МАТЕМАТИКА

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STRONGLY AND SOLIDLY ω_1-WEAK $p^{\omega\cdot 2+n}$-PROJECTIVE ABELIAN p-GROUPS[1]

We define the classes of *strongly* ω_1-*weak* $p^{\omega\cdot 2+n}$-*projective*, *solidly* ω_1-*weak* $p^{\omega\cdot 2+n}$-*projective* and *nicely* ω_1-*weak* $p^{\omega\cdot 2+n}$-*projective* abelian p-groups and study their crucial properties. This continues our recent investigations of this branch, published in Hacettepe J. Math. Stat. (2013) and Bull. Malaysian Math. Sci. Soc. (2014), respectively.

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1. Introduction and terminology

Let all groups into consideration be p-primary abelian, where p is a fixed prime integer, written additively as it is customary. As usual, for some ordinal $\alpha \geq 0$ and a group G, we state the α-*th Ulm subgroup* $p^\alpha G$, consisting of all elements of G with height $\geq \alpha$, inductively as follows: $p^0G = G$, $pG = \{pg \mid g \in G\}$, $p^\alpha G = p(p^{\alpha-1}G)$ if $\alpha - 1$ exists (so α is non-limit) and $p^\alpha G = \cap_{\beta<\alpha} p^\beta G$ if $\alpha - 1$ does not exist (so α is limit). The group G is called p^α-*bounded* if $p^\alpha G = \{0\}$; note that these groups are necessarily reduced. We shall say that G is *separable* if it is p^ω-bounded, where ω is the first infinite ordinal. A group G is abbreviated as Σ-*cyclic* if it is a direct sum of cyclic subgroups. Most of the important unexplained here notations and notions will follow mainly those from [7] and [8]. For any non-negative integer $n \geq 0$ recall that a group G is $p^{\omega+n}$-*projective* if G/P is Σ-cyclic for some p^n-bounded $P \leq G$. Note the crucial facts from [1] that if G is either Σ-cyclic or $p^{\omega+n}$-projective, then so is G/F for every finite $F \leq G$.

We continue with two crucial concepts investigated in [3] in details.

- A group G is said to be *weakly* $p^{\omega\cdot 2+n}$-*projective* if there is a $p^{\omega+n}$-projective subgroup $H \leq G$ such that G/H is Σ-cyclic.

It was established in [3] that these groups are $p^{\omega\cdot 2+n}$-bounded.

- A group G is said to be ω_1-*weakly* $p^{\omega\cdot 2+n}$-*projective* if there is a countable subgroup $K \leq G$ such that G/K is weakly $p^{\omega\cdot 2+n}$-projective.

It was proved in [3] that for such a group G its subgroup $p^{\omega\cdot 2+n}G$ is always countable.

However, this class of groups is quite large, and it will be better to consider some its restricted modifications by exploiting in various aspects the ”niceness” property. Recall that a subgroup N of a group G is *nice* if, for each limit ordinal τ, the equality $\cap_{\alpha<\tau} (N + p^\alpha G) = N + p^\tau G$ holds. Standardly, ω_1 means the first uncountable ordinal.

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So, we will now state our pivotal machinery like this:

Definition 1.1. A group G is said to be *strongly* ω_1*-weak* $p^{\omega\cdot 2+n}$*-projective* if it contains a $p^{\omega+n}$-projective nice subgroup N such that G/N is the direct sum of a countable group and a Σ-cyclic group.

Note that in terms of [6] the quotient G/N is ω-totally Σ-cyclic, i.e., it is ω_1-p^{ω}-projective.

Definition 1.2. A group G is said to be *solidly* ω_1*-weak* $p^{\omega\cdot 2+n}$*-projective* if it contains a countable nice subgroup M such that G/M is weakly $p^{\omega\cdot 2+n}$-projective.

Definition 1.3. A group G is said to be *nicely* ω_1*-weak* $p^{\omega\cdot 2+n}$*-projective* if it contains a weakly $p^{\omega\cdot 2+n}$-projective nice subgroup Q such that G/Q is countable.

The goal of the present paper is to give a comprehensive study of these three concepts, thus somewhat enlarging the results from [2], [3], and [4]. The work is organized as follows: in the next two sections, we state some elementary and useful properties of the new group classes. After that, we establish our basic results. In the final section, we list some interesting left-open questions.

And so, we come to our first working section.

