УДК 519.174.7 DOI 10.17223/20710410/54/5

THE PALETTE INDEX OF SIERPIŃSKI TRIANGLE GRAPHS AND SIERPIŃSKI GRAPHS

A. Ghazaryan

Yerevan State University, Yerevan, Armenia

E-mail: ghazaryan.aghasi@gmail.com

The palette of a vertex v of a graph G in a proper edge coloring is the set of colors assigned to the edges which are incident to v. The palette index of G is the minimum number of palettes occurring among all proper edge colorings of G. In this paper, we consider the palette index of Sierpiński graphs S_p^n and Sierpiński triangle graphs $\widehat{S}_3^n$. In particular, we determine the exact value of the palette index of Sierpiński triangle graphs. We also determine the palette index of Sierpiński graphs S_p^n where p is even, $p = 3$, or $n = 2$ and $p = 4l + 3$.

Keywords: *palette index, Sierpiński triangle graph, Sierpiński graph.*

ОБ ИНДЕКСЕ ПАЛИТРЫ ТРЕУГОЛЬНИКА СЕРПИНСКОГО И ГРАФА СЕРПИНСКОГО

А. Газарян

Ереванский государственный университет, г. Ереван, Армения

Палитра вершины v графа G в правильной раскраске рёбер — это набор цветов, присвоенных рёбрам, инцидентным v. Индекс палитры G — это минимальное количество палитр из всех правильных рёберных раскрасок G. Рассматриваются индексы палитры графов Серпинского S_p^n и треугольников Серпинского $\widehat{S}_3^n$. Доказано, что индекс палитры треугольника Серпинского $\widehat{S}_3^n$ равен 3, если n чётное, и 4 иначе; индекс палитры графа S_p^n равен 2 для чётного p и равен 3 для $p = 3$ или $n = 2$ и $p = 4l + 3$.

Ключевые слова: *индекс палитры графа, треугольник Серпинского, граф Серпинского.*

1. Introduction

In this paper, we use the standard notations of graph theory [1]. Graph coloring problems are even more challenging when there are some constraints on them, such as proper edge or proper vertex colorings. Usually, such constraints are naturally motivated by different applications in scheduling theory.

In [2], a new chromatic parameter is called the *palette index* of a graph and is defined as follows: for a given proper edge coloring of a graph G, we define the *palette* of a vertex $v \in V(G)$ as the set of all colors appearing on edges incident to v. The palette index $\check{s}(G)$ of G is the minimum number of distinct palettes occurring in a proper edge coloring of G.

Mainly, the palette index was studied for regular graphs. In [2], it is shown that the palette index of a regular graph is 1 if and only if the graph is of Class 1, and it is different from 2. The palette index of complete graphs is also determined in [2].

Theorem 1 [2]. For every positive integer $n > 1$, we have

$$\check{s}(K_n) = \begin{cases} 1, & \text{if } n \text{ is even}, \\ 3, & \text{if } n = 4l+3, \\ 4, & \text{if } n = 4l+1. \end{cases}$$

In [2], the authors also studied the palette index of cubic graphs. More specifically,

$$\check{s}(G) = \begin{cases} 1, & \text{if } G \text{ is of Class 1}, \\ 3, & \text{if } G \text{ is of Class 2 and has a perfect matching}, \\ 4, & \text{if } G \text{ is of Class 2 and has no perfect matching}. \end{cases}$$

As mentioned in [3], the palette index of a d-regular graph of Class 2 satisfies the inequality $3 \leqslant \check{s}(G) \leqslant d+1$.

The paper [3] investigates 4-regular graphs and proves that $\check{s}(G) \in \{3,4,5\}$ if G is 4-regular and of Class 2, and that all these values are, in fact, attained.

Since the computing of the chromatic index of cubic graphs is NP-complete [4], determining the palette index of a given graph is also NP-complete, even for cubic graphs [2]. This means that even determining, if a given graph has a palette index 1, is an NP-complete problem.

Vizing's edge coloring theorem yields an upper bound for the palette index of a general graph G with the maximum degree Δ, namely $\check{s}(G) \leqslant 2^{\Delta+1} - 2$. It is not hard to construct graphs whose palette index is quite smaller than $2^{\Delta+1} - 2$. In [5], an infinite family of multigraphs is described, whose palette index grows asymptotically as Δ^2; however, it is an open question whether there are such examples without multiple edges. Furthermore, in [5] it is conjectured that there is a polynomial $p(\Delta)$ so that for any graph with the maximum degree Δ, it holds the bound $\check{s}(G) \leqslant p(\Delta)$.

