity* for the Fox derivatives:

$$\sum_{i=1}^{r} D_i(\alpha)(f_i - 1) = \alpha - \alpha\varepsilon,$$

where $\alpha \in \mathbb{Z}F_r$. Conversely, if for any element $f \in F_r$ and $\alpha_i \in \mathbb{Z}[F_r]$ we have equality

$$\sum_{i=1}^{r} \alpha_i(f_i - 1) = f - 1,$$

then $D_i(f) = \alpha_i$ for $i = 1, \ldots, r$.

Let $M_r = F_r/F_r''$ be a free metabelian group of rank r and $A_r = M_r/M_r' \simeq F_r/F_r'$ be a free abelian group of rank r. Further, denote by $\pi : M_r \to A_r$, $\pi' : F_r \to A_r$ and $\pi'' : F_r \to M_r$ the canonical epimorphisms. Let $\{a_1, \ldots, a_r\}$ and $\{x_1, \ldots, x_r\}$ be the bases for A_n and M_n obtained by π' and π''. The maps π, π' and π'' can be extended linearly to $\pi : \mathbb{Z}M_r \to \mathbb{Z}A_r$, $\pi' : \mathbb{Z}F_r \to \mathbb{Z}A_r$ and $\pi'' : \mathbb{Z}F_r \to \mathbb{Z}M_r$. The kernels of π' and π'' are the ideals of $\mathbb{Z}F_r$ generated by the elements $u - 1$ with $u \in F_r'$ and $u \in F_r''$, respectively.

For every $j = 1, \ldots, r$ the free Fox derivative D_j induces a linear map $d_j : \mathbb{Z}M_r \to \mathbb{Z}A_r$. These maps also are called the *free Fox derivatives*.

Magnus embedding

One of the most powerful approaches to study free solvable groups is via the *Magnus embedding*. Originally W. Magnus established in [131] an embedding of a group $\bar{G}$ of type F_r/R' into the group $M(G, T_r) = \begin{pmatrix} G & T_r \\ 0 & 1 \end{pmatrix}$, where $G = F_r/R$ is a finite group, and T_r is a free module over $\mathbb{Z}[F_r]$ with basis $\{t_1, \ldots, t_r\}$. This map is called the *Magnus embedding*. The finiteness restriction on G can be easily eliminated (see [77]). Also the Magnus embedding can be naturally extended to $\bar{G} \to M(G, T)$ where $\bar{G}$ is a group of the type F/R', and $G = F/R$. Here $F = F_{|\Lambda|}$ has a basis $\{f_\lambda | \lambda \in \Lambda\}$ of arbitrary cardinality $|\Lambda|$, and the free module $T = T_{|\Lambda|}$ over $\mathbb{Z}G$ has a basis $\{t_\lambda | \lambda \in \Lambda\}$. In the following usually $\Lambda = \{1, \ldots, r\}$, and $F = F_r$.

A. L. Shmel'kin [246] (see [109]) interpreted the Magnus theorem as an embedding β of the group $\bar{G}$ in the wreath product $W = A_r \mathrm{wr} G$ in the following way.

Let

$$\bar{\beta} : F_r \to W$$

be defined by the map

$$\bar{\beta}(f_i) = a_i \cdot \mu(f_i) \text{ for } i = 1, \ldots, r,$$

where $\mu : F_r \to G$ is the canonical epimorphism, and $\{a_1, \ldots, a_r\}$ is the basis of A_r corresponding to the basis $\{f_1, \ldots, f_r\}$ for F_r.

Then by the Magnus theorem, $\ker(\overline{\beta}) = R'$, hence $\overline{\beta}$ induces an embedding $\beta : \overline{G} \to W$.

Recall that W is isomorphic to $M(G, T_r)$. The embedding β above is defined in this setting by the map

$$\beta(\mu'(f_i)) = \begin{pmatrix} \mu(f_i) & t_i \\ 0 & 1 \end{pmatrix},$$

where $\mu' : F_r \to \overline{G}$ is the canonical epimorphism.

Easy to prove that every matrix $A \in \beta(\overline{G})$ has the form

$$A = \begin{pmatrix} \mu(f) & \sum_{i=1}^{r} \mu(D_i(f))\, t_i \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \bar{\mu}(f') & \sum_{i=1}^{r} d_i(f')\, t_i \\ 0 & 1 \end{pmatrix},$$

where $f' = \mu'(f)$ is arbitrary element of $\overline{G}$, $\overline{\mu} : \overline{G} \to G$ is the canonical epimorphism, and d_i are the induced free Fox derivatives with values in $\mathbb{Z}G$.

It turned out (see [8, 211] for metabelian case and [30]) that the group $\overline{G}$ is *well embedded* in $M(G, T_r)$. Namely, the image of $\overline{G}$ in $M(G, T_r)$ under the Magnus embedding can be described as follows.

A matrix

$$A = \begin{pmatrix} 1 & \sum\limits_{i=1}^{r} \alpha_i t_i \\ 0 & 1 \end{pmatrix} \in M(G, T_r)$$

belongs to the image $\beta(\overline{G})$ if and only if

$$\sum_{i=1}^{r} \alpha_i(\mu(f_i) - 1) = 0.$$

Therefore, a matrix

$$A = \begin{pmatrix} g & \sum\limits_{i=1}^{r} \alpha_i t_i \\ 0 & 1 \end{pmatrix} \in M(G, T_r)$$

belongs to the image $\beta(\overline{G})$ if and only if

$$\sum_{i=1}^{r} \alpha_i(\mu(f_i) - 1) = g - 1.$$

6. Equations

Solvability problem for equations in various classes of groups has been actively researched for many years. First general results on equations in groups appeared in the 1960s in the works of R. Lyndon [122 – 124] and A. I. Mal'cev [144]. In the 1970s G. S. Makanin [137, 138] proved the solvability of the systems equations for free monoids and free groups. In recent years, significant progress has been made in the computational complexity and structure of solution sets.

R. C. Lyndon

For a general survey of the current state of the theory of solvability of equations and systems of equations in groups, see the observing paper by the author [232] and his monograph [239].

The *Diophantine problem* in a group G is the task to determine whether or not a given finite system of equations with constants in G has a solution in G. This problem is decidable if there is an algorithm that given a finite system E of equations with constants in G decides whether or not E has a solution in G.

Equations in nilpotent groups

Denote by $\mathcal{N}$ the class of all nilpotent groups. As above $\mathfrak{N}_c$ denotes the variety of all nilpotent groups of class $\leqslant c$. In particular $\mathfrak{N}_1$ coincides with the class $\mathfrak{A}$ of all abelian groups.

A. I. Mal'cev [140] proved that any equation of the form $x^m = g$ where g is an element of a torsion-free nilpotent group $G \in \mathfrak{N}_c$, $m \in \mathbb{N}$, has a solution in some torsion-free nilpotent group $H \in \mathfrak{N}_c$, $H \geqslant G$. Moreover, there is a divisible torsion-free nilpotent group $\tilde{G} \in \mathfrak{N}_c$, least by inclusion, containing the group G. Such a group $\tilde{G}$ is uniquely defined up to isomorphism and is called the *Mal'cev completion* of G.

Clearly every abelian group embeds into a divisible abelian group.

Since every finitely generated nilpotent group $G \in \mathfrak{N}_c$ embeds into a direct product of a torsion-free nilpotent group G_0 and a finite direct product $\prod_p G_p$ of finite p-groups G_p (p are primes) also of class c, every equation of the form $x^m = g$, as above, has a solution in a some nilpotent overgroup H of G. Indeed, we can embed G_0 in a torsion-free complete nilpotent group H_0 keeping the class c by [140], and embed every p-group G_p into a finite p-group H_p containing a solution of the considering equation. Note that we need only to extend G_p with roots of equations of the form $x^{p^k} = g_p$. Then we set $H = H_0 \times \prod_p H_p$. But in general, H has class greater than c. Therefore, any finitely generated nilpotent group can be embedded into a divisible nilpotent group.

A system of $m = k$ equations is called *unimodular* if the matrix consisting of the sums of exponentials of the unknowns with which they enter the equations, has determinant 1.

A. L. Shmel'kin [247] established that any unimodular system of equations over a nilpotent group G has an unique solution in G.

Theorem 8 (V. A. Roman'kov [225, 226]). The following statements hold:

— Let $N_{r,c}$ be a free nilpotent group of rank $r \geqslant 2$ and class $c \geqslant 9$ with basis $\{x_1, \ldots, x_r\}$. Then there is an algorithm which for every Diophantine equation $D(\zeta_1, \ldots, \zeta_n) = 0$ gives a split equation $g(z_1, \ldots, z_p) = f$ over the group $N_{r,c}$ that has a solution in $N_{r,c}$ if and only if $D(\zeta_1, \ldots, \zeta_n) = 0$ has a solution in integers. An element f can be chosen in the subgroup $\mathrm{gp}(x_1, x_2)$ of $N_{r,c}$.

— Let M_r be the free metabelian group of rank $r \geqslant 2$ with basis $\{x_1, \ldots, x_r\}$. Then there is an algorithm which for every Diophantine equation $D(\zeta_1, \ldots, \zeta_n) = 0$ gives a split equation $g(z_1, \ldots, x_q) = f$ over group M_r that has a solution in M_r if and only if $D(\zeta_1, \ldots, \zeta_n) = 0$ has a solution in integers. An element f can be chosen in subgroup $\mathrm{gp}(x_1, x_2)$ of M_r.

Therefore, the equation problem for any free nilpotent group $N_{r,c}$, $r \geqslant 2$, $c \geqslant 9$, or free metabelian group M_r, $r \geqslant 2$, is algorithmically undecidable.

The method of interpretation of Diophantine equations in free nilpotent and free metabelian groups has been used in a row forthcoming papers. N. N. Repin applied this method for studying the solvability of equations in nilpotent groups.

We record a number results by N. N. Repin on recognizing the solvability of equations in nilpotent groups, see [213, 214]:

— For every finitely generated nilpotent group of class two the problem of recognizing the solvability of one-variable equations is decidable.

