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SERIES OF FAMILIES OF DEGREE SIX CIRCULANT GRAPHS[1]

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An approach for constructing and optimizing graphs of series of analytically described circulant graphs of degree six with general topological properties is proposed. The paper presents three series of families of undirected circulants having the form $C(N(d,p);1,s_2(d,p),s_3(d,p))$, with an arbitrary diameter $d>1$ and a variable parameter $p(d)$, $1\leqslant p(d)\leqslant d$. The orders N of each graph in the families are determined by a cubic polynomial function of the diameter, and generators s_2 and s_3 are defined by polynomials of the diameter of various orders. We have proved that the found series of families include degree six extremal circulant graphs with the largest known orders for all diameters. By specifying the functions $p(d)$, new infinite families of circulant graphs including solutions close to extremal graphs are obtained.

Keywords: *Abelian Cayley graph, degree/diameter problem, families of degree six circulant graphs, triple loop graphs, extremal circulant graphs.*

СЕРИИ СЕМЕЙСТВ ЦИРКУЛЯНТНЫХ ГРАФОВ СТЕПЕНИ ШЕСТЬ

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Предложен подход к построению и оптимизации графов серий аналитически описываемых циркулянтных графов степени шесть с общими топологическими свойствами. Представлены три серии семейств неориентированных циркулянтов вида $C(N(d,p);1,s_2(d,p),s_3(d,p))$ произвольного диаметра $d>1$ с переменным параметром $p(d)$, $1\leqslant p(d)\leqslant d$. Порядки N каждого графа в семействах определяются кубическим полиномом от диаметра, а образующие s_2 — полиномами от диаметра различных порядков. Доказано, что найденные серии семейств включают экстремальные циркулянтные графы степени 6 с самыми большими известными порядками для всех диаметров. Посредством задания функций $p(d)$ построены новые бесконечные семейства циркулянтных графов, включая решения, близкие к экстремальным графам.

Ключевые слова: *граф Кэли абелевой группы, проблема d/k графов, семейства циркулянтных графов степени 6, трёхмерные кольцевые циркулянтные графы, экстремальные циркулянтные графы.*

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1. Introduction

Circulant graphs are Cayley graphs of Abelian groups. They are widely studied in graph theory and discrete mathematics and play an important role in various applications including network design, see surveys [1–4] and references in them.

Let $s_1, s_2, \ldots, s_k, N$ be a set of integers such that $1 \leqslant s_1 < s_2 < \ldots < s_k < N$, and let $S = (s_1, s_2, \ldots, s_k)$ be a generator set. An undirected graph $C(N; S)$ with sets of vertices $V = \{0, 1, \ldots, N-1\}$ and edges $E = \{(v, (v \pm s_l) \bmod N) : v \in V, l = 1, \ldots, k\}$, is called *circulant graph*, k is the dimension, N is the order of the graph. The diameter of C is $\mathrm{dia}(C) = \max\limits_{i,j \in V} D(i,j)$, where $D(i,j)$ is the length of a shortest path from vertex i to vertex j. In the paper, we consider the class of even-degree multi-loop circulants when $s_1 = 1$ and $s_k \neq N/2$ for even N [1–7].

For any given k and d, the function $M(d,k)$ defines the maximum number of vertices that can be reached from any vertex of a circulant graph of dimension k in at most d steps. It is known [8] that

$$M(d,k) = \sum_{i=0}^{k} 2^i \binom{k}{i} \binom{d}{i},$$

where the value of $M(d,k)$ can be viewed as the Moore bound for circulant graphs of dimension k. This is a polynomial in d of order k:

$$M(d,k) = \frac{2^k}{k!} d^k + \frac{2^{k-1}}{(k-1)!} d^{k-1} + O(d^{k-2}).$$

Following [5], we define for any natural d and k the extremal function $P(d,k)$ as maximum possible (attainable) natural number N such that there is a set of generators $S = (1, s_2, \ldots, s_k)$ for which $\mathrm{dia}(C(N;S)) \leqslant d$. We have $P(d,k) = M(d,k)$ for $k \leqslant 2$, namely,

$$P(d,1) = 2d + 1, \quad P(d,2) = 2d^2 + 2d + 1,$$

and $P(d,k) < M(d,k)$ for $k > 2$ [5, 9–11], including

$$P(d,3) < M(d,3) = \frac{4d^3}{3} + 2d^2 + \frac{8d}{3} + 1.$$

It is difficult to obtain the exact value of $P(d,k)$ for $k \geqslant 3$ and large d. This is a solution to the degree/diameter problem [12–14] in the class of circulant networks and it leads to an exhaustive computer search. The lower bounds for $P(d,k)$ in each individual case may depend on the diameter under consideration and are usually obtained by finding infinite families of graphs with these estimates (see also the table of the orders of the largest known circulant graphs [15]). Note that for a given dimension and diameter, circulant graphs with the maximum possible order (or the nearest to it), taken as a model of interconnection networks of multiprocessor systems, have the maximum reliability and connectivity and the minimum number of steps in the implementation of routing algorithms. Using circulant graphs as a network-on-chip (NoC) topology [16, 17] becomes relevant due to their better structural properties and high scalability compared to standard NoC topologies (2D-mesh, 2D-torus). Three-dimensional circulant topologies for NoCs [18, 19] are a promising alternative to the classic 3D-mesh and 3D-torus topologies [20, 21] due to the best characteristics of the diameter and average distance between nodes. An urgent task for NoCs with three-dimensional circulant topology is the development of effective routing algorithms related to the peculiarities of the requirements for the use of chip resources in NoCs.