There are few results about the palette index of non-regular graphs. In [6], M. Horňák and J. Hudák have completely determined the palette index of the complete bipartite graphs $K_{a,b}$ with $\min\{a,b\} \leqslant 5$.

In [7], C. Casselgren and P. Petrosyan studied the palette index of bipartite graphs. In particular, they have determined the exact value of the palette index of grids and characterized the class of graphs whose palette index equals the number of vertices.

In [8], the palette index of trees is considered. In particular, the authors have proved that $\check{s}(T) \leqslant \sum\limits_{i=1}^{\Delta} \left\lceil \frac{\Delta}{i} \right\rceil$. Moreover, they also have showed the sharpness of the bound by constructing trees T^{Δ} for which the palette index is is equal to this value.

One of the fascinating graph families is *Sierpiński graphs* S_p^n. It has been introduced in [9] as a result of the study of the topological properties of the Lipscomb space, and it is still being studied extensively. An interesting fact is that S_3^n is isomorphic to the graph of the Tower of Hanoi with n disks [9].

Sierpiński triangle graphs $\widehat{S}_3^n$ are quite similar to Sierpiński graphs S_3^{n+1} and are obtained from S_3^{n+1} by a finite number of steps. These graphs are usually studied in conjunction with Sierpiński graphs and have quite interesting properties. Sierpiński triangle graphs are fractals of dimension $d = \log 3/\log 2 \approx 1.585$ and have been introduced in [10].

For Sierpiński graphs and other Sierpiński-type graphs, [11] is a good survey.

In [12], S. Klavžar has introduced an explicit labeling of the vertices of $\widehat{S}_3^n$. Also, he has proved that $\widehat{S}_3^n$ is uniquely 3-colorable and S_3^n is uniquely 3-edge-colorable.

In [13], the authors have studied and summarized vertex, edge, and total colorings of the Sierpiński triangle graphs $\widehat{S}_3^n$ and Sierpiński graphs S_p^n.

In [14], some properties of graphs $\widehat{S}_3^n$ are given, including their cycle structure, domination number, and pebbling number.

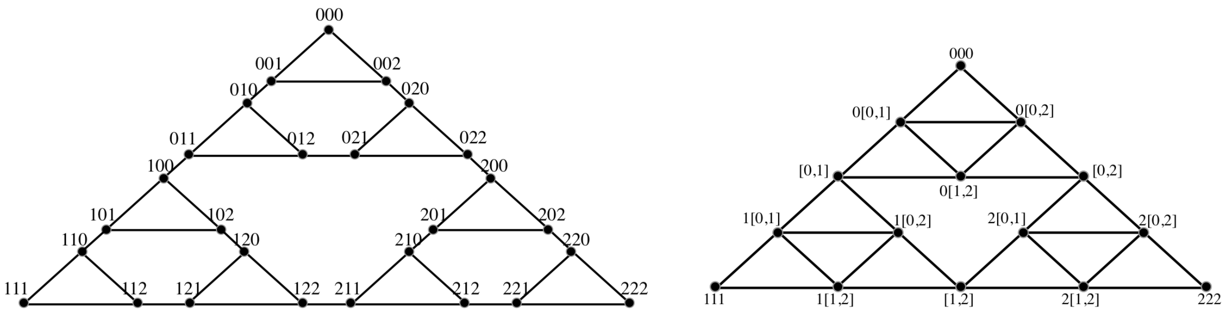
These graphs are beneficial to other theories such as probability theory [15], dynamical systems theory [16], topology [17], etc.

Let us now give explicit definitions of Sierpiński graphs S_p^n and Sierpiński triangle graphs $\widehat{S}_3^n$, where $n \geqslant 0$ and $p > 0$ are integers. The graphs are defined as follows. The vertex set of S_p^n is the set of all n-tuples of integers $0, 1, \ldots, p-1$, namely, $V(S_p^n) = \{0, 1, \ldots, p-1\}^n$. Two different vertices $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ are adjacent if and only if there exists an $h \in \{1, \ldots, n\}$ such that:

1) $u_t = v_t$ for $t = 1, \ldots, h-1$;
2) $u_h \neq v_h$;
3) $u_t = v_h$ and $v_t = u_h$ for $t = h+1, \ldots, n$.