2. Elementary properties

Here we shall quote some elementary but helpful properties like these:

(1) Strongly ω_1-weak $p^{\omega\cdot 2+n}$-projective groups are ω_1-weakly $p^{\omega\cdot 2+n}$-projective.

(2) Solidly ω_1-weak $p^{\omega\cdot 2+n}$-projective groups are ω_1-weakly $p^{\omega\cdot 2+n}$-projective.

(3) Nicely ω_1-weak $p^{\omega\cdot 2+n}$-projective groups are ω_1-weakly $p^{\omega\cdot 2+n}$-projective (this follows from Theorem 2.2 (e) of [3]).

(4) Weakly $p^{\omega\cdot 2+n}$-projective groups are both strongly ω_1-weak $p^{\omega\cdot 2+n}$-projective and solidly ω_1-weak $p^{\omega\cdot 2+n}$-projective.

(5) Strongly ω_1-$p^{\omega+n}$-projective groups are strongly ω_1-weak $p^{\omega\cdot 2+n}$-projective.

(6) Nicely ω_1-$p^{\omega+n}$-projective groups are nicely ω_1-weak $p^{\omega\cdot 2+n}$-projective.

(7) If $p^{\omega}G = \{0\}$, then G is strongly ω_1-weak $p^{\omega\cdot 2+n}$-projective $\Longleftrightarrow$ G is solidly ω_1-weak $p^{\omega\cdot 2+n}$-projective $\Longleftrightarrow$ G is nicely ω_1-weak $p^{\omega\cdot 2+n}$-projective $\Longleftrightarrow$ G is weakly $p^{\omega\cdot 2+n}$-projective.

In fact, in [3] was showed even that p^{ω}-bounded ω_1-weak $p^{\omega\cdot 2+n}$-projective groups are weakly $p^{\omega\cdot 2+n}$-projective.

The following relationship sounds interesting.

Proposition 2.1. *If G is a strongly* ω_1*-weak* $p^{\omega\cdot 2+n}$*-projective group, then G is nicely* ω_1*-weak* $p^{\omega\cdot 2+n}$*-projective.*

Proof. Write $G/N = (K/N) \oplus (S/N)$ for some $p^{\omega+n}$-projective nice subgroup N of G with $N \leq K$ and $N \leq S$, where the first term of the direct decomposition is countable whereas the second one is Σ-cyclic. Thus $G/S \cong (G/N)/(S/N) \cong K/N$ is countable. But S/N is nice in G/N as a direct summand, whence by virtue of [7] we derive that S is nice in G. Moreover, S is by definition weakly $p^{\omega\cdot 2+n}$- projective, as required. ■

3. Some useful preliminaries

The following three affirmations, dealing with weakly $p^{\omega\cdot 2+n}$-projective groups, seem not to appear in [3], and so we will document them here.

Proposition 3.1. (i) *The group G is weakly* $p^{\omega\cdot 2}$*-projective if and only if* p^nG *is weakly* $p^{\omega\cdot 2}$*-projective for some* $n \in \mathbb{N}$.

(ii) *If G is weakly* $p^{\omega\cdot 2}$*-projective and* $T \leq G$ *with* $p^nT = \{0\}$*, then G/T is weakly* $p^{\omega\cdot 2+n}$*-projective.*

Proof. (i) (⇒) Suppose there is a Σ-cyclic subgroup X of G such that G/X is Σ-cyclic. Hence $p^n(G/X) = (p^nG + X)/X \cong p^nG/(p^nG \cap X)$ and $p^nC \cap X$ are both Σ-cyclic groups being subgroups of G/X and X, respectively (see, e.g., [7]). (⇐) Suppose that there exists a Σ-cyclic subgroup Y of p^nG, and hence of G, with $p^nG/Y = p^n(G/Y)$ also a Σ-cyclic group. But consulting with [7], the quotient $p^n(G/Y)$ being Σ-cyclic implies the same for G/Y as well. This gives the wanted result.

(ii) Let G/U be Σ-cyclic for some Σ-cyclic subgroup U. Thus $U + T$ is also Σ-cyclic (see [7]) and $U' = (U + T)/T \cong U/(U \cap T)$ is therefore $p^{\omega+n}$-projective. But

$$(G/T)/((U + T)/T) \cong G/(U + T) \cong (G/U)/((U + T)/U)$$

is $p^{\omega+n}$-projective, because $(U + T)/U$ is p^n-bounded. Denote $G/T = G'$. Since G'/U' is $p^{\omega+n}$-projective, there is $Z' \leq G'$ with $U' \subseteq Z'$ and $p^nZ' \subseteq U'$ such that $(G'/U')/(Z'/U') \cong$ $\cong G'/Z'$ is Σ-cyclic. But p^nZ' is $p^{\omega+n}$-projective, whence so is Z'.