The following infinite families of degree 6 circulant graphs with an analytical description are known in the literature. The family of circulant graphs of diameter d

$$C(3d^2 + 3d + 1; 1, 3d + 1, 3d + 2), \quad d \geqslant 1,$$

was obtained in [22] as a solution to the optimization problem for hexagonal tessellation on the plane of circulants of the form $C(N; s_1, s_2, s_1 + s_2)$ with diameter d. Note that the family is a subset of the family of Eisenstein — Jacobi graphs [23] introduced for constructing perfect codes. Shortest path search algorithms for this family were given in [24, 25]. In [26], the family of three-dimensional circulants has been found as a solution to the optimization problem when considering the Kronecker product of two circulants of degrees 2 and 3. The family has diameter $d \equiv 3, 5 \pmod 6$ and order $N = 4d^2 - 2d - 2$.

Families of the so-called multiplicative circulant graphs of dimension $k \geqslant 3$ have been obtained in [5, 6, 8, 27]. In [6, 27], the diameter and shortest path search algorithms were also given for the families. Below for $k = 3$, we give the orders of graphs of these families represented as a function of diameter d; for diameter $d \geqslant 3$, $d \equiv 0 \pmod 3$ [8, 27]:

$$N = \left(\frac{2d + 3}{3}\right)^3 = \frac{8}{27}d^3 + O(d^2);$$

for diameter $d \geqslant 5$, $d \equiv 2 \pmod 3$ [6]:

$$N = \left(\frac{2d + 2}{3}\right)^3 = \frac{8}{27}d^3 + O(d^2);$$

and for diameter $d \geqslant 3$, $d \equiv 0 \pmod 3$ [5]:

$$N = \frac{32}{27}d^3 + \frac{8}{9}d^2 + \frac{2}{3}d.$$

The families of the largest known degree 6 circulant graphs (extremal circulants) were discovered by E. A. Monakhova [9] and in [28] (for Cayley graphs of Abelian groups).

For dimension $k > 3$, R. R. Lewis have recently obtained families of the largest known circulant graphs of degrees 8 and 10 with an analytical description [10, 11]. Formulas for the order N in diameter $d \geqslant 3$ of degree 8 graphs are shown below:

$$N = \begin{cases} d^4/2 + d^3 + 3d^2 + 2d & \text{for} \quad d \equiv 0 \pmod 2, \\ d^4/2 + d^3 + 3d^2 + 3d + 1/2 & \text{for} \quad d \equiv 1 \pmod 2. \end{cases}$$

As we can see, only individual families of circulants have been obtained earlier, and their topological properties and the possibility of obtaining good routing algorithms for some of them have been studied.

In [29], the author introduced the concept of a series of families of three-dimensional circulant graphs defined by two parameters, one of which is their common diameter and the second is a function of the diameter. This paper is an extension of the previous one. We introduce three series of parametrically described infinite families of circulant graphs that differ in the set of graph generators. We prove that these series include extremal triple loop graphs with the maximum number of vertices for any diameter and have common topological and communicative properties (Sections 2–4). This result makes it possible to create previously unknown infinite families of dense circulant graphs with varying the diameter and also to construct a series of d graphs for a fixed diameter $d > 1$. Section 5 presents examples of constructing new infinite families of triple loop networks in all the series based on specifying functions $p = p(d)$. The results of constructing the three series of circulants are presented in Section 6 and the Appendix. Another important property is the existence of a general structure of the graphs of the found families, which leads to the construction of a general analytical method for finding the shortest paths and routing algorithm for them.

2. A series of circulants with generators $s_2 = f(d)$

Consider a set of circulant graphs of the form $C(N; 1, s_2, s_3)$ with $s_1 = 1$ and $1 < s_2 < < s_3 \leqslant \lfloor N/2 \rfloor$. Denote $\Delta = s_3 - s_2$. We place the vertices of a graph C on the line, as shown in Fig. 1 and 2. They are labeled from 0 to N (the vertex N is understood to be the same vertex as 0). Define $+s_3$- and $-s_3$-jumps from i, if we travel to $(i + s_3) \bmod N$ or $(i - s_3) \bmod N$, respectively. Similarly, define $+s_2$- and $-s_2$-jumps and $+1$- and -1-jumps. In the paper, we also use $\pm\Delta$-jumps of length 2. Since the circulant graphs are vertex transitive, it is sufficient to consider 0 as the initial vertex. For simplicity, we will use the notation $D(v)$ instead of $D(0, v)$.

The following slightly modified Lemma from [29] is used in the proofs of Theorems.

Lemma 1 [29]. In a triple loop graph $C(N; 1, s_2, s_3)$, let $0 \leqslant i < j < N$ be any two vertices, and let $D(i)$, $D(j)$ be the values of their distances from 0. Then, using $+1$-jumps from i and -1-jumps from j, we have

$$\max_{i \leqslant v \leqslant j} D(v) = \lfloor (j - i + D(i) + D(j))/2 \rfloor.$$

The maximum is reached at the vertex $v = \lfloor (j + i + D(j) - D(i))/2 \rfloor$.

At first, we will consider the case when generators s_2 and s_3 of $C(N; 1, s_2, s_3)$ are polynomials in d of orders 1 and 2, respectively. We have the following result.

Theorem 1. Let $1 \leqslant p \leqslant d$ for any integer $d > 1$. Then any circulant graph $C(N; 1, s_2, s_3)$, where

$$\begin{cases} N = p^3 - (4d + 1/2)p^2 + 4d^2 p + 2d^2 + 2d + 1 & \text{for even } p, \\ N = p^3 - (4d + 1/2)p^2 + 4d^2 p + 2d^2 + 2d + 1/2 & \text{for odd } p, \\ s_2 = 2(d - \lceil p/2 \rceil) + 1, \\ s_3 = 2(d - \lfloor p/2 \rfloor) s_2 + 1, \end{cases} \tag{1}$$

has diameter d.

Proof. Consider a graph $C(N; 1, s_2, s_3)$, where N, s_2 and s_3 are given by (1), $d > 1$ and $1 \leqslant p \leqslant d$. The representation of vertices of C is shown in Fig. 1.