We will denote a vertex $(u_1, u_2, \ldots, u_n)$ by $\langle u_1 u_2 \ldots u_n \rangle$ or even $u_1 u_2 \ldots u_n$. The vertices $\langle i \ldots i \rangle$, $i \in \{0, \ldots, p-1\}$, are called the *extreme vertices* of S_p^n. We will denote by $iS_p^n = S_p^n[\{v : v = \langle i \ldots i \rangle\}]$ the subgraph of S_p^n, where $i = 0, 1, \ldots, p-1$. Obviously, iS_p^{n+1} is isomorphic to S_p^n. Consequently, S_p^n contains p^{n-1} copies of the graph $S_p^1 = K_p$. We will call *link edges* all the edges of S_p^n that do not belong to the above-mentioned K_p.

As a result of contracting all the link edges of S_3^{n+1}, we will get the Sierpiński triangle graph $\widehat{S}_3^n$ where $n \geqslant 0$. We label the vertices of $\widehat{S}_3^n$ as follows. Let $\langle u_1 \ldots u_r ij \ldots j \rangle$ and $\langle u_1 \ldots u_r ji \ldots i \rangle$ be the endvertices of a link edge of S_3^{n+1} that is contracted to a vertex x of $\widehat{S}_3^n$. Then we label x with $\langle u_1 \ldots u_r \rangle [i, j]$ or $u_1 \ldots u_r [i, j]$ where $r \leqslant n-2$. $\widehat{S}_3^{n+1}$ contains three isomorphic copies of $\widehat{S}_3^n$, and we denote these copies with $i\widehat{S}_3^{n+1}$, where $i\widehat{S}_3^{n+1}$ is the subgraph which contains $\langle i \ldots i \rangle$, $0 \leqslant i \leqslant 2$; see Fig. 1.



a *b*

Fig. 1. Labeling of S_3^3 (a) and of $\widehat{S}_3^2$ (b)

In this paper, the palette indices of Sierpiński triangle graphs $\widehat{S}_3^n$ and Sierpiński graphs S_3^n are determined. Next, the palette index of S_p^n is determined, where p is even, or $n = 2$ and $p = 4l + 3$.

2. The palette index of Sierpiński triangle graphs

Let $\widehat{S}_3^n$ be the Sierpiński triangle graph. When $n = 0$, then $\widehat{S}_3^n$ is isomorphic to K_3, whose palette index is determined (Theorem 1). Below we consider the case $n > 0$. The graph has two kinds of vertices in terms of vertex degree: three vertices with degree 2 and the remaining vertices with degree 4. To color $\widehat{S}_3^n$, we need at least four colors as $\Delta = 4$.

Let us define edge coloring functions ϕ_n for the graph $\widehat{S}_3^1$ as follows:

$$\phi_n(e) = \begin{cases} n, & \text{if } e \in \{[0,1]00, [1,2]11, [0,2]22\}, \\ n \bmod 4 + 1, & \text{if } e \in \{[0,1]11, [1,2][0,2]\}, \\ (n+1) \bmod 4 + 1, & \text{if } e \in \{[1,2]22, [0,1][0,2]\}, \\ (n+2) \bmod 4 + 1, & \text{if } e \in \{[0,2]00, [0,1][1,2]\}, \end{cases}$$

where $n \in \{1, 2, 3, 4\}$. Thus, for all colorings ϕ_n, we have 4 different sets of palettes and we call this group A:

1) $\{\{1,2,3,4\}, \{1,2\}, \{1,3\}, \{1,4\}\}$;
2) $\{\{1,2,3,4\}, \{2,1\}, \{2,3\}, \{2,4\}\}$;
3) $\{\{1,2,3,4\}, \{3,1\}, \{3,2\}, \{3,4\}\}$;
4) $\{\{1,2,3,4\}, \{1,4\}, \{2,4\}, \{3,4\}\}$.

We will identify two edge colorings if one is obtained by rotating or reflecting the other. Clearly, if we color $\widehat{S}_3^1$ in a manner that all vertices with cardinality 4 have the palette $\{1, 2, 3, 4\}$, then we will get a coloring ϕ_n. After this, let us color the graph $\widehat{S}_3^2$ so that all 4 degree vertices are colored with the palette $\{1, 2, 3, 4\}$. That means we must use ϕ_n colorings to color $i\widehat{S}_3^2$, where $i = 0, 1, 2$. By considering all possible cases, we have four different sets of palettes and we call that group B:

1) $\{\{1,2,3,4\}, \{1,2\}, \{1,3\}, \{2,3\}\}$;
2) $\{\{1,2,3,4\}, \{1,2\}, \{1,4\}, \{2,4\}\}$;
3) $\{\{1,2,3,4\}, \{1,3\}, \{1,4\}, \{3,4\}\}$;
4) $\{\{1,2,3,4\}, \{2,3\}, \{2,4\}, \{3,4\}\}$.

