УДК 62-50
DOI: 10.17223/19988605/55/8

A.I. Rouban

THE SENSITIVITY COEFFICIENTS FOR MULTIVARIATE DYNAMIC SYSTEMS DESCRIBED BY ORDINARY DIFFERENCE EQUATIONS WITH VARIABLE AND CONSTANT PARAMETERS

The variational method of calculation of sensitivity coefficients connecting first variation of quality functionals with variations of variable and constant parameters for multivariate non-linear dynamic systems described by ordinary difference equations is developed. Sensitivity coefficients are components of sensitivity functionals and they are before a variations of variable and constant parameters. The base of calculation is the decision of corresponding difference conjugate equations for Lagrange's multipliers in the opposite direction of discrete time.
Keywords: variational method; sensitivity coefficient; difference equation; conjugate equation; Lagrange's multiplier.

The sensitivity functional connect the first variation of quality functional with variations of variable and constant parameters and the sensitivity coefficients (SC) are components of vector gradient from quality functional according to parameters.

The problem of calculation of SC for dynamic systems is principal in the analysis and syntheses of control laws, identification, optimization, stability [1–16]. The first-order sensitivity characteristics are mostly used. Later on we shall examine only SC of the first-order.

Consider a vector output $x(t)$ of dynamic object model under discrete time $t \in [0,1,...,N+1]$ implicitly depending on vectors parameters α and functional $I(\alpha)$ constructed on $x(t)$ under $t \in [0,1,...,N+1]$ and on α:

$$I(\alpha) = \sum_{t=0}^{N+1} f_0(x(t),\alpha,t) .$$

SC with respect to constant parameters α are called a gradient of $I(\alpha)$ on α: $(dI(\alpha)/d\alpha)^T \equiv \nabla_\alpha I(\alpha)$. SC are a coefficients of single-line relationship between the first variation of functional $\delta_\alpha I(\alpha)$ and the variations $\delta\alpha$ of constant parameters α:

$$\delta_\alpha I(\alpha) = (\nabla_\alpha I(\alpha))^T \delta\alpha = \frac{dI(\alpha)}{d\alpha}\delta\alpha \equiv \sum_{j=1}^{m} \frac{\partial I(\alpha)}{\partial \alpha_j}\delta\alpha_j .$$

The direct method of SC calculation (by means of the differentiation of quality functional with respect to constant parameters) inevitably requires a solution of cumbersome sensitivity equations to sensitivity functions $W(t)$. $W(t)$ is the matrix of single-line relationship of the first variation of dynamic model output with parameter variations: $\delta x(t) = W(t)\delta\alpha$. For instance, for functional $I(\alpha)$ we have following SC vector (row vector):

$$\frac{dI(\alpha)}{d\alpha} = \sum_{t=0}^{N+1}\left[\frac{\partial f_0(x(t),\alpha,t)}{\partial x(t)} W(t) + \frac{\partial f_0(x(t),\alpha,t)}{\partial \alpha}\right].$$

For obtaining the matrix $W(t)$ it is necessary to decide a bulky system equations – sensitivity equations. The j-th column of matrix $W(t)$ is made of the sensitivity functions $dx(t)/d\alpha_j$ with respect to component α_j of vector α. They satisfy a vector equation (if x is a vector) resulting from dynamic model (for x) by derivation on a parameter α_j.

For variable parameters such method essentially becomes complicated and practically is not applicable.

At a choice of good initial constant parameters at identification of objects and also at consecutive calculation of control actions on object often apply a gradient algorithms. It appears that for calculation of components of a gradient from an optimized functional to required variables and constant parameters, it is convenient to apply the conjugate equations (in relation to the dynamic equations of object).

Variational method [6], ascending to Lagrange's, Hamilton's, Euler's memoirs, makes possible to simplify the process of determination of conjugate equations and formulas of account of SC. On the basis of this method it is an extension of quality functional by means of inclusion into it dynamic equations of object by means of Lagrange's multipliers and obtaining the first variation of extended functional on phase coordinates of object and on interesting parameters. Dynamic equations for Lagrange's multipliers are obtained due to set equal to a zero (in the first variation of extended functional) the functions before the first variations of phase coordinates. Given simplification first variation of extended functional brings at presence in the right part only parameter variations, i.e. it is got the sensitivity functional concerning parameters.