— There is a finitely generated nilpotent group of class 3 in which the problem of recognizing the solvability of one-variable equations is undecidable.

— For every free nilpotent group $N_{r,c}$ of rank $r \geqslant 600$ and class $c \geqslant 3$ the problem of recognizing the solvability of equations is udecidable.

— For every free nilpotent group $N_{r,c}$ of rank $r \geqslant 2$ and class $c \geqslant 5 \cdot 10^{10}$ the problem of recognizing the solvability of one-variable equations is undecidable.

In another setting, the interpretation of Diophantine equations was used by Yu. G. Kleiman to prove that the identity problem is undecidable for some relatively free solvable groups (see Theorem 7 above).

Theorem 9 (V. A. Roman'kov [236]). For every Diophantine polynomial $D(\zeta_1, \ldots, \zeta_n)$ there exists a finitely generated nilpotent group G of class 2 with the following property. For every equation of the form $D(\zeta_1, \ldots, \zeta_n) = c$, $c \in \mathbb{Z}$, there is an element $u = u(c) \in G$ such that u is a commutator in G (in other words, the equation $[x, y] = u$ is solvable in G), if and only if the equation $D(\zeta_1, \ldots, \zeta_n) = c$ is decidable over $\mathbb{Z}$. The group G and each element $u(c)$ can be effectively constructed. By the famous Matijasevich's theorem there is a Diophantine polynomial D for which the equation problem is undecidable for the class of equations of the form $\{D = c : c \in \mathbb{Z}\}$. Therefore, the commutator problem is undecidable for G.

Moreover, G is the first example of a finitely generated nilpotent group with undecidable equation problem for the class of quadratic equations. In [236], a finitely generated nilpotent group H of class 2 is also presented for which the endomorphism problem is undecidable. It also has been proved that the retract problem (i.e., question whether the given finitely generated subgroup is a retract of the whole group) is undecidable for the class of finitely generated 2-step nilpotent groups. On the other hand, there is an algorithm which for a given element $u \in N_{r,2}$ determines whether or not u is a commutator.

A. G. Makanin proved in [136] that every split equation $w(x_1, \ldots, x_k) = g$, $g \in G$, over a finitely generated torsion-free nilpotent group G, where $w(x_1, \ldots, x_k)$ does not belong to the derived subgroup $F(X)'$, i.e., $w(x_1, \ldots, x_k)$ is a *non-commutator* word, is finitely approximable.

In [49], M. Duchin et al. show that there exists an algorithm to decide any single equation in the Heisenberg group. The method works for all nilpotent groups of class 2 with rank-one derived subgroup, which includes the higher Heisenberg groups.

Equations in metabelian case

The metabelian Baumslag — Solitar groups are defined by one-relator presentations $\mathrm{BS}(1, k) = \langle a, b | b^{-1}ab = a^k \rangle$, where $k \in \mathbb{N}$. If $k = 1$, then $\mathrm{BS}(1, 1)$ is free abelian of rank 2, so the Diophantine problem in this group is decidable (it reduces to solving finite systems of linear equations over the ring of integers $\mathbb{Z}$).

O. Kharlampovich, L. Lopéz and A. Myasnikov proved in [100]) that the Diophantine problem is decidable in $G = Awr\mathbb{Z}$, where A is a finitely generated abelian group. Equations in the Baumslag — Solitar group $\mathrm{BS}(1, k)$ are also decidable.

I. Lysenok and A. Ushakov [126] proved that the equation problem for spherical quadratic equations in free metabelian groups is solvable and, moreover, NP-complete. E. I. Timoshenko [258] proved the first (solvability) result by using the Magnus embedding.

By the *spherical quadratic equation* over group G with unknowns $X = \{x_1, \ldots, x_t, \ldots\}$ one means an equation of the form

$$\prod_{i=1}^{n} x_i^{-1} c_1 x_i = 1, \quad c_i \in G.$$

V. N. Remeslennikov and N. S. Romanovskii [209] study into algebraic geometry over a non-commutative u-group G, that is, a finitely generated metabelian group whose universal theory is the same as is one for a free metabelian group of rank at least two. They present the construction for a u-product $G_{1,2} = G_1 \circ G_2$ of two u-groups G_1 and G_2, and prove that $G_{1,2}$ is also a u-group and that every u-group, which contains G_1 and G_2 and is generated by these, is a homomorphic image of $G_{1,2}$. They prove that the coordinate group of an affine space G^n is equal to $G \circ M_n$. In [209] irreducible algebraic sets in G are treated for the case where G is a free metabelian group or wreath product of two free abelian groups of finite ranks.

Interpretation of Diophantine equations

The author [225, 226] derived the undecidability of EqP and EndoP in the classes of free nilpotent and free metabelian groups. He based on the famous results by Yu. V. Matijasevich on undecidability of the Diophantine problem [148, 149].

He proved that if $N = N_{r,c}$ is the free nilpotent group of sufficiently large rank r and class $c \geqslant 9$, then for any Diophantine polynomial $D(z_1, \ldots, z_n) \in \mathbb{Z}[z_1, \ldots, z_n]$, we can effectively construct two elements $g, f \in N$ such, that there is an endomorphism $\varphi \in \mathrm{End}(N)$ that $\varphi(g) = f$ if and only if the equation $D(z_1, \ldots, z_n) = c$, where $c \in \mathbb{Z}$, has a solution in $\mathbb{Z}$. Moreover, if such φ exists, we can effectively find it if and only if we can effectively solve the corresponding Diophantine equation. We can also fix the left side of equation $D(z_1, \ldots, z_n) = c$ to obtain non-decidable class of Diophantine equations. Hence, we can fix the element g above to obtain non-decidability of the EqP and EndoP in N. The second element f we choose in a specific cyclic subgroup.

By definition, the *relation matrix* $M(G)$ of the presentation $G = \langle x_1, \ldots, x_n : r_1, \ldots, r_m \rangle$ is an integral $m \times n$ matrix whose ij-th entry is the sum of the exponents of the x_j's that occur in r_i. Recall, that a matrix is said to have *full rank* if its rank equals the largest possible for a matrix of the same dimensions, which is the lesser of the number of rows and columns.

The authors of [64] study metabelian groups G given by a full rank finite presentations $\langle x_1, \ldots, x_n : r_1, \ldots, r_m; \mathfrak{A}^2 \rangle$ in the variety $\mathfrak{A}^2$. They prove that G is a product of a free metabelian subgroup of rank $\max(0, n-m)$ and a virtually abelian normal subgroup, and that if $m \leqslant n-2$, then the Diophantine problem for G is undecidable, while it is decidable if $m \geqslant n$. They also prove that if $m \leqslant n-1$, then, in any direct decomposition of G, all factors, except one, are virtually abelian. Since finite presentations have full rank asymptotically almost surely, metabelian groups finitely presented in the variety of metabelian groups satisfy all the aforementioned properties asymptotically almost surely.

7. Post correspondence problem

Different versions of PCP

There are the bounded versions of PCP. We consider two sorts of a bound: the bound on the solution length n ($\mathrm{BPCP}_{sl}(n)$) and the bound on the size s ($\mathrm{BPCP}_s(s)$).

The following statements are true:

— BPCP_{sl} is NP-complete [62];
— $\mathrm{BPCP}_s(2)$ is decidable [51] (but it remains unknown whether the PCP is solvable for 3–6 pairs of words);
— $\mathrm{BPCP}_s(l)$ for $l \geqslant 7$ is undecidable [150, 151].

Let A be an algebraic system and let $F(A)$ be a free algebraic system in the variety $\mathrm{Var}(A)$ generated by A. For two arbitrary homomorphisms $\varphi, \psi \in \mathrm{Hom}(F(A), A)$ the subset

$$\mathrm{Eq}_A(\varphi, \psi) = \{a \in F(A) : \varphi(a) = \psi(a)\}$$

of $F(A)$ is said to be the *equalizer* of φ and ψ, that is obviously a subsystem of $F(A)$.

The following problem arises: $\mathrm{PCP}(A)$: $\mathrm{Eq}_A(\varphi, \psi) \neq 0$?

The Post correspondence and related problems for groups

Further we will talk only about groups.

Let $\bar{G}, G$ be a pair of groups, and let $\varphi, \psi \in \mathrm{Hom}(\bar{G}, G)$ be a pair of homomorphisms. We denote by

$$\mathrm{Eq}_G(\varphi, \psi) = \{g \in \bar{G} : \varphi(g) = \psi(g)\}$$

the equalizer of φ and ψ, that is obviously a subgroup of $\bar{G}$.

Equalization presentation problem (EPP): Let $\mathcal{C}$ be a class of finitely generated groups. We say that EPP is decidable in $\mathcal{C}$ if for any pair of groups $\bar{G}, G \in \mathcal{C}$ and any pair of homomorphisms $\varphi, \psi \in \mathrm{Hom}(\bar{G}, G)$ we can find effectively a presentation of $\mathrm{Eq}_G(\varphi, \psi)$.

A form of presentation depends of $\mathcal{C}$. It should be explicit for $\mathcal{C}$. In particular case, when $\bar{G} = G$, $\varphi, \psi \in \mathrm{End}(G)$, we get EPP for G.

Equalization problem (EP): We say that EP is decidable in the class $\mathcal{C}$ if for any pair of groups $\bar{G}, G \in \mathcal{C}$ and any pair of homomorphisms $\varphi, \psi \in \mathrm{Hom}(\bar{G}, G)$ there is an algorithm that determines non-triviality of $\mathrm{Eq}_G(\varphi, \psi)$.

In particular case, when $\bar{G} = G$, $\varphi, \psi \in \mathrm{End}(G)$, we consider EP for G.

Let we formulate the Post correspondence problem PCP(G) for a group G in the corresponding variety. Namely, when $\bar{G} = F(\mathrm{Var}(G))$ be a relatively free group in the variety $\mathrm{Var}(G)$, generated by G, we get PCP for G.

PCP(G): $\mathrm{Eq}_G(\varphi, \psi) \neq 1$?