Proposition 1. For every positive integer n, we have

$$\check{s}(\widehat{S}_3^n) \leqslant 4.$$

Proof. If n is 1 or 2, then we can color $\widehat{S}_3^n$ using a set of palettes from group A or B. If we try to color $\widehat{S}_3^3$ so that each $i\widehat{S}_3^3$ is colored the same as we colored $\widehat{S}_3^2$, where $i = 0, 1, 2$, and the vertices $[0,1], [0,2], [1,2]$ have the palette $\{1, 2, 3, 4\}$, then we will have all the possible sets of palettes of group A. We now prove the following stronger lemma:

Lemma 1. For every positive integer n, we can color $\widehat{S}_3^n$ using every set of palettes from group A, if n is odd, and every set of palettes from group B, if n is even.

As we have seen, the lemma is true for $\widehat{S}_3^n$, where $n = 1, 2, 3$. Assume the lemma holds for $n > 0$. We wish to find a satisfying coloring of $\widehat{S}_3^{n+1}$. By the induction assumption, we can color $\widehat{S}_3^n$ using every set of palettes from group A or B depending on whether n is odd or not. In general, if we have colorings for $i\widehat{S}_3^m$, then the palettes of vertices of degree two of $i\widehat{S}_3^m$ are the only essentials for coloring $\widehat{S}_3^m$, where $i = 0, 1, 2$, $m > 1$. As we saw above in the process of coloring $\widehat{S}_3^2$ ($\widehat{S}_3^3$), if we can color $i\widehat{S}_3^2$ ($i\widehat{S}_3^3$) using every set of palettes from group A (B), then we can color $\widehat{S}_3^2$($\widehat{S}_3^3$) using every set of palettes from group B (A). So we can color $\widehat{S}_3^{n+1}$ by using $\widehat{S}_3^n$ colorings as we did for $\widehat{S}_3^2$ or $\widehat{S}_3^3$. ∎

Proposition 2. For every positive integer n, we have

$$\check{s}(\widehat{S}_3^n) \geqslant 3.$$

Proof. Clearly, $\check{s}(\widehat{S}_3^n) \geqslant 2$, since there are vertices with only two different degrees. If we try to color $\widehat{S}_3^1$ with one palette with cardinality 4 ($\{1, 2, 3, 4\}$), then we will get a coloring ϕ_n. In Proposition 1 for coloring $\widehat{S}_3^n$, we used only colorings ϕ_n for any subgraph $\widehat{S}_3^1$ and tried all possible cases. That means that if we want to have only one palette with cardinality 4, we must use a set of palettes from group A or B, where each set of palettes contains three palettes with cardinality 2. Hence, $\check{s}(\widehat{S}_3^n) \geqslant 3$. ■

Proposition 2 means that if we are looking for a coloring with three palettes, then we should have one palette with cardinality 2 and two palettes with cardinality 4.

Proposition 3. For every positive integer n, if $\check{s}(\widehat{S}_3^n) = 3$, then the palettes for vertices with degree 4 should differ in only one color.

Proof. All possible cases for two palettes with degree 4 are described below:

1) $\{1,2,3,4\}$ and $\{1,2,3,5\}$;
2) $\{1,2,3,4\}$ and $\{1,2,5,6\}$;
3) $\{1,2,3,4\}$ and $\{1,5,6,7\}$;
4) $\{1,2,3,4\}$ and $\{5,6,7,8\}$.

In Fig. 2, we show an example of the coloring of $\widehat{S}_3^2$ with three palettes, where the palettes of cardinality 4 correspond to the first case.

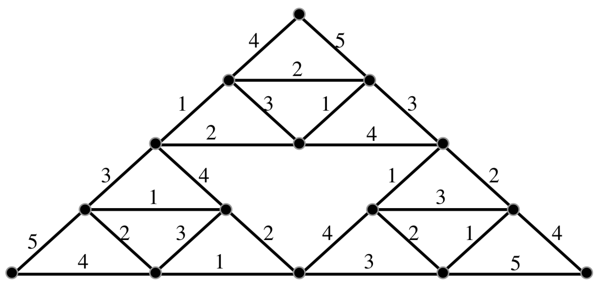


Fig. 2. A coloring of $\widehat{S}_3^2$ with three palettes

We now show that the last three cases are impossible. Suppose we have colored $\widehat{S}_3^n$ with three palettes. If $\check{s}(\widehat{S}_3^n) = 3$, then we have only one palette with cardinality 2. Let's denote that palette by $\{a, b\}$, where $1 \leqslant a < b \leqslant 8$. We will call the colors $5, 6, 7, 8$ the *extra colors*. Now, we change each extra color c with the color $9 - c$ from the set $\{1, 2, 3, 4\}$. The proper edge coloring of $\widehat{S}_3^n$ might be broken only on edges adjacent to the vertices of degree 2 if b is an extra color and $9 - b = a$. But in cases 2–4, we can change the color b with another color different from a. So for the cases 2–4, we can color $\widehat{S}_3^n$ with two palettes, which is a contradiction to Proposition 2. ■