From (1) it follows that $r = N \bmod s_3 = (s_3 + s_2)/2$ and hence

$$N = p s_3 + (s_3 + s_2)/2. \tag{2}$$

We need to prove that the graphs given by (1) have the diameter d. We will divide all the vertices $0 \leqslant v \leqslant N$ into the following segments $A_i = [i s_3, (i + 1) s_3]$, $i = 0, 1, \ldots, p$, which in turn are divided into the subsegments:

$$A_i^l = [i s_3, N - (p - i) s_3], \quad A_i^r = [N - (p - i) s_3, (i + 1) s_3].$$

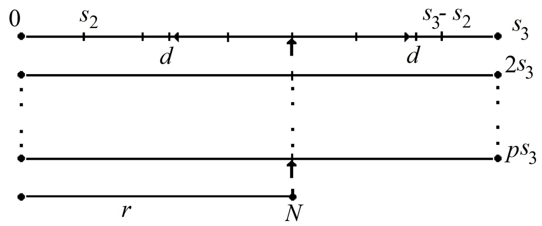

Fig. 1. The representation of vertices of a graph defined by (1)

We have

$$D(is_3) = i, \quad D((i+1)s_3) = i+1, \quad D(N-(p-i)s_3) = p-i.$$

All vertices in A_i^l can be reached from is_3 by $+s_2$-jumps and ± 1-jumps or from $N-(p-i)s_3$ by $-s_2$-jumps and ± 1-jumps. Similarly, all vertices in A_i^r can be reached from $N-(p-i)s_3$ by $+s_2$-jumps and ± 1-jumps or from $(i+1)s_3$ by $-s_2$-jumps and ± 1-jumps. From the structure of the graphs, it follows that $|A_i^l| = |A_i^r| + s_2$, and in A_i^r distances $D(v)$ of vertices v are the same as in $[is_3 + s_2, N-(p-i)s_3]$. Hence, it is sufficient only to consider the segments A_i^l. In turn, they are divided into the subsegments

$$A_{ik}^l = [is_3 + ks_2, is_3 + ks_2 + \lceil s_2/2 \rceil],$$
$$A_{ik}^r = [is_3 + ks_2 + \lceil s_2/2 \rceil, is_3 + (k+1)s_2],$$

where $k = 0, 1, \ldots, d - \lfloor p/2 \rfloor$, since we have $\lfloor (s_3+s_2)/2s_2 \rfloor = d - \lfloor p/2 \rfloor$ by (1). Note that for $k = d - \lfloor p/2 \rfloor$ the segment A_{ik}^r is not considered. We have $|A_{ik}^l| = \lceil s_2/2 \rceil = |A_{ik}^r| + 1$ and accordingly in A_{ik}^r the values of $D(v)$ of the vertices v are the same as in $[is_3 + ks_2 + 1, is_3 + ks_2 + \lceil s_2/2 \rceil]$. Hence, it is sufficient only to consider the segments A_{ik}^l to determine the maximum distance in the graph. We have

$$D(is_3+ks_2) = i+k, \quad D(is_3+ks_2+\lceil s_2/2 \rceil) = d+\lceil p/2 \rceil - i - k, \quad D(is_3+(k+1)s_2) = i+k+1.$$

To obtain in A_{ik}^l the maximum distance of vertices from 0, we use Lemma 1:

$$\max_{v \in A_{ik}^l} D(v) = \lfloor (i+k+d+\lceil p/2 \rceil - (i+k) + \lceil s_2/2 \rceil)/2 \rfloor = d.$$

Now we will show that there is a vertex v at the distance $D(v) = d$ from 0. Let $v = N - ps_3 - (d-p)s_2$. For it, as we can see, there is a path with the number of steps d using $-s_3$-jumps and $-s_2$-jumps. On the other hand, using (2) and substituting s_3 from (1) into v, we obtain $v = \lfloor s_3/2 \rfloor + \lceil s_2/2 \rceil - (d-p)s_2 = \lceil p/2 \rceil s_2 + d - \lceil p/2 \rceil + 1$. It follows that the length of the path from 0 to v through the nearest vertex of the form $\lceil p/2 \rceil s_2$ is $d+1$. Similarly, we have $v = (\lceil p/2 \rceil + 1)s_2 - (d - \lceil p/2 \rceil)$ and hence the length of the path from 0 to v through another nearest vertex $(\lceil p/2 \rceil + 1)s_2$ is also equal to $d+1$. The lengths of other paths from 0 to v are greater than d. Therefore, the graphs prescribed by (1) have diameter d. ∎

We introduce the concept of a series of graphs of Γ. For any integer $d > 1$, let $p = 1, 2, \ldots, d$. Then we have an infinitive set (*series*) Γ of triple loop graphs with diameters $d = 2, 3, \ldots$:

$$\Gamma = \bigcup_{(d>1)\wedge(1 \leqslant p \leqslant d)} C(N; 1, s_2, s_3),$$

where N, s_2 and s_3 are defined using (1).

Now consider the optimization problem for Γ: to find a function $p = p(d)$ giving the maximum of $N = N(p)$ for all $d > 1$ among graphs of Γ with diameter $d > 1$.

Theorem 2. Let $d > 1$ be an integer. The maximum order of circulant graphs $C(N; 1, s_2, s_3) \in \Gamma$ with diameter d is

$$N = \begin{cases} 32d^3/27 + 16d^2/9 + 2d + 1 & \text{for } d \equiv 0 \pmod 3, \\ 32\lfloor d/3\rfloor^3 + 48\lfloor d/3\rfloor^2 + 26\lfloor d/3\rfloor + 5 & \text{for } d \equiv 1 \pmod 3, \\ 32\lfloor d/3\rfloor^3 + 80\lfloor d/3\rfloor^2 + 70\lfloor d/3\rfloor + 21 & \text{for } d \equiv 2 \pmod 3. \end{cases} \quad (3)$$

The maximum is achieved with the following generators:

$$(s_2, s_3) = \begin{cases} (4d/3 + 1, 16d^2/9 + 4d/3 + 1) & \text{for } d \equiv 0 \pmod 3, \\ (4\lfloor d/3\rfloor + 2 \pm 1, 16\lfloor d/3\rfloor^2 + 16\lfloor d/3\rfloor + 5 \pm (4\lfloor d/3\rfloor + 2)) & \text{for } d \equiv 1 \pmod 3, \\ (4\lfloor d/3\rfloor + 3, 16\lfloor d/3\rfloor^2 + 28\lfloor d/3\rfloor + 13) & \text{for } d \equiv 2 \pmod 3. \end{cases} \quad (4)$$

Proof.