Generalized equalization problem (GEP). Also, we say that GEP is decidable in the class $\mathcal{C}$ if for any pair of groups $\bar{G}, G \in \mathcal{C}$, any pair of homomorphisms $\varphi, \psi \in \mathrm{Hom}(\bar{G}, G)$ and given a nontrivial element $v \in G$ we can decide effectively whether there is $g \in \bar{G}$ such that

$$\varphi(g) = v \cdot \psi(g).$$

We consider it as an equation with unknown element $g \in \bar{G}$.

In particular case, when $\bar{G} = G$, $\varphi \in \mathrm{End}(G)$, $\psi = id$, we get TCP for G. If $\psi \in \mathrm{End}(G)$, then we get BTCP for G.

The generalized Post correspondence problem (GPCP)

When $\bar{G} = F(\mathrm{Var}(G))$ is a relatively free group in the variety $\mathrm{Var}(G)$, generated by G, we get GPCP for G.

GPCP(G): Given a finite sequence of instances $(g_1, h_1), \ldots, (g_s, h_s)$ and element f in G, determine if there is a word $w = w(x_1, \ldots, x_s)$ such that

$$w(g_1, \ldots, g_s) = f \cdot w(h_1, \ldots, h_s).$$

This problem admits the following equivalent formulation. Let $F_s(G)$ be a free group of rank s in $\mathrm{Var}(G)$.

GPCP(G): Given a pair $\varphi, \psi \in \mathrm{Hom}(F_s(G), G)$, decide if the solution $w \in F_s(G)$ exists or not of the equation

$$\varphi(w) = f \cdot \psi(w).$$

Now we formulate the hereditary word problem (HWP(G)) in a group G. The following problem is the strongest form of the word problem in G:

HWP(G): Given a finite set $R \cup \{f\}$ of words in generators of G, decide whether or not f is trivial in the quotient $G/ncl(R)$.

GPCP can be decidable only in a group with decidable HWP.

The following results are proved in [162]. Let G be a finitely generated group. Then:

— HWP(G)) P-time reduces to GPCP(G).
— If G contains F_2 then GPCP(G) is undecidable.

In [163] the classical knapsack and subset sum problems to arbitrary groups are introduced. The computational complexity of these new problems were studied. It was shown that these problems, as well as the bounded submonoid membership problem, are P-time decidable in hyperbolic groups and give various examples of finitely presented groups where the subset sum problem is NP-complete.

These problems for a group G are formulated as follows:

— Knapsack problem (KP): Given $g_1, \ldots, g_k, g \in G$, decide if

$$g = \prod_{i=1}^{k} (g_i)^{\mu_i}$$

for some non-negative integers $\mu_1, \ldots, \mu_k$.

— The subset sum problem (SSP): Given $g_1, \ldots, g_k, g \in G$, decide if

$$g = \prod_{i=1}^{k} (g_i)^{\epsilon_i}$$

for some $\epsilon_1, \ldots, \epsilon_k \in \{\pm 1\}$.

— Bounded submonoid membership problem (BSMP): Given $g_1, \ldots, g_k, g \in G$ and $1^m \in \mathbb{N}$ (in unary), decide if g is equal in G to a product of the form $g = \prod_{j=1}^{s} g_j$, where $i_j \in \in \{1, \ldots, k\}$ and $s \leqslant m$.

Let G be a finitely generated virtually nilpotent group. Then SSP(G) and BSMP(G), as well as their search and optimization (with respect to number of factors) variations, are in P [163]. Every polycyclic non-virtually-nilpotent group has NP-complete subset sum problem [181].

In [164], a number of algorithmic problems in groups were introduced and studied, modeled after the classical computational lattice problems. Polynomial time solutions for a nilpotent group have been given to problems such as finding a subgroup element closest to a given group element, or finding the shortest nontrivial subgroup element.

Twisted conjugacy problem

Originally the twisted conjugacy problem was posed as following:

Let G be a group, and $u, w \in G$. Given an endomorphism $\xi \in \mathrm{End}(G)$, one says that u and w are *ξ-twisted conjugated*, denoted by $u \sim_\xi w$, if and only if there exists $g \in G$ such that $u = \xi(g)^{-1} \cdot wg$, or equivalently $\xi(g)\, u = wg$. So it is a question if following equation have a solution g in G:

$$\xi(g)\, u = wg.$$

The question about ξ-twisted conjugacy of given elements $u, w \in G$ can be reduced to case where one of the elements is trivial. To do it we change ξ to $\varphi = \xi \circ \sigma_u$, where $\sigma_u : g \mapsto u^{-1}gu, g \in G$, is an inner automorphism. We get

$$\varphi(g) = vg$$

for $v = u^{-1}w$.

We consider finitely generated metabelian and polycyclic groups. We have the following two equations:

$$\varphi(g) = \psi(g)$$

and

$$\varphi(g) = u\psi(g).$$

Following [11], call a subgroup H of a finitely generated metabelian group M *nearly normal* if the intersection $H \cap M'$ is a normal subgroup of M.

Then

$$H = \mathrm{gp}(h_1, \ldots, h_k, \{v_1, \ldots, v_l\}^{\mathbb{Z}[M/M']}$$

is a finite description of H.

The following results were presented in the talk of the author [234] (see also [233]).

The first assertion shows that the question of the decidability of GEP ($\varphi(g) = f\psi(g)$) under certain assumptions can be transformed into a similar question about the subgroup H of a finite index in G:

— Let G be any group and let $H \leqslant G$ be any subgroup of a finite index in G. Let $\bar{G}$ be a group and $\varphi, \psi \in \mathrm{Hom}(\bar{G}, G)$. Suppose that the membership problem in H is decidable for G. Then, if GEP is decidable for H, it is also decidable for G.

Let us present the main technical results for obtaining solutions of the problems under consideration in the class of all finitely generated metabelian groups:

— Let G be a group and A be its abelian normal subgroup. Let $\bar{G} = \mathrm{gp}(f_1, \ldots, f_n)$ be a finitely generated group and let $\varphi, \psi : \bar{G} \to G$ be a pair of homomorphisms such that $G = \mathrm{gp}(\varphi(\bar{G}), \psi(\bar{G}))$. For every $g \in \bar{G}$ denote $a(g) = \varphi(g)(\psi(g))^{-1}$. In particular, denote $a_i = a(f_i)$, $i = 1, \ldots, n$.
Suppose that the following assumptions are true:
1) for every $g \in \bar{G}$ one has $a(g) \in A$;
2) the derived subgroup G' acts identically on A, i.e., $[G', A] = 1$;
3) $\bar{G}' \leqslant \mathrm{Eq}_G(\varphi, \psi)$, i.e., for every $g \in \bar{G}'$ one has $\varphi(g) = \psi(g)$.
Then
$$\mathrm{Eq}_G(\varphi, \psi) \leqslant \psi^{-1}(\mathrm{C}_G(a_1, \ldots, a_n)),$$
where $\mathrm{C}_G(a_1, \ldots , a_n)$ is the centralizer of the elements $a_1, \ldots, a_n$ in G.
Moreover, for every $g \in \psi^{-1}(\mathrm{C}_G(a_1, \ldots, a_n))$ one has $a(g) \in \zeta_1 G$;
4) hence, if the center $\zeta_1(G)$ of G is trivial, then
$$\mathrm{Eq}_G(\varphi, \psi) = \psi^{-1}(\mathrm{C}_G(a_1, \ldots, a_n)).$$
In general case there is a homomorphism
$$\rho : \psi^{-1}(\mathrm{C}_G(a_1, \ldots, a_n)) \to \zeta_1 G, g \mapsto a(g),$$
and
$$\mathrm{Eq}_G(\varphi, \psi) = \ker(\rho).$$
Let G be a group and A be its abelian normal subgroup. Let $\bar{G}$ be a group and let $\varphi, \psi : \bar{G} \to G$ be a pair of homomorphisms such that $G = \mathrm{gp}(\varphi(\bar{G}), \psi(\bar{G}))$. Let $a(g) = \varphi(g)(\psi(g))^{-1}$. Suppose that the following assumptions are true:
1) for every $g \in \bar{G}$ one has $a(g) \in A$;
2) the derived subgroup G' acts identically on A, i.e., $[G', A] = 1$.
Then
$$a(\bar{G}') = \{a(g) : g \in \bar{G}'\}$$
is a normal subgroup of G.
Moreover, if $\bar{G}'$ is generated as a normal subgroup by a set of elements $\{u_i : i \in I\}$, then $a(\bar{G}')$ is generated as a normal subgroup by the set $\{a(u_i) : i \in I\}$.

Let all the previous notation and assumptions be satisfied. Let $G_1 = G/a(\bar{G})$, and $G \to G_1$ be the standard homomorphism. For simplicity, we do not change the notation for the compositions φ and ψ with this standard homomorphism. Also, we do not change the designation of the images of elements of G in G_1.

Then

$$\mathrm{Eq}_{G_1}(\varphi, \psi) \leqslant \psi^{-1}(\mathrm{C}_{G_1}(a_1, \ldots, a_n)).$$

Moreover, if $\zeta_1 G_1 = 1$, then

$$\mathrm{Eq}_{G_1}(\varphi, \psi) = \psi^{-1}(\mathrm{C}_{G_1}(a_1, \ldots, a_n)).$$

Main results for metabelian and polycyclic groups

Theorem 10. Let M be a finitely generated metabelian group, and let $\bar{M}$ be a finitely generated metabelian group with generating set $\{f_1, \ldots, f_n\}$. Let N be an abelian normal subgroup of M, containing M'.

Then EPP and EP are solvable for any pair of homomorphisms $\varphi, \psi \in \mathrm{Hom}(\bar{M}, M)$ satisfying the assumption that, for any $g \in \bar{M}$, $g\varphi(g\psi)^{-1} \in N$. Equalizer $\mathrm{Eq}_M(\varphi, \psi)$ is described as a nearly normal subgroup of M, i.e.,

$$\mathrm{Eq}_M(\varphi, \psi) = \mathrm{gp}\left(h_1, \ldots, h_k, \{v_1, \ldots, v_l\}^{\mathbb{Z}M/N}\right),$$

where $h_1, \ldots, h_k, v_1, \ldots, v_l$ are given by the algorithm.