In difference from other papers devoted to calculation of SC in given paper the generalized difference models are used. Thus variables and constant parameters enter into the right parts of ordinary difference equations of dynamic object, in an indicator of quality of system work, in the measuring device model and initial values of phase coordinates depend on constant parameters. It is proved that both methods to calculation of SC (with use of Lagrange's functions or with use of sensitivity functions) yield the same result, but the first method it is essential more simple in the computing relation.

1. Problem statement

We suppose that the dynamic object is described by system of non-linear ordinary difference equations [13]

$$x(t+1) = f(x(t), \tilde{\alpha}(t), \alpha, t), \;\; t = 0, 1, 2, \ldots, N, \;\; x(0) = x_0(\alpha) . \tag{1}$$

Here: $\tilde{\alpha}(t), \alpha$ are a vector-columns of interesting variable and constant parameters; x is a vector-column of phase coordinates; $f(\cdot)$ is known continuously differentiated limited vector-functions.

The quality of functioning of system it is characterised of functional

$$I(\tilde{\alpha}, \alpha) = \sum_{t=0}^{N} f_0(x(t), \tilde{\alpha}(t), \alpha, t) + f_0(x(N+1), \tilde{\alpha}(N+1), \alpha, N+1) \tag{2}$$

depending on $\tilde{\alpha}(t)$ and α. The conditions for function $f_0(\cdot), I(\cdot)$ are the same as for $f(\cdot)$. With use of a functional (2) the optimization problem (in the theory of optimal control) are named as the Bolts's problem. From it as the individual variants follow: Lagrange's problem (when there is only the first group of the summands) and Mayer's problem (when there is only the last summand – function from phase coordinates at a finishing point).

With the purpose of simplification of appropriate deductions with preservation of a generality in all transformations (1), (2) there are two vectors of parameters $\tilde{\alpha}(t), \alpha$. If in the equations (1), (2) parameters are different then it is possible formally to unit them in two vectors $\tilde{\alpha}(t), \alpha$, to use obtained outcomes and then to make appropriate simplifications, taking into account a structure of a vectors $\tilde{\alpha}(t), \alpha$.

It shown also that the variation method without basic modifications allows to receive SC in relation to variable and constant parameters:

$$\delta I(\tilde{\alpha}(t), \alpha) = \sum_{t=0}^{N+1} \frac{\partial I(\tilde{\alpha}(t), \alpha)}{\partial \tilde{\alpha}(t)} \delta\tilde{\alpha}(t) + \frac{\partial I(\tilde{\alpha}(t), \alpha)}{\partial \alpha} \delta\alpha . \tag{3}$$

$$\nabla_{\tilde{\alpha}(t)} I(\tilde{\alpha}, \alpha) = \left(\frac{\partial I(\tilde{\alpha}, \alpha)}{\partial \tilde{\alpha}_1(t)} \;\; \cdots \;\; \frac{\partial I(\tilde{\alpha}, \alpha)}{\partial \tilde{\alpha}_{m_1}(t)} \right)^{\mathrm{T}}, \;\; t = 0, 1, 2, \ldots, N, N+1,$$

$$\nabla_{\alpha} I(\tilde{\alpha}, \alpha) = \left(\frac{\partial I(\tilde{\alpha}, \alpha)}{\partial \alpha_1} \;\; \cdots \;\; \frac{\partial I(\tilde{\alpha}, \alpha)}{\partial \alpha_{m_2}} \right)^{\mathrm{T}} .$$

By obtaining of results the obvious designations:

$$f(t) \equiv f(x(t), \tilde{\alpha}(t), \alpha, t), \; t = 0, 1, 2, \ldots, N, \tag{4}$$
$$f_0(t) \equiv f_0(x(t), \tilde{\alpha}(t), \alpha, t), \; t = 0, 1, 2, \ldots, N+1$$

are used.