Finally, $G' = G/T$ is weakly $p^{\omega\cdot2+n}$-projective, as claimed. ■

Theorem 3.2. *The following four points are equivalent*:

(a) *G is weakly $p^{\omega\cdot2+n}$-projective*;

(b) *there exists a $p^{\omega+n}$-projective subgroup $P \leq G$ such that G/P is Σ-cyclic*;

(c) *there exists a p^n-bounded subgroup $T \leq G$ such that G/T is weakly $p^{\omega\cdot2}$-projective*;

(d) *there exist a p^n-bounded subgroup L and a weakly $p^{\omega\cdot2}$-projective group S such that $G \cong S/L$.*

Proof. (a) ⇐⇒ (b) is just the definition.

(b) ⇒ (c). Assume P/X is Σ-cyclic for some $p^nX = \{0\}$. Thus $G/P \cong (G/X)/(P/X)$ is Σ-cyclic, whence G/X is by definition weakly $p^{\omega\cdot2}$-projective, as expected.

(c) ⇒ (b). Let A/T be Σ-cyclic for some $A \leq G$ containing T such that $(G/T)/(A/T) \cong G/A$ is also Σ-cyclic. But it is plainly seen that A is $p^{\omega+n}$-projective, as required.

The implication (d) ⇒ (a), or its equivalence (d) ⇒ (b), follows from Proposition 3.1 (ii). So, we consider the reverse implication (a) ⇒ (d) or its tantamount relationship (c) ⇒ (d). To that aim, if X is a group with $p^nX = G$, then let $S = X/T$. Consequently, $p^nS = p^nX/T = G/T$ is weakly $p^{\omega\cdot2}$-projective by hypothesis. Referring to Proposition 3.1 (i), the last condition forces that S is weakly $p^{\omega\cdot2}$-projective. Letting $L = X[p^n]/T \subseteq S[p^n]$, we deduce that $S/L \cong X/X[p^n] \cong p^nX = G$, proving the desired relation. ■

Lemma 3.3. *If A is a weakly $p^{\omega\cdot2+n}$-projective group and $F \leq A$ is finite, then A/F is also weakly $p^{\omega\cdot2+n}$-projective.*

Proof. Write A/B is Σ-cyclic for some $p^{\omega+n}$-projective subgroup B. Since $(F + B)/B \cong$ $\cong F/(F\cap B)$ is obviously finite, one sees that $(A/B)/(F+B)/B \cong A/(F+B) \cong (A/F)/(F+B)/F$ is Σ-cyclic. However, $(F + B)/F \cong B/(B \cap F)$ is $p^{\omega+n}$-projective too (see [1]), as required. ■

4. Main Results

The following two assertions strengthen point (5) listed above.

Proposition 4.1. *If G is a strongly ω_1-weak $p^{\omega\cdot2+n}$-projective group and $p^{\omega+n}G = \{0\}$, then G is weakly $p^{\omega\cdot2+n}$-projective.*

Proof. Write G/N is the direct sum of a countable group and a Σ-cyclic group for some nice $p^{\omega+n}$-projective subgroup N of G. Since $p^{\omega+n}(G/N) = (p^{\omega+n}G + N)/N = \{0\}$, we have that G/N is $p^{\omega+n}$-projective. Hence there is a subgroup $X \leq G$ containing N such that $p^nX \subseteq N$ and $(G/N)/(X/N) \cong G/X$ is Σ-cyclic.

Since p^nX is $p^{\omega+n}$-projective, we infer that so is X, as required. ■

Proposition 4.2. *If G is a strongly ω_1-weak $p^{\omega\cdot 2+n}$-projective group and $p^{\omega}G$ is finite, then G is weakly $p^{\omega\cdot 2+n}$-projective.*

Proof. Let G/N be the direct sum of a countable group and a Σ-cyclic group for some $p^{\omega+n}$-projective nice subgroup N of G. Therefore, $(G/N)/p^{\omega}(G/N) = (G/N)/(p^{\omega}G+N)/N \cong \cong G/(p^{\omega}G+N)$ is Σ-cyclic. Denote $V = p^{\omega}G+N$, and hence $V/T = (p^{\omega}G + N)/T = = [(p^{\omega}G + T)/T] + [N/T]$, where $T \leq N$ with the property that $p^nT = \{0\}$ and N/T is Σ-cyclic. Since $(p^{\omega}G + T)/T \cong p^{\omega}G/(p^{\omega}G \cap T)$ is finite, it follows that V/T is Σ-cyclic. Thus V is $p^{\omega+n}$-projective and G/V is Σ-cyclic, as required. ■