Corollary 2. Let M be a finitely generated metabelian group, and let $\bar{M} = F(\mathrm{Var}(M))$ be a relatively free metabelian group in the variety $\mathrm{Var}(M)$ generated by M with basis $\{f_1, \ldots, f_n\}$, $n \geqslant 2$.

Then PCP is solvable for any pair of instances $\bar{c} = (c_1, \ldots, c_n)$ and $\bar{d} = (d_1, \ldots, d_n)$ such that for the corresponding homomorphisms $\varphi : f_i \to c_i$ and $\psi : f_i \to d_i$, respectively, one has $a_i = \varphi(f_i)(\psi(f_i))^{-1} \in N$, $i = 1, \ldots, n$.

Theorem 11.

1. Let M be a metabelian polycyclic group, and let $\bar{M}$ be a metabelian polycyclic group with generating set $\{f_1, \ldots, f_n\}$, $n \geqslant 2$. Then EPP and EP are decidable for any pair of homomorphisms φ, ψ of $\bar{M}$ to M.
2. Let M be a metabelian polycyclic group, and let $\bar{M} = F(\mathrm{Var}(M))$ be a relatively free group in the variety $\mathrm{Var}(M)$ generated by M with basis $\{f_1, \ldots, f_n\}$, $n \geqslant 2$. Then PCP_n is decidable for any pair of instances $\bar{c} = (c_1, \ldots, c_n), \bar{d} = (d_1, \ldots, d_n) \in \bar{M}^n$.
3. Let M be a finitely generated metabelian group, and let $\bar{M}$ be a finitely generated metabelian group generated by $f_1, \ldots, f_n$, $n \geqslant 2$. Let N be an abelian normal subgroup of M, that contains M'. Then GEP is solvable for any pair of homomorphisms $\varphi, \psi \in \mathrm{Hom}(\bar{M}, M)$ and any element $a \in N$ such that for any $g \in \bar{M}$ one has $\varphi(g)(\psi(g))^{-1} = a(g) \in N$.

Corollary 3.

1. Let M be a finitely generated metabelian group and N a normal abelian subgroup of M containing M'. Let φ, ψ be a pair of endomorphisms in $\mathrm{End}(M)$ such that, for each $g \in M$, $\varphi(g)(\psi(g))^{-1} \in N$. Then the bi-twisted conjugacy problem is solvable for φ, ψ.
   In particular, the bi-twisted conjugacy problem is solvable for any pair of endomorphisms $\varphi, \psi \in \mathrm{End}(M)$, each of which induces an identical map onto M/M'. This generalizes the main result of paper [264], where φ induces an identical map onto M/M' and $\psi = id$.
2. Let M be a finitely generated metabelian group, and let $\bar{M} = F(\mathrm{Var}(M))$ be a relatively free metabelian group in the variety $\mathrm{Var}(M)$ with basis $\{f_1, \ldots, f_n\}$, $n \geqslant 2$. Let N be an abelian normal subgroup of M, that contains M'.

Then GPCP_n is decidable for every pair of instances $\bar{c} = (c_1, \ldots, c_n), \bar{d} = (d_1, \ldots, d_n) \in M^n$ such that for the corresponding to these instances homomorphisms $\varphi, \psi \in \mathrm{Hom}(\bar{M}, M)$ and every element $g \in \bar{M}$ we have $\varphi(g)(\psi(g))^{-1} = a(g) \in N$.

Theorem 12. Let M be a metabelian polycyclic group. Let N be an abelian normal subgroup of M, that contains M'. Let $\bar{M}$ be a metabelian polycyclic group generated by $f_1, \ldots, f_n$, $n \geqslant 2$. Then GEP is decidable for every pair of homomorphisms φ and ψ of $\bar{M}$ to M.

Corollary 4. Let M be a polycyclic metabelian group and N be a normal abelian subgroup of M containing M'. Then the bi-twisted conjugacy problem is solvable for any pair of endomorphisms φ, ψ of M. Thus, the bi-twisted conjugacy problem is solvable for M. This generalize the result [264] where φ is arbitrary endomorphism and $\psi = id$.

Theorem 13. Let M be a metabelian polycyclic group, and let $\bar{M} = F(\mathrm{Var}(M))$ be a relatively free group in $\mathrm{Var}(M)$, $n \geqslant 2$. Then GPCP_n is decidable for any pair of instances $\bar{c} = (c_1, \ldots, c_n), \bar{d} = (d_1, \ldots, d_n) \in \bar{M}^n$.

Theorem 14. All the problems just considered are solvable in the class of polycyclic groups.

8. Elementary and universal theories

The elementary theory $\mathrm{Th}(G)$ of a group G (or a ring, or an arbitrary structure) in a language L is the set of all first-order sentences in L that are true in G.

We restrict ourselves to considering only the group-theoretical case. Usually L is the standard group-theoretic language $\langle \cdot, ^{-1}, =, 1\rangle$. Sometimes L includes predicates or other than 1 constants. If the group A is elementarily equivalent to the group B, i.e., if $\mathrm{Th}(A) = \mathrm{Th}(B)$, then we write $A \equiv B$.

One of the main results of W. Szmielew [249] is the determination of group theoretic invariants $I(A)$ which characterize abelian groups A up to elementary equivalence. The decidability of the theory of abelian groups follows relatively easily from this result: $A \equiv B \leftrightarrow I(A) = I(B)$. More exactly, $\mathrm{Th}(A)$ is decidable if the sequence of Szmielew invariants of A is computable. Finitely generated abelian groups have decidable elementary theories. This assertion easily carries over to their finite extensions, i.e., almost abelian finitely generated groups. Two finitely generated abelian groups are elementary equivalent if and only if they are isomorphic, that is, $A \equiv B \leftrightarrow A \simeq B$.

A compretiensive survey of the first-order properties of abelian groups is given by P. C. Eklof and E. R. Fisher in [54]. Their principal method is the investigation of saturated abelian groups. They gave a new model-theoretic proof results of Szmielew and obtained new results on the existence of saturated models of complete theories of abelian groups. It turned out that elementarily equivalent saturated abelian groups of the same cardinality are isomorphic.

There are several main results on elementary theories of nilpotent groups. Examples of finitely generated nilpotent groups with undecidable elementary theories were first given by A. I. Mal'cev. In his pioneering paper [146], A. I. Mal'cev showed that the ring R with unity can be defined by first-order formulas in the group $\mathrm{UT}_3(R)$ of unitriangular matrices over R (considered as an abstract group). In particular, the ring of integers $\mathbb{Z}$ is definable in the group $\mathrm{UT}_3(\mathbb{Z})$, which is a free nilpotent of rank 2 and class 2. Yu. L. Ershov [57] proved that the group $\mathrm{UT}_3((Z))$ (hence the ring $\mathbb{Z}$) is definable in any finitely generated nilpotent group G, which is not virtually abelian. Therefore, the elementary theory of G is undecidable (see more general statement of theorem 15 below).

In [143], A. I. Mal'cev proved that the elementary theory of any free solvable group $S_{r,d}$ of rank $r \geqslant 2$ and class $d \geqslant 2$ is undecidable. All members of the derived series are definable in $S_{r,d}$. In [145], he established fundamental results on linear groups. In particular, he proved the following theorem: Let $G =$GL (or PGL, SL, PSL), let $n, m \geqslant 3$, and let K and L be commutative rings of characteristic zero, then $\mathrm{GL}_m(K) \equiv \mathrm{GL}_n(L)$ if and only if $m = n$ and $K \equiv L$. In the case of GL and PGL the result holds for $n, m \geqslant 2$.

In [108], M. I. Kargapolov posed the following Question 1.26: Does elementary equivalence of two finitely generated nilpotent groups imply that they are isomorphic?

In [268], B. I. Zil'ber constructed an example of two finitely generated nilpotent of class 2 groups that are elementary equivalent but nonisomorphic.

M. I. Kargapolov was the initiator of many studies on solvable groups and algorithms

A. G. Myasnikov in the series of papers [158, 160, 161] studied the elementary theories of bilinear mappings. In particular, he gave a description of abstract isomorphisms of bilinear mappings.

If G is torsion free finitely generated nilpotent group and R is binomial domain, then G^R means the P. Hall R-completion of G.

In the papers [164–166] A. G. Myasnikov and V. N. Remeslennikov proved that the Kargapolov's conjecture holds "essentially" true in the class of nilpotent $\mathbb{Q}$-groups (i.e., divisible torsion-free nilpotent groups). Indeed, it turned out that two such groups G and H are elementarily equivalent if their cores $\tilde{G}$ and $\tilde{H}$ are isomorphic and G and H either simultaneously coincide with their cores or they do not. Here the *core* of G is uniquely defined as a subgroup $\tilde{G} \leqslant G$ such that $\mathrm{C}(\tilde{G}) \leqslant \tilde{G}'$ and $G = \tilde{G} \times G_0$, for some abelian $\mathbb{Q}$-group G_0. Developing this approach, A. G. Myasnikov described in [157, 159] all groups elementarily equivalent to a given finitely generated nilpotent K-group G over an arbitrary field K of characteristic zero.

In a series of papers [22–24] O. V. Belegradek completely characterized groups which are elementarily equivalent to a unitriangular matrix group $\mathrm{UT}_n(\mathbb{Z})$ for $n \geqslant 3$. In particular, he showed in [23, 24] that there are groups elementarily equivalent to $\mathrm{UT}_n(\mathbb{Z})$ which are not isomorphic to any group of the type $\mathrm{UT}_n(R)$ as above (he called them *quasi-unitriangular groups*).

A. G. Myasnikov

The paper [174] gives a complete algebraic description of the groups G that are elementarily equivalent to the P. Hall completion N^R of a given free nilpotent group N of finite rank over an arbitrary binomial domain R. In particular, all groups elementarily equivalent to a free nilpotent group N of finite rank are characterized. F. Oger [189] studied special circumstances under which elementary equivalence of two finitely generated finite-by-nilpotent groups implies isomorphism. Finally, F. Oger showed in [190] that two finitely generated nilpotent groups G and H are elementarily equivalent if and only if they are essentially isomorphic, i.e., $G \times \mathbb{Z} \simeq H \times \mathbb{Z}$. However, the full classification problem for finitely generated nilpotent groups is currently wide open.