C a s e 1. Let p be an even number. Our goal is to find the maximum function $N = N(p)$ for a given $d > 1$. The function N is a cubic polynomial in p with the following coefficients: $a = 1 > 0$, $b = -4d - 1/2$, $c = 4d^2$. We will find a valued-integer function $p(d)$ such that N defined by (1) has a maximum for any $d > 1$. The discriminant $D = 3ac - b^2 = -4d^2 - 4d - 1/4 < 0$. Hence, $N(p)$ has one maximum when

$$p = -\frac{b + \sqrt{-D}}{3a} = \frac{4}{3}d - \frac{2}{3}\sqrt{d^2 + d + \frac{1}{16}} + \frac{1}{6}.$$

Since $d + 1/4 < \sqrt{d^2 + d + 1/16} < d + 1/2$, we have $\frac{2}{3}d - \frac{1}{6} < p < \frac{2}{3}d$. Since $p(d)$ is a valued-integer function with even values, we take the nearest even integer and obtain

$$p = 2\lfloor d/3\rfloor, \quad \text{if} \quad d \equiv 0, 1 \pmod 3.$$

Substituting the found p into (1) we obtain s_2, s_3 and N for $d \equiv 0, 1 \pmod 3$ (the version with "+").

C a s e 2. Let p be an odd number. Similarly to case 1, we have the following: the maximum of $N = N(p)$ for a given d is reached when

$$p = 2\lfloor d/3\rfloor + 1, \quad \text{if} \quad d \equiv 1, 2 \pmod 3.$$

Substituting the found p into (1), we obtain s_2, s_3, and N for $d \equiv 1, 2 \pmod 3$ (the version with "−"). ∎

3. A series of circulants with generators $s_2 = f(d^2)$

We will now study the case when generators s_2 and s_3 of a graph are polynomials in d of order 2. We have the following result.

Theorem 3. Let $1 \leqslant p < d$ for any integer $d > 1$. Then any circulant graph $C(N; 1, s_2, s_3)$, where

$$\begin{cases} N = -8p^3 + (8d + 4)p^2 + 2d + 1, \\ s_2 = -4p^2 + 4dp + 4p - 2d, \\ s_3 = s_2 + 4d - 4p + 2, \end{cases} \quad (5)$$

has diameter d.

We introduce the concept of a series of graphs of Φ. For any integer $d > 1$, let $p = 1, 2, \ldots, d-1$. Then we have an infinite set (series) Φ of triple loop graphs with diameters $d = 2, 3, \ldots$:

$$\Phi = \bigcup_{(d>1)\wedge(1\leqslant p<d)} C(N; 1, s_2, s_3),$$

where N, s_2 and s_3 are defined by (5).

First, we prove Lemma 2, which gives a general analytical method for obtaining the distance function $D(v)$ for all graphs in Φ. We can restrict our consideration to the values $0 \leqslant v \leqslant \lfloor N/2 \rfloor$, since $D(v) = D(N - v)$ in any circulant graph.

Lemma 2. Let $C(N; 1, s_2, s_3) \in \Phi$ be a triple loop graph and $\delta = 2(d + 1 - p)$. For any vertex v, $0 \leqslant v \leqslant \lfloor N/2 \rfloor$ of the graph C let $i = \lfloor v/s_3 \rfloor$. If $v \leqslant (i + 1)s_2$, then

$$D(v) = \begin{cases} i + 2j + \delta - |k|, & \text{if } (0 \leqslant i + 2j \leqslant d - \delta) \text{ and } (-\delta \leqslant k < \delta - 2), \\ & \text{or if } \ (d - \delta < i + 2j \leqslant d) \text{ and} \\ & (-\delta \leqslant k \leqslant d - \delta - (i + 2j)), \text{ or} \\ & i + 2j - (d - \delta) \leqslant k < \delta - 2, \\ 2d + 1 - (i + 2j + \delta - |k|) & \text{otherwise,} \end{cases} \tag{6}$$

where

$$j = \lfloor (v - is_3)/(s_3 - s_2) \rfloor, \quad k = v - is_3 - j(s_3 - s_2) - \delta. \tag{7}$$

If $v > (i + 1)s_2$, then

$$D(v) = \begin{cases} i + \delta - |k|, & \text{if } (0 \leqslant i \leqslant d - \delta) \text{ and } (-\delta < k < \delta - 1), \\ & \text{or if } \ (d - \delta < i \leqslant d - \delta/2) \text{ and} \\ & (-\delta < k \leqslant d - \delta - i), \text{ or } i - (d - \delta) \leqslant k < \delta - 1, \\ 2d + 1 - (i + \delta - |k|) & \text{otherwise,} \end{cases} \tag{8}$$

where

$$j = \lfloor (v - (i + 1)s_2)/(s_3 - s_2) \rfloor, \quad k = v - (i + 1)s_2 - j(s_3 - s_2) - \delta + 1. \tag{9}$$

Proof. Consider a graph $C(N; 1, s_2, s_3) \in \Phi$, where $d > 1$ and $p = 1, 2, \ldots, d - 1$. For simplicity, we define $\Delta = s_3 - s_2 = 4(d - p) + 2$, $\delta = \Delta/2 + 1$. Hence, $p = \lfloor s_3/\Delta \rfloor$, $N = (2p - 1)s_3 + \Delta + 1$, and

$$\delta = 2(d + 1 - p). \tag{10}$$

The representation of vertices of C is shown in Fig. 2, here $q = 2p - 1$, $r = \Delta + 1$. We will divide the vertices $0 \leqslant v \leqslant N/2$ into the following segments:

$$A_i = [is_3, (i + 1)s_2], \quad B_i = [(i + 1)s_2, (i + 1)s_3], \quad 0 \leqslant i < p.$$

Note that $v = \lfloor N/2 \rfloor \in B_{p-1}$. We have $|A_i| = (p - i - 1)\Delta + \Delta/2 + 1$, $|B_i| = (i + 1)\Delta$, $D(is_2) = D(is_3) = i$.