We continue with

Proposition 4.3. *If G is a strongly ω_1-weak $p^{\omega\cdot 2+n}$-projective group and $p^{\omega}G$ is countable, then $G/p^{\omega}G$ is weakly $p^{\omega\cdot 2+n}$-projective.*

Proof. Observe that $(N +p^{\omega}G)/p^{\omega}G \cong N/(N \cap p^{\omega}G) \cong [N/p^{\omega}N]/[(N \cap p^{\omega}G)/p^{\omega}N]$ is separable being embedded in $G/p^{\omega}G$, and thus it is $p^{\omega+n}$-projective according to Theorem 4.2 of [5]. But $p^{\omega}(G/N)$ is countable being contained in a direct summand of G/N, whence it easily follows that

$$(G/N)/p^{\omega}(G/N) = (G/N)/((p^{\omega}G + N)/N) \cong G/(p^{\omega}G + N) \cong$$
$$\cong (G/p^{\omega}G)/((p^{\omega}G + N)/p^{\omega}G)$$

is Σ-cyclic, as required for the factor-group $G/p^{\omega}G$ to be weakly $p^{\omega\cdot 2+n}$-projective. ■

Remark 1. The last statement follows also from Theorem 2.4 in [3], but the stated above argument gives a new more simple and conceptual proof. Analyzing the corresponding definitions, especially Definition 1.2, and again utilizing the same theorem, we then can say even a little more:

Theorem 4.4. *If G is a group such that $p^{\omega}G$ is countable, then the following points are equivalent*:

(i)*G is ω_1-weakly $p^{\omega\cdot 2+n}$-projective*;

(ii) *G is solidly ω_1-weak $p^{\omega\cdot 2+n}$-projective*;

(iii) *$G/p^{\omega}G$ is weakly $p^{\omega\cdot 2+n}$-projective.*

Thus Proposition 4.3 can be extended to nicely ω_1-weak $p^{\omega\cdot 2+n}$-projective groups (compare with Proposition 2.1 quoted above).

However, when the subgroups $p^{\alpha}G$ are finite for some infinite ordinal α, we obtain the following strengthening.

Proposition 4.5. *If G is a strongly ω_1-weak $p^{\omega\cdot 2+n}$-projective group and $p^{\alpha}G$ is finite for some $\alpha \geq \omega$, then $G/p^{\alpha}G$ is strongly ω_1-weak $p^{\omega\cdot 2+n}$-projective.*

Proof. Given $G/N = (C/N)\oplus(S/N)$, where C/N is countable and S/N is Σ-cyclic for some $p^{\omega+n}$-projective subgroup N of G which is nice in G. But $p^{\alpha}(G/N) = p^{\alpha}(C/N)$ and thus

$$(G/N)/p^{\alpha}(G/N) \cong [(C/N)/p^{\alpha}(C/N)] \oplus (S/N)$$

and therefore, since $p^{\alpha}(G/N) = (p^{\alpha}G + N)/N$, we obtain that

$$G/(p^{\alpha}G + N) \cong (G/p^{\alpha}G)/(p^{\alpha}G + N)/p^{\alpha}G$$

is the direct sum of a countable group and a Σ-cyclic group. However, $(p^{\alpha}G + N)/p^{\alpha}G$ is nice in $G/p^{\alpha}G$ because $p^{\alpha}G+N$ is so in G (see, e.g., [7]), and moreover $(p^{\alpha}G + N)/p^{\alpha}G \cong \cong N/(N \cap p^{\alpha}G)$ is $p^{\omega+n}$-projective since $N \cap p^{\alpha}G$ is finite (see, for instance, [1]). ■

Proposition 4.6. *If G is a solidly ω_1-weak $p^{\omega\cdot 2+n}$-projective group and $p^{\alpha}G$ is finite for some $\alpha \geq \omega$, then $G/p^{\alpha}G$ is solidly ω_1-weak $p^{\omega\cdot 2+n}$-projective.*