The index t in functions $f(x(t), \tilde{\alpha}(t), \alpha, t)$ and $f_0(x(t), \tilde{\alpha}(t), \alpha, t)$ also reflects not only obvious dependence on step number, but also that the kind of functions from a step to a step can change.

Let's receive the conjugate equations for calculation of Lagrange's multipliers and on the basis of them formulas for calculation of SC.

2. Conjugate equations

To an initial indicator of optimality $I(\tilde{\alpha}, \alpha)$ the dynamic equations (1) (written down in the form of restrictions of type of equalities) by means of Lagrange's multipliers $\lambda(t)$ are added. The size of the expanded indicator of optimality always coincides with size initial functional on which judge an optimality of work of system. SC for both functionals coincide also – section 4 of given paper see. Then for the received expanded indicator the first variation leaves and the dynamic equations for $\lambda(t)$ from an additional condition that factors before variations of phase coordinates $\delta x(N+1), \ldots, \delta x(1)$ addressed in a zero are worked out. Factors before variations of parameters $\delta\tilde{\alpha}(t), \delta\alpha$ in the first variation of the expanded indicator I represent required SC (3).

Complement a quality functional (2) by restrictions-equalities (1) by means of Lagrange's multipliers $\lambda(t), \; t = 0, 1, 2, \ldots, N+1$ (column vectors) and get the expended functional

$$I = I(\tilde{\alpha}, \alpha) + \lambda^{\mathrm{T}}(0)[-x(0) + x_0(\alpha)] + \sum_{t=0}^{N} \lambda^{\mathrm{T}}(t+1)\left[-x(t+1) + f(t)\right] =$$
$$= \left[-\lambda^{\mathrm{T}}(N+1)x(N+1) + f_0(N+1)\right] +$$
$$+ \sum_{0}^{t=N} \left[-\lambda^{\mathrm{T}}(t)x(t) + \lambda^{\mathrm{T}}(t+1) f(t) + f_0(t)\right] + \lambda^{\mathrm{T}}(0) x_0(\alpha), \tag{5}$$

which complies with $I(\tilde{\alpha}, \alpha)$ when (1) is fulfilled.

We calculate the first variation of extended functional, caused by a variation of phase coordinates, and also a variation of variables and constant parameters:

$$\delta I = \sum_{t=0}^{N+1} \frac{\partial I}{\partial x(t)} \delta x(t) + \sum_{t=0}^{N+1} \frac{\partial I}{\partial \tilde{\alpha}(t)} \delta\tilde{\alpha}(t) + \frac{\partial I}{\partial \alpha} \delta\alpha. \tag{6}$$

The factors standing in the formula (6) before variations of phase coordinates look like:

$$\frac{\partial I}{\partial x(N+1)} = -\lambda^T(N+1) + \frac{\partial f_0(N+1)}{\partial x(N+1)}, \tag{7}$$
$$\frac{\partial I}{\partial x(t)} = -\lambda^T(t) + \lambda^T(t+1)\frac{\partial f(t)}{\partial x(t)} + \frac{\partial f_0(t)}{\partial x(t)}, \; t = N, N-1, \ldots, 1, 0.$$

We equate to their zero and it is received the conjugate equations for Lagrange's multipliers:

$$\lambda^T(N+1) = \frac{\partial f_0(N+1)}{\partial x(N+1)}, \tag{8}$$
$$\lambda^T(t) = \lambda^T(t+1)\frac{\partial f(t)}{\partial x(t)} + \frac{\partial f_0(t)}{\partial x(t)}, \; t = N, N-1, \ldots, 1, 0,$$

These equations are decided in the opposite direction changes of an independent integer variable t.