By Proposition 3 and its proof, we may assume that if we have a coloring with three palettes, then the palettes are $\{1,2,3,4\}$, $\{1,2,3,5\}$, and $\{4,5\}$. Moreover, it is possible to use the palette $\{1,2,3,5\}$ exactly three times.

Let us now expose all possible colorings of $\widehat{S}_3^1$, where the palettes $\{1,2,3,4\}$, $\{1,2,3,5\}$, and $\{4,5\}$ are used. After this, we will have this group of sets of palettes, and we will call this group C:

1) $\{\{1,2,3,4\},\{1,2,3,5\},\{4,5\},\{1,2\},\{3,4\}\}$;
2) $\{\{1,2,3,4\},\{1,2,3,5\},\{4,5\},\{1,3\},\{2,4\}\}$;
3) $\{\{1,2,3,4\},\{1,2,3,5\},\{4,5\},\{1,4\},\{2,3\}\}$.

To find colorings with three palettes of $\widehat{S}_3^n$, as a set of palettes of $\widehat{S}_3^1$, it is enough for us to use only sets of palettes of groups A and C, as we can use the palette $\{1,2,3,5\}$ three times.

There is no coloring of $\widehat{S}_3^1$ with three palettes; thus, $\check{s}(\widehat{S}_3^1) = 4$. As shown in Fig. 2, we can color $\widehat{S}_3^2$ by coloring each $i\widehat{S}_3^2$ using a set of palettes from group C, where $i = 0, 1, 2$.

Let us see what sets of palettes we can have for $\widehat{S}_3^2$ by coloring each $i\widehat{S}_3^2$ using a set of palettes from group A and C, where $i = 0, 1, 2$. We have three cases shown in Fig. 3.

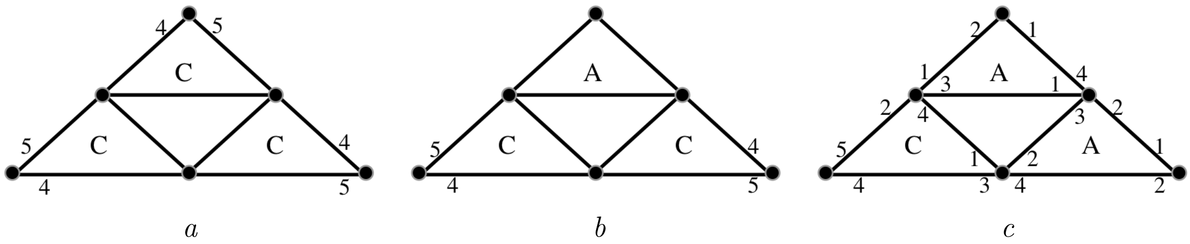


Fig. 3. Cases a, b, c

For the case a, the possible combination is the following: all $i\widehat{S}_3^2$ have the same set of palettes from group C, where $i = 0, 1, 2$, and that gives us a coloring with three palettes. Let us notice that we can not use this coloring of $\widehat{S}_3^2$ for coloring a subgraph of $\widehat{S}_3^n$, where $n > 2$, because we do not have a palette with cardinality 4 that has colors 4 and 5. The case b does not give a coloring with three palettes, and we can not use it for coloring $\widehat{S}_3^n$, where $n > 2$, because there are two vertices with the palette $\{4, 5\}$. The case c gives a new group of sets of palettes, and we will call this group D:

1) $\{\{1,2,3,4\}, \{1,2,3,5\}, \{4,5\}, \{1,2\}\}$;
2) $\{\{1,2,3,4\}, \{1,2,3,5\}, \{4,5\}, \{1,3\}\}$;
3) $\{\{1,2,3,4\}, \{1,2,3,5\}, \{4,5\}, \{1,4\}\}$;
4) $\{\{1,2,3,4\}, \{1,2,3,5\}, \{4,5\}, \{2,3\}\}$;
5) $\{\{1,2,3,4\}, \{1,2,3,5\}, \{4,5\}, \{2,4\}\}$;
6) $\{\{1,2,3,4\}, \{1,2,3,5\}, \{4,5\}, \{3,4\}\}$.