Proof. Let M be a countable nice subgroup of G such that G/M is weakly $p^{\omega\cdot 2+n}$-projective. Since $p^{\alpha}(G/M) = (p^{\alpha}G+M)/M \cong p^{\alpha}G/(p^{\alpha}G\cap M)$ is finite, according to Lemma 3.3, we deduce that

$$(G/M)/p^\alpha(G/M) \cong G/(p^\alpha G + M) \cong (G/p^\alpha G)/(p^\alpha G + M)/p^\alpha G$$

is weakly $p^{\omega\cdot 2+n}$-projective as well. Moreover, $(p^\alpha G + M)/p^\alpha G \cong M/(M \cap p^\alpha G)$ is countable and nice in $G/p^\alpha G$ (cf. [7]), as desired. ■

Proposition 4.7. *If G is a nicely ω_1-weak $p^{\omega\cdot 2+n}$-projective group and $p^\alpha G$ is finite for some $\alpha \geq \omega$, then $G/p^\alpha G$ is nicely ω_1-weak $p^{\omega\cdot 2+n}$-projective.*

Proof. Write G/Q is countable for some nice weakly $p^{\omega\cdot 2+n}$-projective subgroup Q. Likewise, in virtue of Lemma 3.3, the quotient $(Q + p^\alpha G)/p^\alpha G \cong Q/(Q \cap p^\alpha G)$ is again weakly $p^{\omega\cdot 2+n}$-projective. We also derive that $G/(Q + p^\alpha G) \cong (G/p^\alpha G)/(Q + p^\alpha G)/p^\alpha G$ is countable. But $(Q + p^\alpha G)/p^\alpha G$ is nice in $G/p^\alpha G$ by [7], as wanted. ■

The following technicality is well-known, but we list and prove it here only for the sake of completeness and for the convenience of the reader.

Lemma 4.8. *If A is a Σ-cyclic group and $C \leq A$ is its countable subgroup, then A/C is a direct sum of a countable group and a Σ-cyclic group. In particular, if C is nice in A, then A/C is also a Σ-cyclic group.*

Proof. Since C is countable, there exists a countable subgroup K of A with the property that $K \supseteq C$ and $A = K \oplus T$ for some $T \leq A$. Therefore, $A/C \cong (K/C) \oplus T$, where C is nice in K. Thus, K/C is a separable countable group and hence a Σ-cyclic group, as wanted. ■

For arbitrary infinite ordinals α and countable Ulm subgroups $p^\alpha G$, we have the following:

Proposition 4.9. *If $\alpha \geq \omega$ and G is a strongly ω_1-weak $p^{\omega\cdot 2}$-projective group such that $p^\alpha G$ is countable, then $G/p^\alpha G$ modulo a countable subgroup is nicely ω_1-weak $p^{\omega\cdot 2}$-projective.*

Proof. Suppose that there is a nice Σ-cyclic subgroup N of G such that G/N is the direct sum of a countable group and a Σ-cyclic group. Since $(p^\alpha G + N)/N = p^\alpha(G/N)$ is countable and contained in the countable direct summand of G/N, it easily follows that

$$G/(p^\alpha G + N) \cong (G/N)/p^\alpha(G/N) \cong (G/p^\alpha G)/(p^\alpha G + N)/p^\alpha G$$

is again the direct sum of a countable group and a Σ-cyclic group. Moreover, $(p^\alpha G + N)/p^\alpha G$ is nice in $G/p^\alpha G$ (see [7]) and, because $N \cap p^\alpha G$ is countable, one can infer by Lemma 4.8 that $(p^\alpha G + N)/p^\alpha G \cong N/(N \cap p^\alpha G)$ is the direct sum of a countable group and a Σ-cyclic group, say $(p^\alpha G + N)/p^\alpha G = K \oplus S$, where the first term K is countable and the second one S is Σ-cyclic. Consequently, denoting $G_\alpha = G/p^\alpha G$, we write

$$G_\alpha/(K \oplus S) = [R/(K \oplus S)] \oplus [V/(K \oplus S)],$$

where $R/(K \oplus S)$ is countable while $V/(K \oplus S)$ is Σ-cyclic. Since $K \oplus S$ is nice in G_α, it follows from [7] that V is also nice in G_α. Besides, G_α/V is countable.

On the other hand, both $(V/K)/(K \oplus S)/K \cong V/(K \oplus S)$ and $(K \oplus S)/K \cong S$ must be Σ-cyclic, whence V/K must be weakly $p^{\omega\cdot 2}$-projective. So V is ω_1-weakly $p^{\omega\cdot 2}$-projective. Finally, one sees that $(G_\alpha/K)/(V/K) \cong G_\alpha/V$, as required. ■

We will now consider how the three new properties are inherited by the action on Ulm subgroups.