3. Sensitivity coefficients

In the equation (6) SC for variables and constant parameters look like:

$$\frac{\partial I}{\partial\tilde{\alpha}(N+1)}=\frac{\partial f_0(N+1)}{\partial\tilde{\alpha}(N+1)}, \tag{9}$$

$$\frac{\partial I}{\partial\tilde{\alpha}(t)}=\lambda^T(t+1)\frac{\partial f(t)}{\partial\tilde{\alpha}(t)}+\frac{\partial f_0(t)}{\partial\tilde{\alpha}(t)},\ t=N,N-1,\ ...\ ,1,0,$$

$$\frac{\partial I}{\partial\alpha}=\frac{\partial f_0(N+1)}{\partial\alpha}+\sum_{t=0}^{N}\left[\lambda^T(t+1)\frac{\partial f(t)}{\partial\alpha}+\frac{\partial f_0(t)}{\partial\alpha}\right]+\lambda^T(0)\frac{dx_0(\alpha)}{d\alpha}.$$

This result is more common in relation to appropriate results of monograph [13].

4. Equivalence of sensitivity coefficient for initial (2) and expended (5) functionals

We take expanded functional, presented in an initial part of the formula (5):

$$I=I(\tilde{\alpha},\alpha)+\lambda^T(0)[-x(0)+x_0(\alpha)]+\sum_{t=0}^{N}\lambda^T(t+1)\big[-x(t+1)+f(t)\big].$$

Before $\lambda^T(\cdot)$ in square brackets there are the dynamic equations of the object which has been written down in the form of the equation of equality type. Hence, values of functions in square brackets are always equal to zero.

Let's calculate from both parts of the previous equation derivatives in the beginning on a vector of constant parameters α :

$$\frac{\partial I}{\partial\alpha}=\frac{\partial I(\tilde{\alpha},\alpha)}{\partial\alpha}+\lambda^T(0)\left[-W_\alpha(0)+\frac{dx_0(\alpha)}{d\alpha}\right]+$$

$$+\sum_{t=0}^{N}\lambda^T(t+1)\left[-W_\alpha(t+1)+\frac{\partial f(t)}{\partial x(t)}W_\alpha(t)+\frac{\partial f(t)}{\partial\alpha}\right].$$

Before $\lambda^T(\cdot)$ now there are sensitivity equations for a matrix of sensitivity functions. These equations are written down as in the form of restriction of equality type. Values of functions in square brackets also are always equal to zero.

Hence, SC rather both for initial functional and for its expanded variant have identical values.

That the sensitivity equation had the specified appearance, it is necessary (1) to impose a condition of differentiability of $f(t)$ on phase coordinates and on considered parameters on the right member of equation of movement of dynamic object (1).

We receive the same result and for SC in relation to variable parameters. The sensitivity equations for each fixed value of argument of variable parameters $\tilde{\alpha}(j),\ j=0,1,...,N+1$ look like:

$$W_{\tilde{\alpha}(j)}(t+1)=\frac{\partial f(t)}{\partial x(t)}W_{\tilde{\alpha}(j)}(t),\ t=j+1,j+2,...,N;$$

$$W_{\tilde{\alpha}(j)}(j+1)=\frac{\partial f(j)}{\partial\tilde{\alpha}(j)};\quad W_{\tilde{\alpha}(j)}(t)=0,\ t=0,1,...,j;$$

$$j=0,1,...,N-1.$$

At $j=N$

$$W_{\tilde{\alpha}(N)}(N+1)=\frac{\partial f(N)}{\partial\tilde{\alpha}(N)};\quad W_{\tilde{\alpha}(N)}(t)=0,\ t=0,1,...,N.$$

5. Example

We suppose that directed by a problem (1), (2) there are only constant parameters. These parameters are initial values of phase coordinates in the equation (1). Then

$$f(t) \equiv f(x(t),t),\ t = 0, 1, 2, \ldots, N, \quad x_0(\alpha) = \alpha, \tag{10}$$

$$f_0(t) \equiv f_0(x(t),t),\ t = 0, 1, 2, \ldots, N+1.$$

Let's calculate a vector of SC to parameters α. As $f(t)$ and $f_0(t)$ do not depend on α, that $\partial I/\partial\alpha \equiv dI/d\alpha = \lambda^T(0)$. From the decision of the conjugate equations (8) it is received SC:

$$\frac{dI}{d\alpha} = \lambda^T(0) = \frac{\partial f_0(N+1)}{\partial x(N+1)} \frac{\partial f(N)}{\partial x(N)} \cdots \frac{\partial f(0)}{\partial x(0)} + \frac{\partial f_0(N)}{\partial x(N)} \frac{\partial f(N-1)}{\partial x(N-1)} \cdots \frac{\partial f(0)}{\partial x(0)} + \cdots +$$

$$+ \frac{\partial f_0(1)}{\partial x(1)} \frac{\partial f(0)}{\partial x(0)} + \frac{\partial f_0(0)}{\partial x(0)}.$$