A *universal formula* is a formula which can be written $\forall x_1 \ldots \forall x_n \Phi(x_1, \ldots, x_n)$ for some quantifier free formula $\Phi(x_1, \ldots, x_n)$. If it has no free variables, a universal formula is called a *universal sentence*. The universal theory $\mathrm{Th}_\forall(G)$ of a group G is the set of universal sentences satisfied by G. If the group G is universally equivalent to the group H, i.e., if $\mathrm{Th}_\forall(G) = \mathrm{Th}_\forall(H)$, then we write $G \equiv_\forall H$.

Similarly, one can define existential formulae and sentences, and the existential theory $\mathrm{Th}_\exists(G)$ of a group G. Note that two groups which have the same universal theory also have the same existential theory since the negation of a universal sentence is equivalent to an existential statement.

It is well known that two nontrivial free abelian groups are universally equivalent. E. I. Timoshenko [252] established that any two free solvable groups $S_{r,d}$ and $S_{q,d}$ of the same length $d \geqslant 1$ and $r, q \geqslant 2$ (in the case $d = 1$, $r, q \geqslant 1$) are universally equivalent ($S_{r,d} \equiv_\forall S_{qd}$). This result has been independently proved in [61]. In [252], E. I. Timoshenko also proved that any two free nilpotent groups $N_{r,c}$ and $N_{q,c}$ where $r \neq q$, of the same class $c \geqslant 2$ are universally equivalent if and only if the following conditions are satisfied: $r, q \geqslant c - 1$ for $c \geqslant 3$; and $r, q \geqslant 2$ for $c = 2$.

E.I. Timoshenko

The first example of a finitely generated nilpotent group G, whose universal theory $\mathrm{Th}_\forall(G)$ is undecidable, was constructed by the author in [227]. This group G is a torsion-free metabelian group of the nilpotency class 4 with 6 generators.

In [251], E. I. Timoshenko considers the problem of preserving elementary and universal equivalence under wreath products. His result is as follows. If the group A is elementarily equivalent to the group B, and K is a finite group, then the wreath product $G = AwrK$ is elementarily equivalent to $H = BwrK$. Universal equivalence is preserved under wreath products, that is $A_1 \equiv_\forall A_2, B_1 \equiv_\forall B_2 \to A_1wrB_1 \equiv_\forall A_2wrB_2$, but elementary equivalence (in the general case) is not, that is $A_1 \equiv A_2, B_1 \equiv B_2 \not\to A_1wrB_1 \equiv A_2wrB_2$ [251].

In [35], O. Chapuis proved his remarkable result: The elementary theory of any free metabelian group is decidable. An explicit description of this theory is given by him in [36]. He also proved that a noncyclic free metabelian group is universally equivalent to the wreath product of any two nontrivial torsion-free abelian groups.

V. Remeslennikov and R. Stöhr [212] characterizated the finitely generated groups in the quasivariety generated by a noncyclic free metabelian group from three different points of view: In terms of wreath products, in terms of module theoretic properties of their Fitting subgroups, and in terms of quasi-identities.

In [37], O. Chapuis proved that the terms of the derived series of a free solvable group are definable by existential formulae. He used this result to prove that if Hilbert's 10th problem has a negative answer for the field of the rationals, then the universal theory of a noncyclic free solvable group of class $\geqslant 3$ is undecidable. N. S. Romanovskii [221] proved that a free solvable group of derived length at least 4 has an algorithmically undecidable universal theory.

E. I. Timoshenko [257] proved that the universal theory of a free polynilpotent group $\mathfrak{N}_{c_1} \cdots \mathfrak{N}_{c_s}$, $s \geqslant 2$, $c_i \geqslant 1$, for $i = 1, \ldots, s-1$, $c_s \geqslant 2$, is undecidable.

The following result has been proved in [254]. Let $F(\mathfrak{V})$ be a free group of a variety $\mathfrak{V}$, approximable by finite p-groups for an infinite sequeuce of primes p. If the subgroup G of $F(\mathfrak{V})$ generates the same variety as $F(\mathfrak{V})$, then $G \equiv_\forall F(\mathfrak{V})$.

In [112], an algebraic characterization of elementary equivalence for polycyclic-by-finite groups was established. This characterization allowed to give the relations between their elementary equivalence and the elementary equivalence of the factors in their decompositions in direct products of indecomposable groups. In particular, it has been proved that the elementary equivalence of two such groups $G \equiv H$ is equivalent to each of the following properties: (1) $G \times \cdots \times G$ (k times G) for an integer $k \geqslant 1$; (2) $A \times G \equiv B \times H$

for two polycyclic-by-finite groups A, B such that $A \equiv B$. It is not presently known if (1) implies $G \equiv H$ for any groups G, H.

N. S. Romanovskii and E. I. Timoshenko found in [223] conditions for the universal equivalence of the metabelian group G with few relations to the free metabelian group M_r of rank r. They also proved that if an n-generated solvable group G is elementarily equivalent to a free solvable group S_{rd} of rank r and derived length d, then for $d = 2$ or $d > 2$ and $n = r$, the groups G and $S_{r,d}$ are isomorphic. In [260], E. I. Timoshenko studies elementary and universal theories of relatively free solvable groups in a group signature expanded by one predicate distinguishing primitive or annihilating systems of elements. In [259], he proved the following results. Let be the set of all primitive elements of M_2. Then there is a countable set of existential formulas that determines P, however, no finite subset of these formulas does. He also proved that two elements $g, f \in M_2'$ conjugate by some automorphism of M_2 if and only if they satisfy the same existential formulas.

The concept of a rigid (solvable) group was introduced by N. S. Romanovskii about 10 years ago. The rigid group class turned out to be quite interesting and noteworthy. At present, a number of results have been obtained for it, both group-theoretical and model-theoretic. Most of these results were obtained by the discoverer of this class. For these reasons, rigid groups can be called *Romanovskii' groups.* A group G is said to be m-rigid, where m is a natural number, if it has a normal series of the form $G = G_1 > \ldots > > G_m > G_{m+1} = 1$, whose quotients G_i/G_{i+1} are abelian and are torsion free when treated as $\mathbb{Z}[G/G_i]$-modules. Examples of rigid groups are free soluble groups. A. G. Myasnikov and N. S. Romanovskii [169] gave a recursive system of universal axioms distinguishing m-rigid groups in the class of solvable groups of length m. They proved that if G is an arbitrary m-rigid group, and W is an iterated wreath product of m infinite cyclic groups, then the universal theories for these groups satisfy the inclusions $\mathrm{Th}_\forall(W) \subseteq \mathrm{Th}_\forall(\mathrm{G}) \subseteq \subseteq \mathrm{Th}_\forall(\mathrm{S_{r,m}})$, where $r \geqslant 2$. An $\exists$-axiom is given that distinguish among m-rigid groups those that are universally equivalent to W. An arbitrary m-rigid group embeds in a divisible decomposed m-rigid group M, the semidirect product of m abelian groups. A recursive system of axioms distinguishing among M-groups those that are universally equivalent to M. As a consequence, it is stated that the universal theory of M with constants is decidable. By contrast, the universal theory of W with constants is undecidable.

Let $\Gamma = (X, E)$ be a finite simple graph. The right-angled Artin group (in other terminology, a partially commutative group) $G(\Gamma)$, corresponding to Γ, has the specification $\langle X,\, xy = yx\, (x, y) \in E\rangle$. If $\mathfrak{V}$ is a variety of groups, then the partially commutative $\mathfrak{V}$-group, corresponding to Γ, has the specification $\langle X,\, xy = yx\, (x, y) \in E; \mathfrak{V}\rangle$.

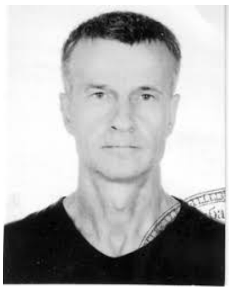
The paper [74] proves that two partially commutative metabelian groups have equal elementary theories if and only if their defining graphs are isomorphic, and that every partially commutative metabelian group is embeddable in a finitely generated metabelian group with decidable universal theory. In [224], N. S. Romanovskii and E. I. Timoshenko proved the following statement: Let the variety $\mathfrak{V}$ contain the variety $\mathfrak{N}_2$, and the finitely generated group H is elementarily equivalent to the partially free group $G = F(\Gamma, \mathfrak{V})$, then $G \simeq H$.

In [255], necessary and sufficient conditions are given for two partially commutative metabelian groups defined by trees to be universally equivalent. In [75], further properties of partially commutative metabelian groups and of their universal theories are described. In particular, it is shown that two partially commutative metabelian groups defined by cycles are universally equivalent if and only if the cycles are isomorphic. It is proved also that the metabelian product of two non-trivial free abelian groups is universally equivalent to any free noncyclic metabelian group.

In [155] some necessary and sufficient conditions of the universal equivalence of the nilpotent R-groups of class 2 defined by trees, with R a binomial Euclidean ring are determined. Partially commutative nilpotent metabelian groups are considered in [76]. Universal theories for partially commutative nilpotent metabelian groups are compared: conditions on defining graphs of two partially commutative nilpotent metabelian groups are presented which are sufficient for the two groups to have equal universal theories; conditions on defining graphs of two partially commutative metabelian groups are specified which are sufficient for the two groups to be universally equivalent; a criterion is given that decides whether two partially commutative nilpotent metabelian groups defined by trees are universally equivalent.

A description of solvable groups with solvable elementary theory is known.

Theorem 15 (Yu. L. Ershov [57], N. S. Romanovskii [219], G. A. Noskov [185]). The elementary theory of a finitely generated solvable group is decidable if and only if the group is virtually abelian.



G. A. Noskov

The corresponding problem has been posed in [95]. Yu. L. Ershov proved this statement [57] in the nilpotent case, N. S. Romanovskii [219] generalized it to the polycyclic case, and finally, G. A. Noskov [185] established the most general statement for the case of a finitely generated solvable group.