Let us now see what sets of palettes we can have for $\widehat{S}_3^3$ by coloring each $i\widehat{S}_3^3$ using a set of palettes from group B and D, where $i = 0, 1, 2$. We have three cases (Fig. 4).

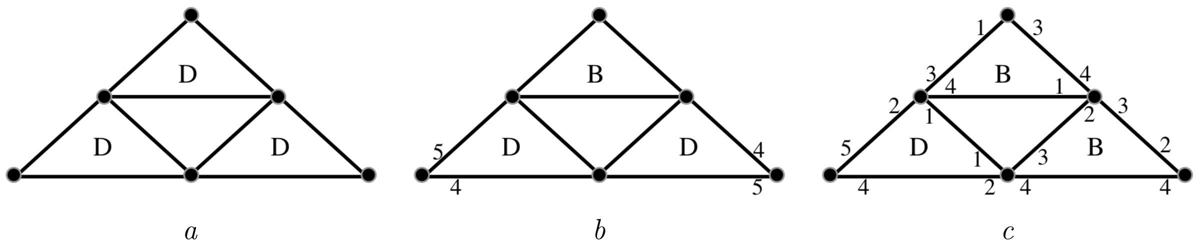


Fig. 4. Cases a, b, c

For the case a, there is no possibility to construct a proper edge coloring. The case b does not give a coloring with three palettes, and we can not use it for coloring $\widehat{S}_3^n$, where $n > 3$, because there are two vertices with the palette $\{4, 5\}$. And finally, the case c gives the same sets of palettes of group C that close the chain. Thus, depending on whether the number $n > 0$ is even or odd, we can color $\widehat{S}_3^n$ with the palettes of any set of groups C or D, respectively.

Theorem 2. For every non-negative integer n, the palette index of $\widehat{S}_3^n$ is determined by the formula

$$\check{s}(\widehat{S}_3^n) = \begin{cases} 3, & \text{if } n \text{ is even,} \\ 4, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Clearly, $\check{s}(\widehat{S}_3^0) = 3$, so we will consider the cases when $n > 0$.

We have shown that we can color $\widehat{S}_3^n$ using any set of palettes from groups A and C if n is odd, and from groups B and D if n is even. Also, we have shown that we can color $\widehat{S}_3^n$ with three palettes if each $i\widehat{S}_3^n$ is colored with palettes of a set of group C, where $i = 0, 1, 2$. Since we have considered all possible cases of coloring $\widehat{S}_3^n$ with three palettes, and it is possible only if $i\widehat{S}_3^n$ is colored using a set of palettes from group C, then $\check{s}(\widehat{S}_3^n) > 3$ for odd n. Consequently, by Proposition 1, $\check{s}(\widehat{S}_3^n) = 4$ if n is odd. ■

3. The palette index of Sierpiński graphs

In this section, we will examine the palette index of Sierpiński graphs S_p^n, where $p > 1$.

If $n = 1$, then S_p^n is isomorphic to K_p, whose palette index was determined (Theorem 1). Now, let us consider the cases when $n > 1$. Here, we have two kinds of vertices in terms of vertex degree: extreme vertices with degree $n - 1$ and the remaining vertices with degree n. So $\check{s}(S_p^n) \geqslant 2$.

Theorem 3. For every even integer $p > 1$ and every integer $n > 1$, we have

$$\check{s}(S_p^n) = 2.$$

Proof. Since $\check{s}(S_p^n) \geqslant 2$, we just need a coloring of S_p^n with two palettes. We color all p^{n-1} copies of $S_p^1 = K_p$ in S_p^n with the palette $\{1, \ldots, p-1\}$. Then we color all link edges with the color p. In this way all extreme vertices have the palette $\{1, \ldots, p-1\}$, and the other vertices have the palette $\{1, \ldots, p\}$. ■

Now, consider the case when p is odd.