Consider two types of vertices in A_i reached from vertices is_3 by $+\Delta$-jumps and from vertices $(i + 1)s_2$ by $-\Delta$-jumps:

$$x_{ij} = is_3 + j\Delta = (i + j)s_3 - js_2,$$
$$x'_{ij} = x_{ij} + \delta = (i + 1)s_2 - (p - 1 - i - j)\Delta = (p - j)s_2 - (p - 1 - i - j)s_3,$$

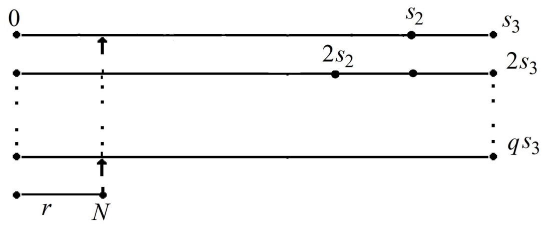

Fig. 2. The representation of vertices of a graph defined by (5)

where $0 \leqslant i < p$, $0 \leqslant j < p - i$. Using (10) and taking into account that there is a path from x'_{ij} to x_{ij} with length $2p - 1 - (i + 2j) + \delta = 2d + 1 - (i + 2j)$, we obtain $D(x_{ij}) = \min\{i + 2j, 2d + 1 - (i + 2j)\}$. Therefore,

$$D(x_{ij}) = \begin{cases} i + 2j, & \text{if } i + 2j \leqslant d, \\ 2d + 1 - (i + 2j), & \text{if } i + 2j > d. \end{cases} \tag{11}$$

Similarly, we have $D(x'_{ij}) = \min\{2d + 1 - i - 2j - \delta, i + 2j + \delta\}$ and

$$D(x'_{ij}) = \begin{cases} i + 2j + \delta, & \text{if } i + 2j \leqslant d - \delta, \\ 2d + 1 - (i + 2j + \delta), & \text{if } i + 2j > d - \delta. \end{cases} \tag{12}$$

Consider the vertices in B_i reached from vertices $(i + 1)s_2$ by $+\Delta$-jumps:

$$y_{ij} = (i + 1)s_2 + j\Delta = (i + 1 - j)s_2 + js_3, \quad 0 \leqslant i < p, \quad 0 \leqslant j \leqslant i + 1.$$

From here we have

$$D(y_{ij}) = i + 1. \tag{13}$$

Note that $B_i = \bigcup_{j=0}^{i} [y_{ij}, y_{ij+1}]$.

Let $0 \leqslant v \leqslant \lfloor N/2 \rfloor$ be a vertex of the graph. Compute $i = \lfloor v/s_3 \rfloor$.

C a s e 1. Let $v \leqslant (i + 1)s_2$, hence $v \in A_i$, $0 \leqslant i < p$. Compute j and k by (7). Having $0 \leqslant j < p - i$, $-\delta \leqslant k < \delta - 2$, and using (10), we obtain $0 \leqslant i + 2j \leqslant 2d - \delta - i$. Taking into account that either $0 \leqslant i + 2j \leqslant d - \delta$, or $d - \delta < i + 2j \leqslant d$, or $d < i + 2j \leqslant 2d - \delta - i$, and using (11) and (12), we obtain (6).

C a s e 2. Let $v > (i + 1)s_2$, hence $v \in B_i$, $0 \leqslant i < p$. Compute j and k by (9).

2a) Let $0 \leqslant i \leqslant d - \delta$. Using (13), we obtain $D(v) = i + \delta - |k|$ when $-\delta < k < \delta - 1$.

2b) Let $i > d - \delta$. In the graph, the movement from N to 0 alone generators $-s_3$ and $-s_2$ decreases the function $D(v)$ for vertices $v \in [y_{ij}, y_{ij+1}]$, $d - \delta < i < p$, $0 \leqslant j \leqslant i$. Consider the vertices m_{ij} lying in the middle of segments $[y_{ij}, y_{ij+1}]$:

$$m_{ij} = y_{ij} + \delta - 1 = (i + 1)s_2 + j\Delta + \delta - 1, \quad d - \delta < i < p, \quad 0 \leqslant j \leqslant i.$$

The movement from N to 0 alone generators $-s_3$ and $-s_2$ gives

$$D(m_{ij}) = 2d + 1 - i - \delta, \tag{14}$$

since $m_{ij} = N - (d + 1 - \delta/2 - j)s_3 - (d - \delta/2 - i + j)s_2$. Taking into account that either $-\delta < k \leqslant d - i - \delta$, or $|k| < i + \delta - d$, or $i + \delta - d \leqslant k < \delta - 1$, and using (13) and (14) and taking the minimum path from 0 to v, we obtain (8). ∎

We can now prove Theorem 3.