9. Rational subsets

The class $\mathrm{Rat}(G)$ of *rational subsets* of a group G is the smallest class that contains all finite subsets of G and that is closed with respect to the following *rational* operations:

— union $= A, B \in \mathrm{Rat}(G) \to A \cup B \in \mathrm{Rat}(G)$;
— product $= A, B \in \mathrm{Rat}(G) \to A \cdot B \in \mathrm{Rat}(G)$;
— taking the monoid generated by a set (Kleeny operation) $= A \in \mathrm{Rat}(G) \to A^* = \{1\} \cup \bigcup_{i=1}^{\infty} A^i$.

This concept generalizes the classical notion of a regular subset of the free monoid Σ.

There is an analogue of Kleene's theorem on the definition of regular subsets of a free monoid by finite automata: a subset R of a group G is rational if and only if R is the output set of a finite automaton over G.

Recall that a finite automaton A over an alphabet σ consists of:

— a finite directed graph with edges labeled by elements of Σ;
— a distinguished initial vertex v_0;
— a set of final vertices $v_1, \ldots, v_t$.

The language $L(A)$ of the automaton consists of all words labeling a path from the initial vertex to a final vertex. A language is called *rational* if it is accepted by some finite automaton.

For definitions and basic properties of rational subsets in groups, see [66, 67, 235].

By well-known theorem of A. Anissimov and A. W. Seifert [5], a subgroup H of G belongs to $\mathrm{Rat}(G)$ if and only if H is finitely generated.

Rational submonoids need not be finitely generated. Rational subsets are not in general closed under complement and intersection.

Rational subset theory has many applications:

— V. Diekert, C. Gutierrez, and C. Hagenah [45] showed solving equations with rational constraints over free groups is PSPACE-complete.
— V. Diekert and M. Lohrey [46] used this to solve equations and decide the positive theory for right-angled Artin groups.
— F. Dahmani and V. Guirardel [40] solved equations over hyperbolic groups with special rational constraints. They gave an algorithm for solving equations and inequations with rational constraints in virtually free groups. This algorithm is based on E. Rip's classification of measured band complexes. Using canonical representatives, they deduced an algorithm for solving equations and inequations in hyperbolic groups (maybe with torsion).
— F. Dahmani and J. Groves [39] used rational subsets in their solution to the isomorphism problem for toral relatively hyperbolic groups.
— The order of g is finite if and only if $g^{-1} \in \{g\}^*$, so decidability of submonoid membership gives decidability of order.

The *rational subset membership problem* for a finitely generated group G is the decision problem, where for a given rational subset A of G and a group element g it is asked whether $g \in A$.

This section presents a survey on known decidability and undecidability results for the rational subset membership problem for groups. The membership problems for finitely generated submonoids and finitely generated subgroups will be discussed as well.

We list some of known results on the rational subset problem.

Positive results:

— (M. Benois [25]). Rational subset membership is decidable for free groups. (The proof uses an automata theoretic analogue of Stallings folding.)
— (C. Eilenberg and M. P. Schutzenberger [53]). Rational subset membership is decidable in abelian groups.
— (Z. Grunschlag [73]). Decidability of rational subset membership is a virtual property. (A property is called *virtual* if its execution for a subgroup of the finite index entails its execution on the entire group.)
— (M. Yu. Nedbai [177]). The decidability of rational subset membership passes through free products.
— (M. Cadilhac, D. Chistikov, and G. Zetzsche [38]). Rational subset membership is decidable for the Baumslag — Solitar groups $\mathrm{BS}(1, q)$ for $q \geqslant 2$.
— (M. Kambites, P. W. Silva, and B. Steinberg [91]). Decidability of rational subset membership is preserved by free products with amalgamation and HNN-extensions with finite edge groups. More generally, if G is a fundamental group of a graph of groups with finite edge groups and for each vertex group the rational set membership problem is solvable, then this problem is also solvable for G.

Let $\mathcal{C}$ be the smallest class of groups containing the trivial group and closed under:

— taking finitely generated subgroups;
— taking finite index overgroups;
— free products with amalgamation and HNN-extensions with finite edge groups;
— direct product with $\mathbb{Z}$.

Theorem 16 (M. Lohrey and B. Steinberg [118]). Every group in the class $\mathcal{C}$ has decidable rational subset membership problem.

There is no need to talk about the solvability of the membership problem for finitely generated submonoids of the group G if the classical problem of the membership for finitely generated subgroups of the group G is unsolvable. Note that direct products do not preserve the solvability of the occurrence problem. It was shown by K. A. Mikhailova [154], that the direct product $F_2 \times F_2$ of two copies of the free group of rank 2 contains a fixed finitely generated subgroup with an undecidable membership problem. In particular, $F_2 \times F_2$ has an undecidable subgroup membership problem. Hence, also the submonoid membership problem and the rational subset membership problem for $F_2 \times F_2$ are undecidable. This result is remarkable since $F_2 \times F_2$ is a very natural group.

The above result of M. Benois cannot be generalized to hyperbolic groups. Indeed, E. Rips [215] proved the existence of hyperbolic torsion-free groups, in particular, groups with small cancellation, on which condition $C'_{1/6}$ is satisfied, and in which the membership problem is unsolvable.

Let $\Gamma = (X, E)$ be a finite simple graph. Recall, that the right-angled Artin group (in other terminology, a partially commutative group) $G(\Gamma)$, corresponding to Γ, has the specification $\langle X,\, xy = yx\, (x, y) \in E\rangle$. It is said that the graph $\Gamma_1 = (X, E)$ contains a *induced graph* Γ_2 if there is a subset of vertices $U \subseteq V$ such that the graph Γ_2 is isomorphic to the graph $(U, E \cap (U \times U))$.

The subgroup membership problem is solvable in any group $G(\Gamma)$ when the graph Γ does not contain an induced cycle C_4 of length 4 [93]. On the other hand, the group $G(C_4)$ contains the direct product $F_2 \times F_2$, therefore, by the above-mentioned theorem of K. A. Mikhailova, there is a finitely generated subgroup in it with unsolvable membership problem.

M. Lohrey and B. Steinberg [118] show that the membership problem in a finitely generated submonoid of a right-angled Artin group is decidable if and only if the independence graph (commutation graph) is a transitive forest, i.e., it does not contain induced subgraphs of type C_4 or P_4, where P_4 denotes a straight line segment consisting of four vertices and three edges. Moreover, in the unsolvable case, one can indicate a fixed finitely generated submonoid of the group $G(\Gamma)$, the membrtship problem for which is unsolvable.

It is shown in [118] that membership in rational subsets of wreath products $HwrV$ with H a finite group and V a virtually free group is decidable. On the other hand, it is shown that there exists a fixed finitely generated submonoid in the wreath product $\mathbb{Z}wr\mathbb{Z}$ with an undecidable membership problem.

The author proved in [237] that any verbal subset $w[G]$ of a finitely generated nilpotent group G with respect to a word w of positive exponent is rational. Examples of verbal subsets of finitely generated metabelian groups that are not rational are given. Recall that the *verbal subset* of a group G is the set of all values of the group word w in this group.

Negative results:

— (V. A. Roman'kov [229]). There exists a number r such that the free nilpotent group $N_{r,2}$ of class 2 generated by r elements has an undecidable rational subset membership problem.

— M. Lohrey and B. Steinberg show in [119]that the free metabelian group M_2 of rank 2 contains a fixed finitely generated submonoid with an undecidable membership problem.

This result is shown via a reduction from the membership problem for finitely generated subsemimodules of free $(\mathbb{Z}\times\mathbb{Z})$-modules of finite rank. This considered problem is shown to be undecidable in by interpreting it as a particular tiling problem of the Euclidean plane that in turn is shown to be undecidable via a direct encoding of a Turing machine.

- M. Lohrey, B. Steinberg, and G. Zetzsche [120] prove that the submonoid membership problem is undecidable for $\mathbb{Z}\mathrm{wr}\mathbb{Z}$.
- U. U. Umirbaev [261] show that the free solvable group $S_{2,3}$ of derived length 3 and rank 2 has an undecidable subgroup membership problem.
- M. Lohrey [117] prove that there are numbers $n, l \geqslant 3$ and a sequence of cyclic subgroups $C_1, \ldots, C_l$ of the unitriangular matrix group $\mathrm{UT}_n(\mathbb{Z})$ over integers such that the membership problem with respect to the product $C_1 \cdots C_l$ is unsolvable.

The submonoid membership problem is the most important fragment of the rational subset problem. The well-known submonoid membership problem for nilpotent groups was recently solved by the author.

Theorem 17 (V. A. Roman'kov [240]). There is a finitely generated submonoid M of a free nilpotent group $N_{r,l}$ of class $l \geqslant 2$ of sufficiently large rank r, the membership problem for which is algorithmically unsolvable.

A. G. Myasnikov and the author [170] established that a verbal subset $w[F_r]$ of a free group F_r of finite rank $r \geqslant 2$ is rational in F_r if and only if $w[F_r] = 1$ or $w[F_r] = F_r$. The last two cases are easily recognized by the form of the word w. This statement is generalized to a wide class of free products of groups.

Rational subsets in nilpotent groups were also studied by G. A. Bazhenova [19]. She proved that the rational subsets of a finitely generated nilpotent group G are a Boolean algebra if and only if G is virtually abelian. Other results on the characterization of finitely generated groups G in which the set of rational subsets $\mathrm{Rat}(G)$ is a Boolean algebra, that is, a family of subsets closed under union, intersection, and complement operations are given in [20, 235, 238].

See [147] for a connection between the submonoid membership problem for a group G and the geometric properties of this group.

It is worth noting that the submonoid membership problem of entering for a free abelian group $A_r \simeq \mathbb{Z}^r$ of rank r is related to the following integer linear programming problem: For a given matrix $A \in \mathrm{M}_{m\times r}$ and vector $b \in \mathbb{Z}^r$ determine whether there exists a solution $x \in \mathbb{N}^m$ of the equation $xA = b$.