Proposition 4.10. *If G is strongly ω_1-weak $p^{\omega\cdot 2+n}$-projective, then so is $p^\alpha G$ for any ordinal α.*

Proof. Let there exist a nice $p^{\omega+n}$-projective subgroup N of G such that G/N is the direct sum of a countable group and a Σ-cyclic group. Therefore, $N \cap p^\alpha G$ is by [7] nice in $p^\alpha G$ and is also $p^{\omega+n}$-projective being a subgroup of N. Moreover, $p^\alpha G/(p^\alpha G \cap N) \cong$ $\cong (p^\alpha G + N)/N \subseteq G/N$ is ω-totally Σ-cyclic as well (cf. [6]), thus proving the assertion, as promised. ■

Proposition 4.11. *If G is solidly ω_1-weak $p^{\omega\cdot 2+n}$-projective, then so is $p^\alpha G$ for any ordinal* α.

Proof. There is a countable nice subgroup M of G such that G/M is weakly $p^{\omega\cdot 2+n}$-projective. Thus, in view of [3], we have that $p^\alpha(G/M) = (p^\alpha G + M)/M \cong p^\alpha G/(p^\alpha G \cap M)$ is also weakly $p^{\omega\cdot 2+n}$-projective. But $M \cap p^\alpha G$ is countable and nice in $p^\alpha G$ (cf. [7]), as required. ■

Proposition 4.12. *If G is nicely ω_1-weak $p^{\omega\cdot 2+n}$-projective, then so is $p^\alpha G$ for any ordinal* α.

Proof. Write G/Q is countable for some nice $p^{\omega\cdot 2+n}$-projective subgroup Q of G. Therefore, $p^\alpha(G/Q) = (p^\alpha G + Q)/Q \cong p^\alpha G/(p^\alpha G \cap Q)$ is countable, where $p^\alpha G \cap Q$ is nice in $p^\alpha G$ (see [7]) and $p^\alpha G \cap Q$ is $p^{\omega\cdot 2+n}$-projective being a subgroup of Q (see [3]). ■

The next technicality is well-known but we, however, will give a proof for the sake of completeness and for the readers' convenience.

Lemma 4.13. *If C is a countable group and L is a Σ-cyclic group, then there are a countable group K and a Σ-cyclic group S such that $C + L = K \oplus S$.*

Proof. Since $C \cap L \subseteq L'$ for some countable subgroup L' of L with $L = L' \oplus L''$, we have that $C + L = C + L' + L'' = (C + L') \oplus L''$. In fact, $x \in (C + L') \cap L''$ gives that $x = c + b$, where $b \in L'$ and $c \in C$. Thus $x - b = c \in C \cap L$ and hence $x - b \in L'$ which forces that $x \in L'' \cap L' = 0$, as required. Putting now $C + L' = K$ and $L'' = S$, we are set. ■

The following two technical claims are pivotal (see also [4]).

Lemma 4.14. *Suppose that α is an ordinal, and that G and F are groups where F is finite. Then the following formula is fulfilled*:

$$p^\alpha(G + F) = p^\alpha G + F \cap p^\alpha(G + F).$$

Proof. We will use a transfinite induction on α. First, if $\alpha - 1$ exists, we have

$$p^\alpha(G + F) = p(p^{\alpha-1}(G + F)) = p(p^{\alpha-1}G + F \cap p^{\alpha-1}(G + F)) =$$
$$p(p^{\alpha-1}G) + p(F \cap p^{\alpha-1}(G + F)) \subseteq p^\alpha G + F \cap p(p^{\alpha-1}(G + F)) =$$
$$p^\alpha G + F \cap p^\alpha(G + F).$$

Since the reverse inclusion "$\supseteq$" is obvious, we obtain the desired equality.

If now $\alpha-1$ does not exist, we have that $p^\alpha(G+F) = \cap_{\beta<\alpha}(p^\beta(G+F)) \subseteq$ $\subseteq \cap_{\beta<\alpha}(p^\beta G+F) = \cap_{\beta<\alpha}p^\beta G+F = p^\alpha G+F$. In fact, the second sign " = " follows like this: Given $x \in \cap_{\beta<\alpha}(p^\beta G + F)$, we write that $x = g_{\beta_1} + f_1 = \cdots = g_{\beta_s} + f_s = \cdots$, where $f_1, \cdots, f_s \in F$ are all the elements of F; $g_{\beta_1} \in p^{\beta_1}G, \ldots, g_{\beta_s} \in p^{\beta_s}G$ with $\beta_1 < \cdots < \beta_s < \cdots$.