Proof. Using the formulas (6) and (8), we have the following: the sum of lengths of two possible paths from 0 to v is always equal to $2d+1$ for any vertex v. Therefore, one of the lengths is less than or equal to d. Now it is necessary to show that there is a vertex v of C with $D(v) = d$. Let $v = \lfloor N/2 \rfloor = -4p^3 + (4d+2)p^2 + d$. For even p, there is a path of length d from 0 to v, since $\lfloor N/2 \rfloor = d - p + (p/2)(s_2 + s_3)$. For odd p we have $\lfloor N/2 \rfloor - N = -\lceil N/2 \rceil = d - p - \lfloor p/2 \rfloor s_2 - \lceil p/2 \rceil s_3$ and for v there is also a path of length d from 0. The same result follows from (8). There are no other paths in the graph to v with length less than d. ■

Now, let us consider the optimization problem for Φ: to find a function $p = p(d)$ giving the maximum of $N = N(p)$ for all $d > 1$ among graphs of Φ with diameter $d > 1$.

Theorem 4. Let $d > 1$ be an integer. The maximum order of circulant graphs $C(N; 1, s_2, s_3) \in \Phi$ with diameter d is

$$N = \begin{cases} 32d^3/27 + 16d^2/9 + 2d + 1 & \text{for } d \equiv 0 \pmod 3, \\ 32\lfloor d/3 \rfloor^3 + 48\lfloor d/3 \rfloor^2 + 30\lfloor d/3 \rfloor + 7 & \text{for } d \equiv 1 \pmod 3, \\ 32\lfloor d/3 \rfloor^3 + 80\lfloor d/3 \rfloor^2 + 70\lfloor d/3 \rfloor + 21 & \text{for } d \equiv 2 \pmod 3. \end{cases} \tag{15}$$

The bound is achieved with the following generators:

$$(s_2, s_3) = \begin{cases} (8d^2/9 + 2d/3, 8d^2/9 + 2d + 2) & \text{for } d \equiv 0 \pmod 3, \\ (8\lfloor d/3 \rfloor^2 + 6\lfloor d/3 \rfloor + 2, 8\lfloor d/3 \rfloor^2 + 10\lfloor d/3 \rfloor + 4) & \text{for } d \equiv 1 \pmod 3, \\ (8\lfloor d/3 \rfloor^2 + 10\lfloor d/3 \rfloor + 4, 8\lfloor d/3 \rfloor^2 + 14\lfloor d/3 \rfloor + 6) & \text{for } d \equiv 2 \pmod 3. \end{cases} \tag{16}$$

Proof. Consider a circulant graph $C(N; 1, s_2, s_3) \in \Phi$ with diameter d. The function N is a cubic polynomial in p with the following coefficients: $a = -8$, $b = 8d+4$, $c = 0$. We will find a valued-integer function $p(d)$ such that N defined by (5) has a maximum for any $d > 1$. Take the derivative of $N(p)$ with respect to p: $\dfrac{dN}{dp} = -24p^2 + 2(8d+4)p = 0$. Therefore, N has the maximum when $p = (2d+1)/3$. Since $p(d)$ is a valued-integer function, we take the nearest integer and obtain for any $d \geqslant 1$

$$p(d) = p^* = (2d + d \bmod 3)/3.$$

Substituting p^* into (5), we get that N, s_2 and s_3 are equal to (15) and (16), respectively. ■

The family with $p = d$, namely the graphs $C(4d^2 + 2d + 1; 1, 2d, 2d + 2)$, where $d > 1$, can also be included in Φ since they have diameter d.

4. A series of circulants with generators $s_2 = f(d^3)$

The case of circulant graphs, when all N, s_2 and s_3 are polynomials in d of order 3, was studied in [29].

Theorem 5 [29]. Let $1 \leqslant p < d$ for any integer $d > 1$. Then any circulant graph $C(N; 1, s_2, s_3)$, where

$$\begin{cases} N = 8p^3 - (16d+8)p^2 + (8d^2+8d)p + 2d + 1, \\ s_2 = 4p(d-p)^2 + 2p(d-p) + d - 3p, \\ s_3 = s_2 + 4p, \end{cases} \tag{17}$$

has diameter d.

For any integer $d > 1$, let $p = 1, 2, \ldots, d-1$. Then we obtain an infinite set (series) Ψ of triple loop graphs with diameters $d = 2, 3, \ldots$:

$$\Psi = \bigcup_{(d>1)\wedge(1\leqslant p<d)} C(N; 1, s_2, s_3),$$

where N, s_2 and s_3 are defined by (17). In [29], the optimization problem for Ψ has been also solved.

Theorem 6. Let $d > 1$ be an integer. The maximum order of circulant graphs $C(N; 1, s_2, s_3) \in \Psi$ with diameter d is

$$N = \begin{cases} 32d^3/27 + 16d^2/9 + 2d + 1 & \text{for } d \equiv 0 \pmod 3, \\ 32\lfloor d/3\rfloor^3 + 48\lfloor d/3\rfloor^2 + 22\lfloor d/3\rfloor + 3 & \text{for } d \equiv 1 \pmod 3, \\ 32\lfloor d/3\rfloor^3 + 80\lfloor d/3\rfloor^2 + 70\lfloor d/3\rfloor + 21 & \text{for } d \equiv 2 \pmod 3. \end{cases} \tag{18}$$

The bound is achieved with the following generators:

$$(s_2, s_3) = \begin{cases} (16d^3/27{+}4d^2/9, s_2{+}4d/3) & \text{for } d \equiv 0 \pmod 3, \\ (16d^3/27{+}4d^2/9{-}2d/3{+}17/27, s_2{+}4d/3{-}4/3) & \\ \text{or} & \\ (16d^3/27{+}4d^2/9{-}4d/3{-}46/27, s_2{+}4d/3{+}8/3) & \text{for } d \equiv 1 \pmod 3, \\ (16d^3/27{+}4d^2/9{-}2d/9{-}29/27, s_2{+}4d/3{+}4/3) & \text{for } d \equiv 2 \pmod 3. \end{cases} \tag{19}$$

For all diameters $d \equiv 0, 2 \pmod 3$ the value of N from (18) is equal to the maximum N from (15), but for $d \equiv 1 \pmod 3$ it is less. Note that for $d \equiv 1 \pmod 3$ there are two sets of generators for N, defined by (18). It is easy to check that for $d \equiv 0, 2 \pmod 3$ the generators (19) are obtained by an equivalent transformation of generators (4), and thus the graphs from Theorems 2 and 6 are isomorphic for these diameters.