Proposition 4. For every odd integer $p > 1$ and every integer $n > 1$, we have

$$\check{s}(S_p^n) \geqslant 3.$$

Proof. In [13, Claim in Theorem 4.1], it has been proved for any integer $n \geqslant 1$ and any odd integer $p > 1$, that $\chi'(S_p^n) = p$ and the palettes of extreme vertices of S_p^n are pairwise different. If we try to color S_p^n with two palettes, then we must use a single palette for coloring vertices with degree p, which means that we use only p colors. So we will have p palettes for extreme vertices [13, Claim in Theorem 4.1]. Hence, $\check{s}(S_p^n) \geqslant 3$. ■

Proposition 5. For every odd integer $p > 1$ and every integer $n > 1$, we have

$$\check{s}(S_p^n) \leqslant \begin{cases} \check{s}(K_p), & \text{if } n = 2, \\ \check{s}(K_p) + 1, & \text{if } n > 2. \end{cases}$$

Proof. As shown in [2], if p is odd, then we can always color K_p with three or four palettes, and there is only one vertex s with a unique palette P_s. For a coloring S_p^2, we color each complete graph iS_p^2, and we keep the vertex s as the extreme vertex ($0 \leqslant i \leqslant p-1$). At this moment, we have these palettes: P_s and $P_1, \ldots, P_m$, where m is 2 or 3. Then we color all link edges with a new color c. Thus, we have these $\check{s}(K_p)$ palettes: P_s and $P_1 \cup \{c\}, \ldots, P_m \cup \{c\}$. To color S_p^3, we color each iS_p^3 as mentioned above, and again, we color uncolored link edges with the same color c. Thus, we have $\check{s}(K_p) + 1$ palettes: P_s and $P_s \cup \{c\}, P_1 \cup \{c\}, \ldots, P_m \cup \{c\}$. To color S_p^n, where $n > 3$, we can do the same steps. In that case, we do not create a new palette. ■

Corollary 1. For every integer p, we have

$$\check{s}(S_{4p+3}^2) = 3.$$

Proof. This result follows from Proposition 4, Proposition 5, and Theorem 1. ■

Theorem 4. For every positive integer n, we have

$$\check{s}(S_3^n) = 3.$$

Proof. By Proposition 4, we obtain $\check{s}(S_3^n) \geqslant 3$. So we just need a coloring with three palettes to prove the theorem. If $n = 1$, then we have K_3, whose palette index is 3. If $n = 2$, we can color S_3^2 as shown in Fig. 5.

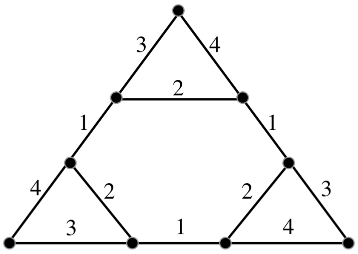


Fig. 5. A coloring of S_3^2 with three palettes

Now, let us consider the two colorings of S_3^2 from Fig. 6.

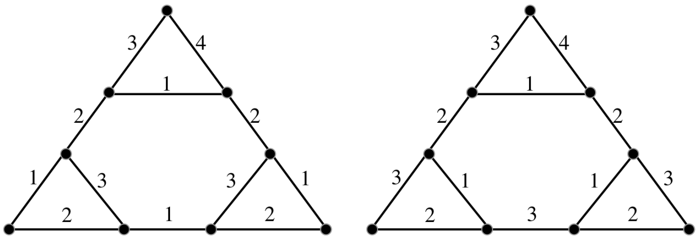
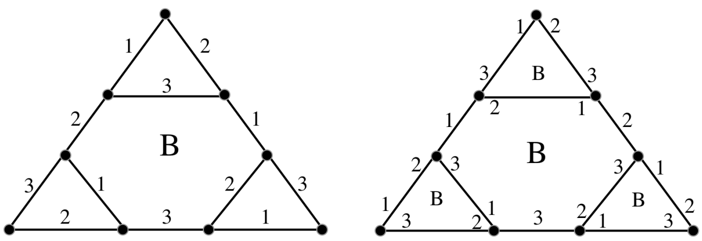


Fig. 6. Group A

Let us denote this group of sets of palettes by A:

1) $\{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 2\}, \{3, 4\}\}$;
2) $\{\{1, 2, 3\}, \{1, 2, 4\}, \{2, 3\}, \{3, 4\}\}$.

Next, let us consider the coloring of S_3^n in Fig. 7.



a *b*

Fig. 7. Group B

Figure 7, a gives a coloring of S_3^2. We denote this single element group of sets of palettes by B:

1) $\{\{1, 2, 3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\}$.

Figure 7, b shows how we can color S_3^n using palettes from group B, where $n > 1$.