In group-theoretic language, this is the submonoid membership problem for the group A_r generated by the rows of the matrix A. It is well known that this version of the integer linear programming problem belongs to the class of NP-complete problems. The submonoid membership problem for an arbitrary group is currently considered as a natural generalization of the problem of integer linear programming. An overview of the related results can be found in [18].

10. Geodesic problems

The computational complexity of the WP in free solvable groups $S_{r,d}$, where $r \geqslant 2$ is the rank and $d \geqslant 2$ is the solvability class of the group, was studied in [171]. Let n be a length of a word (input) $w \in S_{r,d}$.

It is known that the Magnus embedding of $S_{r,d}$ into matrices provides a polynomial time decision algorithm for WP in a fixed group $S_{r,d}$. Unfortunately, the degree of the polynomial grows together with d, so the uniform algorithm is not polynomial in d.

Theorem 18 (A. Myasnikov, V. Roman'kov, A. Ushakov, and A. Vershik [171]).

— The Fox derivatives of elements from $S_{r,d}$ with values in the group ring $\mathbb{Z}S_{r,d-1}$ can be computed in time $O(n^3rd)$.
— The WP has time complexity $O(rn\log_2 n)$ in $S_{r,2}$, and $O(n^3rd)$ in $S_{r,d}$ for $d \geqslant 3$.

In [171], the following algorithmic and decision problems were considered:

— *The Geodesic problem* (GP): Given a word $w \in F(X)$, find a word $u \in F(X)$ which is geodesic in G such that $w =_G u$.
— *The Geodesic length problem* (GLP): Given a word $w \in F(X)$, find $|w|_G$.
— *Bounded geodesic length problem* (BGLP): Given a word $w \in F(X)$ and an integer k, decide if a geodesic representative has length $\leqslant k$.

It has been shown that for free metabelian groups (with standard generating sets) BGLP is NP-complete.

Though GLP seems easier than GP, in practice, to solve GLP one usually solves GP first, and only then computes the geodesic length. It is an interesting question if there exists a group G and a finite set X of generators for G relative to which GP is strictly harder than GLP.

Turing reducibility of the geodesic problems

It has been shown in [171] that a polynomial time solution to any of these problems implies a polynomial time solution to the next, and each implies a polynomial time solution to the word problem for the group.

The algorithmic "hardness" of the problems WP, BGLP, GLP, and GP in a given group G is explained by the following implications: each one is Turing reducible in polynomial time to the next one in the list:

$$WP \preceq_{T,p} BGLP \preceq_{T,p} GLP \preceq_{T,p} GP,$$

and GP is Turing reducible to WP in exponential time:

$$GP \preceq_{T,\exp} WP.$$

M. Elder and A. Rechnitzer [56] established that GP, GLP and BGLP are polynomial time and space reducible to each other.

Complexity of the geodesic problems

Recall the concept of time complexity. Let A be an algorithm with inputs from a set S, $|w|$ is the size of $w \in S$, $TA(w)$ is the number of steps required for A to stop on the input $w \in S$, A is in polynomial time if for some polynomial $p(x)$ means $TA(w) \leqslant p(|w|)$.

If G has polynomial *growth*, i.e., there is a polynomial $p(n)$ such that for each n cardinality of the ball B_n of radius n in the Cayley graph $\Gamma(G, X)$ is at most $p(n)$, then one can easily construct this ball B_n in polynomial time with an oracle for the WP in G. It follows that if a group with polynomial growth has WP decidable in polynomial time, then all the problems above have polynomial time complexity. Observe now, that by famous Gromov's theorem finitely generated groups of polynomial growth are virtually nilpotent. It is also known that the latter have WP decidable in polynomial time (nilpotent finitely generated groups are linear). These two facts together imply that the GP is polynomial time decidable in finitely generated virtually nilpotent groups.

On the other hand, there are many groups of exponential growth where GP is decidable in polynomial time:

— hyperbolic groups — B. A. Epstein et al;

— the Baumslag — Solitar group (metabelian, non polycyclic of exponential growth)

$$BS(1, p) = \langle a, t \mid t^{-1}at = a^p \rangle$$

(M. Elder [55]). An algorithm is presented to convert a word of length n in the standard generators of the solvable Baumslag — Solitar group $\mathrm{BS}(1, p)$ into a geodesic word, which runs in linear time and $O(nlogn)$ space on a random access machine.

In general, if WP in G is polynomially decidable, then BGLP is in the class NP, i.e., it is decidable in polynomial time by a non-deterministic Turing machine. In this case GLP is Turing reducible in polynomial time to an NP problem, but we cannot claim the same for GP. Observe, that BGLP is in NP for any finitely generated metabelian group, since they have WP decidable in polynomial time.

It might happen though, that WP in a group G is polynomial time decidable, but BGLP in G is NP-complete.

W. Parry [191] showed that BGLP is NP-complete in the metabelian group $\mathbb{Z}_2 wr(\mathbb{Z}\times\mathbb{Z})$, the wreath product of $\mathbb{Z}_2$ and $\mathbb{Z} \times \mathbb{Z}$.

It was claimed by C. Droms, J. Lewin, and H. Servatius [48] that in $S_{r,d}$ GLP is decidable in polynomial time. Unfortunately, in this particular case their argument is fallacious. It turned out [171], that BGLP for M_r, $r \geqslant 2$, is NP-complete. Therefore, the search problems SGP and SGLP are NP-hard in non-abelian M_r. To see the NP-completeness, the authors of [171] constructed a polynomial reduction of the rectilinear Steiner tree problem to BGLP in M_r.

Free solvable groups of finite ranks

The conjugacy problem for S_r, d reduces via the Magnus embedding to a similar problem for $A_r wr S_{r,d-1}$ in time $O(n^3 rd)$.

The power problem (PP) in a group G:

$$\exists ?n \in \mathbb{Z} : g^n = f.$$

S. Vassileva [263] proved the following statements.

— The power problem in $S_{r,d}$ is decidable in time $O(n^6 rd)$.
— The conjugacy problem has time complexity $O(n^8 rd)$ in $S_{r,d}$.

A. Ushakov [262] designed new deterministic and randomized algorithms for computational problems in free solvable groups. He improved the results of [171, 263], namely, he proved that:

— There exists a quasi-quadratic time $\tilde{O}(n^2)$ deterministic algorithm solving the word problem in $S_{r,d}$.
— There exists a quasi-quadratic time $\tilde{O}(n^2)$ deterministic algorithm solving the power problem in $S_{r,d}$.
— There exists a quasi-quintic time $\tilde{O}(n^5)$ deterministic algorithm solving the conjugacy problem in $S_{r,d}$.

These results can be improved further if we grant our machine an access to a random number generator. But the result in this approach can be incorrect. Fortunately, the probability of an error is under control: for any fixed polynomial p we can adjust some internal parameter in the algorithm to guarantee that the probability of an error converges to 0 as fast as $O(1/p(n)$. In other words, there exists a quasi-linear time $\tilde{O}(n)$ false-biased randomized algorithm solving the word problem in $S_{r,d}$. There also exists a quasi-linear time $\tilde{O}(n)$ unbiased randomized algorithm solving the power problem in $S_{r,d}$.

Moreover, there exists a quasi-quartic time $\tilde{O}(n^4)$ unbiased randomized algorithm solving the conjugacy problem in $S_{r,d}$.

Thus, A. Ushakov [262] proved that the word problem and the power problem can be solved in quasi-linear time and the conjugacy problem can be solved in quasi-quartic time by Monte Carlo type algorithms.

The origins of computation group theory date back to the late nineteenth and early twentieth centuries. Since then, the field has flourished, particularly during the past 30 to 40 years, and today it remains a lively and active branch of mathematics.

The Handbook of Computational Group Theory offers the first complete treatment of all the fundamental methods and algorithms in CGT presented at a level accessible even to advanced undergraduate students. It develops the theory of algorithms in full detail and highlights the connections between the different aspects of CGT and other areas of computer algebra. While acknowledging the importance of the complexity analysis of CGT algorithms, the authors' primary focus is on algorithms that perform well in practice rather than on those with the best theoretical complexity.

Throughout the book, applications of all the key topics and algorithms to areas both within and outside of mathematics demonstrate how CGT fits into the wider world of mathematics and science. The authors include detailed pseudocode for all of the fundamental algorithms, and provide detailed worked examples that bring the theorems and algorithms to life.

We assume that practical algorithms work with random data. In numerous of cases "random" exclude "the worst" case. The Simplex Method is a very good sample of such algorithm.

Hence, the generic set of data when the algorithm works well became a very important notion.

It is known [65] that the Dehn function $\mathrm{D}(G)$ of a finitely presented group G is recursive if and only if G has decidable word problem. Moreover, for every finitely presented group G with Dehn function $\mathrm{D}(G)$ there exists a nondeterministic Turing machine $\mathrm{M}(G)$ which solves the word problem in G with time function equivalent to $\mathrm{D}(n)$. This machine solves the word problem in every finitely generated subgroup of G as well. Therefore if a finitely generated group G is a subgroup of a finitely presented group with polynomial isoperimetric function then the word problem in G is in NP (i.e., it can be solved by a non-deterministic Turing machine with polynomial time function

J. C. Birget, A. Y. Olshanskii, E. Rips, and M. V. Sapir [29] obtained a general result on the connection between the complexity of the Dehn function of a group and the complexity of the word problem. The word problem of a finitely generated group G is in NP if and only if this group is a subgroup of a finitely presented group H with polynomial isoperimetric function. The embedding can be chosen in such a way that G has bounded distortion in H.

There is a natural concept of the averaged Dehn function $\mathrm{D}_{av}(G)$, introduced by M. Gromov [68]. In [110], E. G. Kukina and the author, answering to the question, posed in [68], proved that $\mathrm{D}(A_r)$ is sub-quadratic (remind that $\mathrm{D}(A_r)$ is quadratic). In [230], the author answered to the another question posed in [68] on the average Dehn function of a free nilpotent group. He showed that this function is asymptotically negligible to the Dehn function in this case.