Since F is finite, while the number of equalities is infinite due to the infinite cardinality of α, we infer that $g_{\beta_s} \in p^\beta G$ for any ordinal $\beta < \alpha$ which means that $g_{\beta_s} \in \cap_{\beta<\alpha} p^\beta G = p^\alpha G$. Thus $x \in \cap_{\beta<\alpha}p^\beta G + F = p^\alpha G + F$, as claimed. Furthermore, $p^\alpha(G + F) \subseteq (p^\alpha G + F) \cap p^\alpha(G + F) = p^\alpha G + F \cap p^\alpha(G + F)$ which is obviously equivalent to an equality, as wanted. ■

Lemma 4.15. *Let N be a nice subgroup of a group G. Then*

(i) *$N + R$ is nice in G for every finite subgroup $R \leq G$*;

(ii) *N is nice in $G + F$ for each finite group F.*

Proof. (i) For any limit ordinal γ, we deduce that $\cap_{\delta<\gamma}(N + R + p^\delta G) \subseteq R + \cap_{\delta<\gamma}(N + p^\delta G) = R + N + p^\gamma G$, as required. Indeed, the relation "$\subseteq$" follows like this: Given $x \in \cap_{\delta<\gamma}(N + R + p^\delta G)$, we write $x = a_1 + r_1 + g_1 = \cdots = a_s + r_s + g_s = \cdots = a_k + r_1 + g_k = \cdots$, where $a_1, \cdots, a_k \in N$; $r_1, \cdots, r_s \in R$; $g_1 \in p^{\delta_1}G, \ldots, g_k \in p^{\delta_k}G$ with $\delta_1 < \cdots < \delta_k$. So $a_1 + g_1 = \cdots = a_k + g_k = \cdots \in \cap_{\delta<\gamma}(N + p^\delta G)$ and hence $x \in R + \cap_{\delta<\gamma}(N + p^\delta G)$, as requested.

(ii) Since N is nice in G, we may write $\cap_{\delta<\gamma}[N + p^{\delta}G] = N + p^{\gamma}G$ for every limit ordinal γ. Furthermore, with Lemma 4.14 at hand, we subsequently deduce that

$$\cap_{\delta<\gamma}[N + p^{\delta}(G + F)] = \cap_{\delta<\gamma}[N + p^{\delta}G + F \cap p^{\delta}(G + F)] \subseteq$$
$$\subseteq \cap_{\delta<\gamma}(N + p^{\delta}G) + F \cap p^{\gamma}(G + F) = N + p^{\gamma}G + F \cap p^{\gamma}(G + F) = N + p^{\gamma}(G + F).$$

In fact, the inclusion "$\subseteq$" follows thus: Given $x \in \cap_{\delta<\gamma}[N + p^{\delta}G + F \cap p^{\delta}(G + F)]$, we write $x = a_1 + g_1 + f_1 = \cdots = a_s + g_s + f_s = \cdots = a_k + g_k + f_1 = \cdots$, where $a_1, \cdots, a_k \in N$; $g_1 \in p^{\delta_1}G, ..., g_k \in p^{\delta_k}G$; $f_1 \in F \cap p^{\delta_1}(G + F), \cdots, f_k = f_1 \in F \cap p^{\delta_k}(G+F)$ with $\delta_1 < \cdots < \delta_k$. Hence $a_1 + g_1 = \cdots = a_k + g_k = \cdots \in \cap_{\delta<\gamma}(N + p^{\delta}G)$ and because the number of the f_i's $(1 \le i \le k)$ is finite whereas the number of equalities is not, we can deduce that $f_1 \in \cap_{\delta<\gamma}(F \cap p^{\delta}(G + F)) = F \cap p^{\gamma}(G + F)$, as needed. ■

We are now in a position to proceed by proving with the following statement concerning finite extensions of the three new group classes:

Proposition 4.16. *Let G be a group with a subgroup H such that G/H is finite. The following three points are true*:

(1) *If H is strongly ω_1-weak $p^{\omega\cdot 2+n}$-projective, then G is strongly ω_1-weak $p^{\omega\cdot 2+n}$-projective.*

(2) *If H is solidly ω_1-weak $p^{\omega\cdot 2+n}$-projective, then G is solidly ω_1-weak $p^{\omega\cdot 2+n}$-projective.*

(3) *If H is nicely ω_1-weak $p^{\omega\cdot 2+n}$-projective, then G is nicely ω_1-weak $p^{\omega\cdot 2+n}$-projective.*

Proof. Write $G = H + F$ for some finite $F \le G$.