In Fig. 8, we can see that, for any integer $n \geqslant 2$, we can color S_3^n using each set of palettes from group A by coloring one of iS_3^n using a set of palettes from group A, two others using a set of palettes from group B, and link edges that connect these iS_3^n with colors $1, 2$ or 3, where $i = 0, 1, 2$. We can also see that for S_3^n we can color it with three palettes by coloring all iS_3^n using a set of palettes from group A, and link edges that connect these iS_3^n with colors 1 or 3, where $i = 0, 1, 2$. The theorem is proved. ■

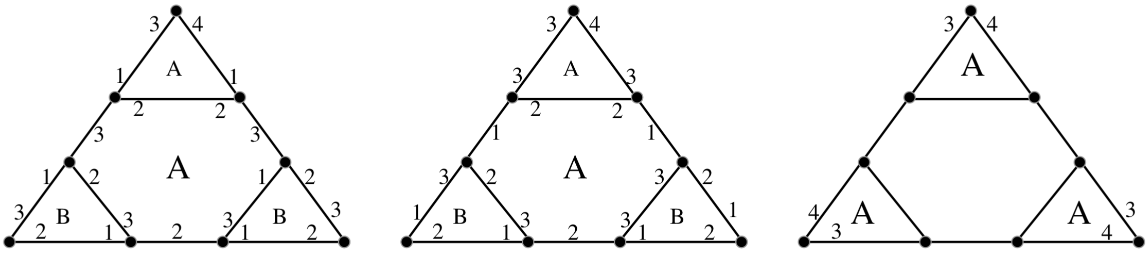


Fig. 8. The coloring of S_3^n with three palettes

Acknowledgement. The author thanks P. A. Petrosyan for many useful comments and suggestions.

REFERENCES

1. *West D.* Introduction to Graph Theory. Prentice-Hall, 2014.
2. *Horňák M., Kalinowski R., Meszka M., and Woźniak M.* Minimum number of palettes in edge colorings. Graphs Combin., 2014, vol. 30, pp. 619–626.
3. *Bonvicini S. and Mazzuoccolo G.* Edge-colorings of 4-regular graphs with the minimum number of palettes. Graphs Combin., 2016, vol. 32, pp. 1293–1311.
4. *Holyer I.* The NP-completeness of edge-coloring. Siam J. Comput., 1981, vol. 10, pp. 718–720.
5. *Avesani M., Bonisoli A., and Mazzuoccolo G.* A family of multigraphs with large palette index. Ars Math. Contemp., 2019, vol. 17, pp. 115–124.
6. *Horňák M. and Hudák J.* On the palette index of complete bipartite graphs. Discussiones Mathematicae Graph Theory, 2018, vol. 38, pp. 463–476.
7. *Casselgren C. and Petrosyan P.* Some results on the palette index of graphs. Discrete Math. Theor. Comput. Sci., 2019, vol. 21, no. 3, A2.
8. *Bonisoli A., Bonvicini S., and Mazzuoccolo G.* On the palette index of a graph: the case of trees. Lecture Notes of Seminario Interdisciplinare di Matematica, 2017, vol. 14, pp. 49–55.
9. *Klavžar S. and Milutinović U.* Graphs $S(n, k)$ and a variant of the Tower of Hanoi problem. Czechoslovak Math. J., 1997, vol. 47, no. 122, pp. 95–104.
10. *Scorer R., Grundy P., and Smith C.* Some binary games. Math. Gaz., 1944, vol. 28, pp. 96–103.
11. *Hinz A., Klavžar S., and Zemljič S.* A survey and classification of Sierpiński-type graphs. Discrete Math., 2017, vol. 217, no. 3, pp. 565–600.
12. *Klavžar S.* Coloring Sierpiński graphs and Sierpiński triangle graphs. Taiwan J. Math., 2008, vol. 12, pp. 513–522.
13. *Jakovac M. and Klavžar S.* Vertex-, edge-, and total-colorings of Sierpiński-like graphs. Discrete Math., 2009, vol. 309, no. 6, pp. 1548–1556.
14. *Teguia A. and Godbole A.* Sierpiński triangle graphs and some of their properties. Australas. J. Combin., 2006, vol. 35, pp. 181–192.

15. *Hinz A. and Schief A.* The average distance on the Sierpiński gasket. Probab. Theory Related Fields, 1990, vol. 87, pp. 129–138.
16. *Kaimanovich V.* Random walks on Sierpiński graphs: hyperbolicity and stochastic homogenization. P. Grabner, W. Woess (eds.) Fractals in Graz 2001. Trends in Mathematics. Birkhäuser, Basel, 2003, pp. 145–183.
17. *Imran M., Hafi S., Gao W., and Farahani M. R.* On topological properties of Sierpiński networks. Chaos Soliton. Fract., 2017, vol. 98, pp. 199–204.