Comparing the values of N obtained in Theorems 2, 4, and 6, we come to the conclusion that (15) gives the largest values of N among graphs of sets Γ, Φ and Ψ for all diameters d. This result is in good agreement with the extremal values of N obtained in [9, 28].

5. New families of three-dimensional circulants

If we take any valued-integer functions $p(d)$ such that $1 \leqslant p(d) \leqslant d$ as a parameter p, then we can generate new infinite families of circulant graphs. We have shown examples of new families of circulants constructed by this method for sets Γ, Φ and Ψ (Theorems 2, 4, and 6). These found families are the best ones in the ratio N/d (order/diameter) among all known families of degree 6 circulants. Some examples of constructing other possible families of sets Γ, Φ, and Ψ are presented below.

Example 1. Let $C(N; 1, s_2, s_3) \in \Gamma$, $d > 1$, and

$$p(d) = \begin{cases} 2\lfloor d/3\rfloor - 1 & \text{for } d \equiv 0 \pmod 3, \\ 2(\lfloor d/3\rfloor + 1) & \text{for } d \equiv 1, 2 \pmod 3. \end{cases}$$

Substituting the value of p in (1), we obtain N, s_2 and s_3 for a new infinite family. The resulting N is less than (3) by $4\lfloor d/3\rfloor + 2$ for $d \equiv 0, 2 \pmod 3$ or by $12\lfloor d/3\rfloor + 2$ for $d \equiv 1 \pmod 3$.

Example 2. Let $C(N; 1, s_2, s_3) \in \Phi$, $d > 1$, and

$$p(d) = \begin{cases} 2d/3 + 1 & \text{for } d \equiv 0 \pmod 3, \\ 2(d-1)/3 & \text{for } d \equiv 1 \pmod 3, \\ (2d-1)/3 & \text{for } d \equiv 2 \pmod 3. \end{cases}$$

Substituting the value of p in (5), we obtain N, s_2, and s_3 for a new infinite family. Resulting N is less than (15) by $8\lfloor d/3 \rfloor + 4$ for $d \equiv 0, 2 \pmod 3$ or by $24\lfloor d/3 \rfloor + 4$ for $d \equiv 1 \pmod 3$.

With respect to N, the resulting graphs $C(N; 1, s_2, s_3)$ in examples 1 and 2 are the nearest to the graphs with the maximum N among all graphs of the sets Γ and Φ, respectively.

For Ψ, the following family was created in [29].

Example 3. Let $C(N; 1, s_2, s_3) \in \Psi$, $d > 1$, and $p(d) = \lceil d/2 \rceil$. Then

$$(N; 1, s_2, s_3) = \begin{cases} (d^3 + 2d^2 + 2d + 1; 1, (d^3 + d^2 - d)/2, (d^3 + d^2 + 3d)/2) & \text{for even } d, \\ (d^3 + d^2 + d; 1, (d^3 - 3)/2 - d, (d^3 - 3)/2 + d + 2) & \text{for odd } d. \end{cases}$$

For Ψ, the following function $p(d)$ generates a new family of circulants.

Example 4. Let $C(N; 1, s_2, s_3) \in \Psi$, $d > 1$, and

$$p(d) = \begin{cases} 2\lfloor d/3 \rfloor & \text{for } d \equiv 0 \pmod 3, \\ 2\lfloor d/3 \rfloor + 1 & \text{for } d \equiv 1, 2 \pmod 3. \end{cases}$$

Substituting the value of p in (17), we obtain N, s_2, and s_3 for a new infinite family.

As for the ratio N/d, the new families of circulants in examples 1 and 2 are better than families in [5, 6, 8, 22, 26], and the family in example 3 is better than families in [6, 8, 22, 26] and, moreover, in contrast to families from [5, 6, 8, 26], all the obtained families exist for any diameter of the graph.

6. Descriptions of graphs series

Using the system Wolfram Mathematica 10, fragments of circulant families for all sets Γ, Φ, and Ψ with diameters $d = 2, \ldots, 25$ have been implemented (see the Appendix).

Tables 1–3 show part of the descriptions of the obtained circulants $C(N; 1, s_2, s_3)$ with diameters d, $2 \leqslant d \leqslant 10$: the diameters of graphs d, the values of p, where $1 \leqslant p \leqslant d$ for Γ, Φ and $1 \leqslant p \leqslant d - 1$ for Ψ, the orders of graphs N, and generators s_2 and s_3.

Figures 3–5 show the dependence of the number of vertices N on diameter d, $2 \leqslant d \leqslant 25$, and parameter p for the fragments of descriptions for sets Γ, Φ, and Ψ. The values of N of graphs corresponding to the families with the maximum orders for the given diameters are marked in blue in Fig. 3–5. Figure 5 additionally shows the orders of graphs of the family from Example 4 in red. Note that parameter p has a different meaning in figures: $p = \lfloor N/s_3 \rfloor$ in Fig. 3, $p = \lfloor s_3/(s_3 - s_2) \rfloor$ in Fig. 4, and $p = (s_3 - s_2)/4$ in Fig. 5.