(1) Suppose that $H/N = (K/N) \oplus (S/N)$, where the first direct summand is countable and the second direct summand is Σ-cyclic, for some nice $p^{\omega+n}$- projective subgroup N of H. By Lemma 4.15, N is nice in G and, moreover, $G/N = (H/N) + (F + N)/N = (L/N) + (S/N)$, where $L = K + F + N$. Clearly, L/N is countable and thus, according to Lemma 4.13, we are set.

(2) We first shall show that if A is a weakly $p^{\omega\cdot 2+n}$-projective group, then $A + F$ is again a weakly $p^{\omega\cdot 2+n}$-projective group. In fact, letting A/T be Σ-cyclic where T is $p^{\omega+n}$-projective, we have that $(A + F)/T = (A/T) + (F + T)/T$. Since the first term is Σ-cyclic and the second term is finite, the sum remains Σ-cyclic appealing to [1], as required.

We are now ready to continue with the proof the major assertion. To that goal, assume that H/M is weakly $p^{\omega\cdot 2+n}$-projective for some countable nice subgroup M of H. By Lemma 4.15, M is nice in G. Furthermore, $G/M = (H/M) + [(F + M)/M]$. Since $(F + M)/M$ is obviously finite, by what we have shown in the preceding paragraph, we are finished.

(3) Suppose H/Q is countable for some weakly $p^{\omega\cdot 2+n}$-projective nice subgroup Q of H. So, $G/Q = (H + F)/Q = (H/Q) + (F + Q)/Q$ is also countable because the second summand is finite. Since by Lemma 4.15 the subgroup Q remains nice in G, we are thus done. ■

Remark 2. It is interesting to know whether or not the converses of these three implications hold.

5. Open problems

In closing, we state here three problems of interest.

Problem 1. Does it follow that a group G is strongly ω_1-weak $p^{\omega\cdot 2+n}$-projective if and only if $p^{\omega\cdot 2+n}G$ is countable and $G/p^{\omega\cdot 2+n}G$ is weakly $p^{\omega\cdot 2+n}$-projective?

Problem 2. If G is strongly ω_1-weak $p^{\omega\cdot 2+n}$-projective and $p^{\omega\cdot 2+n}G = \{0\}$ (in particular, $p^{\omega}G$ is countable with $p^{\omega\cdot 2}G = \{0\}$), is then G weakly $p^{\omega\cdot 2+n}$-projective?

Problem 3. Is it true that G is strongly ω_1-weak $p^{\omega\cdot 2+n}$-projective if and only if G is solidly ω_1-weak $p^{\omega\cdot 2+n}$-projective if and only if G is nicely ω_1-weak $p^{\omega\cdot 2+n}$- projective if and only if G is ω_1-weakly $p^{\omega\cdot 2+n}$-projective?

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Keywords: Σ-cyclic groups, $p^{\omega+n}$-projective groups, ω_1-$p^{\omega\cdot 2+n}$-projective groups, strongly ω_1-$p^{\omega+n}$-projective groups.

We define the classes of *strongly* ω_1-*weak* $p^{\omega\cdot 2+n}$-*projective*, *solidly* ω_1-*weak* $p^{\omega\cdot 2+n}$-*projective* and *nicely* ω_1-*weak* $p^{\omega\cdot 2+n}$-*projective* abelian p-groups and study their crucial properties. This continues our recent investigations of this branch, published in Hacettepe J. Math. Stat. (2013) and Bull. Malaysian Math. Sci. Soc. (2014), respectively.

Ключевые слова: Σ-циклические группы, $p^{\omega+n}$-проективные группы, ω_1-$p^{\omega\cdot 2+n}$-проективные группы, строго ω_1-$p^{\omega+n}$-проективные группы.

Определены *строго* ω_1-*слабо* $p^{\omega\cdot 2+n}$-*проективные, плотно* ω_1-*слабо* $p^{\omega\cdot 2+n}$-*проективные и хорошо* ω_1-*слабо* $p^{\omega\cdot 2+n}$-*проективные* p-группы и изучены важнейшие их свойства. Данная статья является продолжением исследований автора, опубликованных в Hacettepe J. Math. Stat. (2013) и Bull. Malaysian Math. Sci. Soc. (2014) соответственно.

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