For check of correctness of the decision we calculate SC with use of sensitivity functions: $dx(t)/d\alpha \equiv W_\alpha(t)$:

$$\frac{dI(\alpha)}{d\alpha} = \sum_{t=0}^{N+1} \frac{\partial f_0(t)}{\partial x(t)} W_\alpha(t).$$

Sensitivity functions satisfy to the equations:

$$W_\alpha(t+1) = \frac{\partial f(t)}{\partial x(t)} W_\alpha(t),\ t = 0, 1, 2, \ldots, N, W(0) = E.$$

Here E is an unitary matrix. The same result is received.

SC $dI/d\alpha$ (with use expanded functional and Lagrange’s functions) and $dI(\alpha)/d\alpha$ (with use initial functional and sensitivity functions) coincide. With increase in dimension of a vector α the first approach has essential computing advantages.

6. The account of the measuring device model

At additional use of model of the measuring device it is necessary to make changes to problem statement:

$$f_0(t) \equiv f_0(\eta(t), \tilde{\alpha}(t), \alpha, t),\ t = 0, 1, 2, \ldots, N+1;$$

$$\eta(t) \equiv \eta(x(t), \tilde{\alpha}(t), \alpha, t),\ t = 0, 1, 2, \ldots, N+1.$$

In the received results it is necessary to execute small replacements:

$$\frac{\partial f_0(t)}{\partial x(t)} \text{ to replace by } \frac{\partial f_0(t)}{\partial \eta(t)} \frac{\partial \eta(t)}{\partial x(t)};$$

$$\frac{\partial f_0(t)}{\partial \alpha(t)} \text{ to replace by } \frac{\partial f_0(t)}{\partial \alpha(t)} + \frac{\partial f_0(t)}{\partial \eta(t)} \frac{\partial \eta(t)}{\partial \alpha(t)};$$

$$\frac{\partial f_0(t)}{\partial \alpha} \text{ to replace by } \frac{\partial f_0(t)}{\partial \alpha} + \frac{\partial f_0(t)}{\partial \eta(t)} \frac{\partial \eta(t)}{\partial \alpha}.$$

Conclusion

Variational method is applicability for calculation of SC of multivariate non-linear dynamic systems described by ordinary difference equations. Variables and constant parameters are present at object model, at model of the measuring device and at generalized quality functional for system (the Bolts's problem).

In a basis of calculation of SC the decision of the difference equations of object model in a forward direction of time and obtained difference equations for Lagrange's multipliers in the opposite direction of time.

It is proved that both methods to calculation of SC (with use of Lagrange's functions or with use of sensitivity functions) yield the same result, but the first method it is essential more simple in the computing relation.

Variation method of calculation of SC allows to generalize it for objects described by vectorial difference equations with distributed memory for phase coordinates and variables parameters.

Results of present paper are applicable at design of high-precision systems and devices.

This paper continues research in [13, 16].