In [92], I. Kapovich, A. G. Myasnikov, V. Shpilrain, and P. Schupp proposed a generic approach to the theory of computability and computational complexity. Within the framework of this approach, the algorithmic problem is considered not on the entire set of inputs, but on a certain subset of almost all inputs. They showed that for a large class of finitely generated groups the generic time complexity of some classical decision problems from combinatorial group theory, namely the word problem, conjugacy problem and membership problem, are linear. It turns also out that some classical undecidable problems are, in fact, strongly undecidable, i.e., they are undecidable on every strongly generic subset of inputs. A. G. Myasnikov and A. N. Rybalov [172] proved an analog of the Rice theorem for strongly undecidable problems, which provides plenty of examples of strongly undecidable problems. To construct strongly undecidable problems, they introduced a method of generic amplification (an analog of the amplification in complexity theory).

In recent years, interest in the analysis of algorithms from the point of view of complexity theory and practical feasibility has significantly increased. Substitution groups form the most developed part of the computational theory of groups. The basis for this was the corresponding technique for their study, developed by C. Simps back in the 60s of the twentieth century. M. L. Furst, D. Hopcroft, and E. M. Luks [60] showed that the method proposed by Simps works in polynomial time. The time-polynomial theory of linear groups began with a consideration of matrix groups over finite fields. The main problems were the problems of determining the order of a subgroup given by a finite set of generators, and the membership problem for a given group. Even in the case of abelian groups, it is not known how to solve such problems without solving difficult number-theoretic problems, for example, problems of the discrete logarithm and factorization of numbers. The approach to finding a solution using a number-theoretic oracle became natural.

Computing in permutation and in matrix groups

Permutation groups is the most developed subdomain in the Computational Group Theory. Fundamental is a technique first proposed by C. Sims in the 1960's, see monograph [248]. C. Sims introduced many algorithms for working with permutation groups. These were among the first algorithms in CAYLEY and GAP. In 1990s nearly linear algorithms for permutation groups emerged. These are now in GAP and MAGMA. In 2003, Á. Seress published his monograph [245] described the theory behind permutation group algorithms, including developments based on the classification of finite simple groups. He gave rigorous complexity estimates, implementation hints, and advanced exercises. The book fills a significant gap in the symbolic computation literature.

Let $G \leqslant Sym(\Omega)$, where $\Omega = \{\omega_1, \ldots, \omega_n\}$. The tower

$$G = G^{(1)} \geqslant \ldots \geqslant G^{(n+1)} = 1,$$

where $G^{(i)}$ is a pointwise stabilizer of $\{\omega_1, \ldots, \omega_{i-1}\}$, underlines almost all practical algorithms. It was proved in [60] that a variant of Sims' method runs in polynomial time. Now there is the non-substancional polynomial-time library for permutation groups.

Polynomial-time theory of linear groups started with matrix groups over finite fields. Such group is specified by finite list of generators. The two most basic questions are:

— membership in $\mathrm{gp}(U)$;
— the order of the group $\mathrm{gp}(U)$.

Even in the case of abelian groups it is not known how to answer these questions without solving hard number-theoretic problems (factoring and discrete logarithm). So the reasonable question is whether these problems are decidable in randomized polynomial time using number theory oracles.

The first algorithms for computing with finite solvable matrix groups were designed by E. M. Luks [121].

E. M. Luks, L. Babai, R. Beals, Á. Seress et al. study this area for last 25–30 years (see [7]). Let $G \leqslant GL(n, \mathbb{F}_q)$ be a finitely generated matrix group over a finite field $\mathbb{F}_q$.

— One can test in polynomial time whether G is solvable and, if so, whether G is nilpotent.
— If G is solvable, one can also find, for each prime p, the p-part of G. In the nilpotent case it is its (unique) Sylow p-subgroup.
— Also, given a solvable $G \leqslant GL(n, \mathbb{F}_q)$ the following problems can be solved: find $|G|$, decide the MP with respect to G, find a presentation of G via generators and defining relators, find a composition series of G, et cetera.

For polycyclic groups pc-presentation approach was introduced by B. Eick, D. Kahrobaei, G. Ostheimer et al. See [52] for definition and basic properties of pc-presentations. Pc-presentation of a polycyclic group exhibits its polycyclic structure. Pc-presentations allows efficient computations with the groups they define. In particular, the WP is efficiently decidable in a group given by a pc-presentation. GAP package polycyclic is designed for computations with polycyclic groups which are given by a pc-presentations.

Let G be a polycyclic group. Then

$$G \in \mathcal{N}\mathfrak{A}\mathcal{F}.$$

Hence, nilpotent-by-abelian-by finite presentation approach can be applied in this case.

In particular, Bieri — Strebel's invariant is defined for for this type of groups [28].

The solution of BTCP in a finitely generated metabelian group looks more practical than the Noskov's solution in the classical case of the CP. Main feature is that we can reduce the problem changing the group itself. In the polycyclic case we can start with the metabelian image G/G'' and then use induction relative the structure of a polycyclic group as above.

A. Garetta et al. [63] introduce a model of random finitely generated, torsion-free nilpotent groups G of class 2. They prove that for some values of parameters the following holds asymptotically almost surely:
— The ring of integers $\mathbb{Z}$ is definable in G.
— Systems of equations over $\mathbb{Z}$ are reducible to systems over G (and hence they are undecidable).
— The maximal ring of scalars of G is $\mathbb{Z}$.
— G is indecomposable as a direct product of non-abelian factors.

The similar models of random polycyclic groups and random finitely generated nilpotent groups of any nilpotency step, possibly with torsion, were also introduced

For matrix groups over infinite fields, we state the following theorem as the first result.

Theorem 19 (V. M. Kopytov [107]). Let $G \leqslant GL(n, K)$ be a finitely generated matrix group over an algebraic number field K. Then the following problems are decidable:
— determine finiteness of G;
— determine solvability of G;
— MP_{sol} = the membership problem with respect to solvable G.

Most computational problems are known to be decidable for polycyclic matrix groups over number fields.

The WP and MP can be solved [6], many further structural problems have a practical solution.

D. F. Holt, B. Eick, and O'Brien published the monograph "The Handbook of Computational Group Theory" [90] which offers the first complete treatment of all the fundamental methods and algorithms in computational group theory. It develops the theory of algorithms in full detail and highlights the connections between the different aspects of computational group theory and other areas of computer algebra. The monograph focused on algorithms that perform well in practice rather than on those with the best theoretical complexity.

Some methods are developed for computing with matrix groups defined over a range of infinite domains.

A. Detinko, B. Eick, and D. L. Flannery [43] gave a practical nilpotent testing algorithm for finitely generated matrix groups over an infinite field $\mathbb{F}$.

The main algorithms have been implemented in GAP, for groups over $\mathbb{Q}$.

Let $\mathbb{F} = \bar{\mathbb{Q}}$ be an algebraic number field. By the celebrated Tits's theorem a finitely generated subgroup $G \leqslant GL(n, \bar{\mathbb{Q}})$ either contains a nonabelian free subgroup F or has a solvable subgroup H of finite index (*Tits Alternative*).

R. Beals [21] established the following results:

— There is a polynomial time algorithm for deciding which of two conditions of the Tits's Alternative holds for a given G.
— Let G has a solvable subgroup H of finite index. Then one is able in polynomial time to compute a homomorphism φ such that $\varphi(G)$ is a finite matrix group, and $\ker(\varphi)$ is solvable.
  If, in addition, H is nilpotent, then there is efficient method to compute an encoding of elements of G.

Nowadays, it is recognized that there are decision and search variations of algorithmic problems:

— Search word problem (SWP) in G : given $w \in F(X)$, such that $w =_G 1$, find a decomposition $w = \prod_{i=1}^{n} g_i^{-1} r_{i_j} g_i$, where $g_i \in F(X)$, $r_{i_j} \in R^{\pm 1}$.
— Search conjugacy problem (SCP) in G: given two words $u, v \in F(X)$, which define conjugated elements in G, find a conjugator.
— Search membership problem (SMP) in G for a fixed subgroup $H \leqslant G$: given $w \in F(X)$ which belongs to H, find its decomposition as a product of the generators of H.
— Search isomorphism problem (SIP) in a given class $\mathcal{C}$ of presentations: given two presentations in $\mathcal{C}$ of isomorphic groups, find an isomorphism.

In [127], it is proved that the basic algorithmic problems (normal forms, conjugacy of elements, subgroup membership, centralizers, presentation of subgroups, etc.) can be solved by algorithms running in logarithmic space and quasilinear time. Further, if the problems are considered in "compressed" form with each input word provided as a straight-line program, we showed that the problems are solvable in polynomial time. See monograph [116] for the necessary background and detailed exposition of known results on the compressed word problem, emphasizing efficient algorithms for the compressed word problem in various groups.

Basic information about circuit complexity is contained in monograph [265]. This monograph presents a broad and up-to-date view of the computational complexity theory of Boolean circuits. The theory of circuit complexity classes is thoroughly developed.

In [176], the authors pushed the complexity of these problems lower, showing that they may be solved by TC^0 circuits. In [175], it was shown that the conjugacy problem in a wreath product $AwrB$ is uniform-TC^0-Turing-reducible to the conjugacy problem in the factors A and B and the power problem in B. Under certain natural conditions, there is a uniform TC^0 Turing reduction from the power problem in $AwrB$ to the power problems of A and B.

In [128], the authors expand the list of algorithmic problems for nilpotent groups which may be solved in these low complexity conditions to include several fundamental problems concerning subgroups. The following algorithmic problems are solved using TC^0 circuits, or in logspace and quasilinear time, uniformly in the class of nilpotent groups with bounded nilpotency class and rank: subgroup conjugacy, computing the normalizer and isolator of a subgroup, coset intersection, and computing the torsion subgroup. Additionally, if any input words are provided in compressed form as straight-line programs or in Mal'cev coordinates, the algorithms run in quartic time.

A. V. Menshov, A. G. Myasnikov, and A. V. Ushakov [153] study the computational complexity of the fundamental algorithmic problems in finitely generated metabelian groups. They rewrite and streamline some classical algorithms to fit them into the framework of Groebner basis. In many cases this reduction can be done in polynomial time. The algorithmic problems in metabelian groups are classified in terms of logspace and circuit complexities.

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