REFERENCES

1. Ostrovskiy, G.M. & Volin, Yu.M. (1967) *Methods of optimization of chemical reactors*. Moscow: Khimiya.
2. Rosenvasser, E.N. & Yusupov, R.M. (1969) *Sensitivity of Automatic Control Systems.* Leningrad: Energiya.
3. Krutyko, P.D. (1969) The decision of a identification problem by a sensitivity theory method. *News of Sciences Academy of the USSR. Technical Cybernetics.* 6. pp. 146–153.
4. Petrov, B.N. & Krutyko, P.D. (1970) Application of the sensitivity theory in automatic control problems. *News of Sciences Academy of the USSR. Technical Cybernetics.* 2. pp. 202–212.
5. Gorodetskiy, V.I., Zacharin, F.M., Rosenvasser, E.N. & Yusupov, R.M. (1971) *Methods of Sensitivity Theory in Automatic Control.* Leningrad: Energiya.
6. Bryson, A.E. & Ho, Ju-Chi. (1972) *Applied Theory of Optimal Control*. Moscow: Mir.
7. Speedy, C.B., Brown, R.F. & Goodwin, G.C. (1973) *Control Theory: Identification and Optimal Control*. Moscow: Mir.
8. Gekher, K. (1973) *Theory of Sensitivity and Tolerances of Electronic Circuits*. Moscow: Sovetskoe radio.
9. Ruban, A.I. (1975) *Nonlinear Dynamic Objects Identification on the Base of Sensitivity Algorithm.* Tomsk: Tomsk State University.
10. Rosenvasser, E.N. & Yusupov, R.M. (eds) (1977) *Cybernetics Problems. Sensitivity Theory and its Application*. Vol. 23. Moscow: Svyaz.
11. Voronov, A.A. (1979) *Stability, Controllability, Observability*. Moscow: Nauka.
12. Rosenvasser, E.N. & Yusupov, R.M. (1981) *Sensitivity of Control Systems*. Moscow: Nauka.
13. Ruban, A.I. (1982) *Identification and Sensitivity of Complex Systems*. Tomsk: Tomsk State University.
14. Haug, E.J., Choi, K.K. & Komkov, V. (1988) *Design Sensitivity Analysis of Structural Systems*. Moscow: Mir.
15. Afanasyev, V.N., Kolmanovskiy, V.B. & Nosov, V.R. (1998) *The Mathematical Theory of Designing of Control Systems.* Moscow: Vysshaya shkola.
16. Rouban, A.I. (2019) The sensitivity functionals in the Bolts's problem for multivariate dynamic systems described by integral equations with delay time. *Vestnik Tomskogo gosudarstvennogo universiteta. Upravlenie vychislitelnaja tehnika i informatika – Tomsk State University Journal of Control and Computer Science*. 46. pp. 83–92. DOI: 10.17223/19988605/46/10

Received: February, 19, 2021

Rouban A.I. (2021) THE SENSITIVITY COEFFICIENTS FOR MULTIVARIATE DYNAMIC SYSTEMS DESCRIBED BY ORDINARY DIFFERENCE EQUATIONS WITH VARIABLE AND CONSTANT PARAMETERS. *Vestnik Tomskogo gosudarstvennogo universiteta. Upravlenie, vychislitelnaja tehnika i informatika* [Tomsk State University Journal of Control and Computer Science]. 55. pp. 65–70

DOI: 10.17223/19988605/55/8

Рубан А.И. КОЭФФИЦИЕНТЫ ЧУВСТВИТЕЛЬНОСТИ ДЛЯ МНОГОМЕРНЫХ ДИНАМИЧЕСКИХ СИСТЕМ, ОПИСЫВАЕМЫХ ОБЫКНОВЕННЫМИ РАЗНОСТНЫМИ УРАВНЕНИЯМИ С ПЕРЕМЕННЫМИ И ПОСТОЯННЫМИ ПАРАМЕТРАМИ. *Вестник Томского государственного университета. Управление, вычислительная техника и информатика.* 2021. № 55. С. 65–70

Вариационный метод применен для расчета коэффициентов чувствительности, которые связывают первую вариацию функционалов качества работы систем с вариациями переменных и постоянных параметров для многомерных нелинейных динамических систем, описываемых обобщенными разностными уравнениями и обобщенным функционалом качества работы системы (функционалом Больца).

Ключевые слова: вариационный метод; коэффициент чувствительности; разностное уравнение, функционал качества работы системы; сопряженное уравнение; множитель Лагранжа.

ROUBAN Anatoly Ivanovich (Doktor of Technical Sciences, Professor of Computer Science Department of Institute of Space and Information Technologies, Siberian Federal University, Krasnoyarsk, Russian Federation).
E-mail: ai-rouban@mail.ru