Table 1

Descriptions of the graphs of Γ

d	p	N	s_2	s_3	d	p	N	s_2	s_3
2	1	21	3	13	8	1	369	15	241
2	2	19	3	7	8	2	535	15	211
3	1	49	5	31	8	3	647	13	183
3	2	55	5	21	8	4	713	13	157
3	3	47	3	13	8	5	737	11	133
4	1	89	7	57	8	6	727	11	111
4	2	111	7	43	8	7	687	9	91
4	3	111	5	31	8	8	625	9	73
4	4	97	5	21	9	1	469	17	307
5	1	141	9	91	9	2	691	17	273
5	2	187	9	73	9	3	851	15	241
5	3	203	7	57	9	4	957	15	211
5	4	197	7	43	9	5	1013	13	183
5	5	173	5	31	9	6	1027	13	157
6	1	205	11	133	9	7	1003	11	133
6	2	283	11	111	9	8	949	11	111
6	3	323	9	91	9	9	869	9	91
6	4	333	9	73	10	1	581	19	381
6	5	317	7	57	10	2	867	19	343
6	6	283	7	43	10	3	1083	17	307
7	1	281	13	183	10	4	1237	17	273
7	2	399	13	157	10	5	1333	15	241
7	3	471	11	133	10	6	1379	15	211
7	4	505	11	111	10	7	1379	13	183
7	5	505	9	91	10	8	1341	13	157
7	6	479	9	73	10	9	1269	11	133
7	7	431	7	57	10	10	1171	11	111

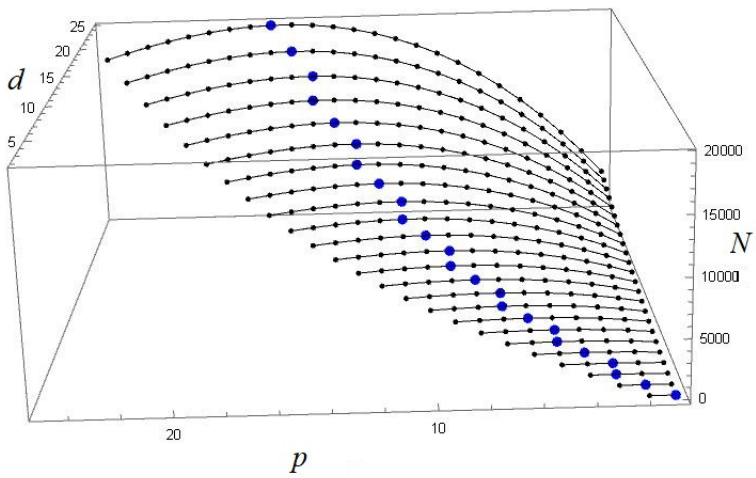

Fig. 3. Number of vertices $N = N(d, p)$ for $C(N; 1, s_2, s_3) \in \Gamma$

Table 2

Descriptions of the graphs of Φ

d	p	N	s_2	s_3	d	p	N	s_2	s_3
2	1	17	4	10	8	1	77	16	46
2	2	21	4	6	8	2	225	40	66
3	1	27	6	16	8	3	413	56	78
3	2	55	10	16	8	4	593	64	82
3	3	43	6	8	8	5	717	64	78
4	1	37	8	22	8	6	737	56	66
4	2	89	16	26	8	7	605	40	46
4	3	117	16	22	8	8	273	16	18
4	4	73	8	10	9	1	87	18	52
5	1	47	10	28	9	2	259	46	76
5	2	123	22	36	9	3	487	66	92
5	3	191	26	36	9	4	723	78	100
5	4	203	22	28	9	5	919	82	100
5	5	111	10	12	9	6	1027	78	92
6	1	57	12	34	9	7	999	66	76
6	2	157	28	46	9	8	787	46	52
6	3	265	36	50	9	9	343	18	20
6	4	333	36	46	10	1	97	20	58
6	5	313	28	34	10	2	293	52	86
6	6	157	12	14	10	3	561	76	106
7	1	67	14	40	10	4	853	92	118
7	2	191	34	56	10	5	1121	100	122
7	3	339	46	64	10	6	1317	100	118
7	4	463	50	64	10	7	1393	92	106
7	5	515	46	56	10	8	1301	76	86
7	6	447	34	40	10	9	993	52	58
7	7	211	14	16	10	10	421	20	22

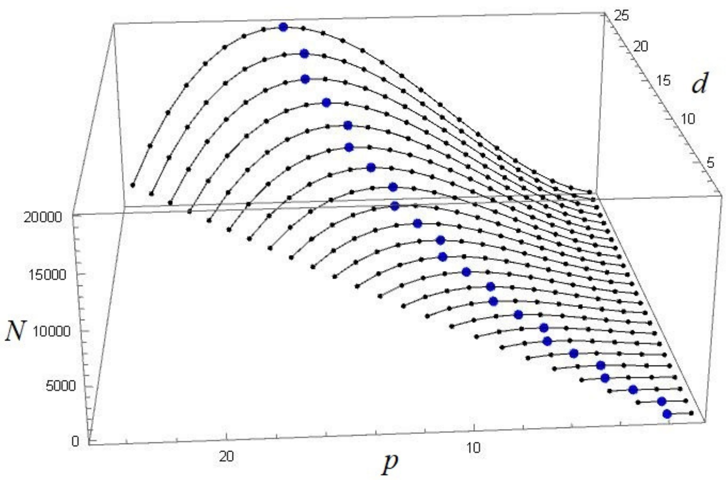

Fig. 4. Number of vertices $N = N(d,p)$ for $C(N;1,s_2,s_3) \in \Phi$

Table 3

Descriptions of the graphs of Ψ

d	p	N	s_2	s_3	d	p	N	s_2	s_3
3	1	55	20	24	8	3	737	329	341
3	2	39	9	17	8	4	657	284	300
4	1	105	43	47	8	5	497	203	223
4	2	105	38	46	8	6	305	110	134
4	3	57	13	25	8	7	129	29	57
5	1	171	74	78	9	1	595	278	282
5	2	203	83	91	9	2	915	423	431
5	3	155	56	68	9	3	1027	468	480
5	4	75	17	33	9	4	979	437	453
6	1	253	113	117	9	5	819	354	374
6	2	333	144	152	9	6	595	243	267
6	3	301	123	135	9	7	355	128	156
6	4	205	74	90	9	8	147	33	65
6	5	93	21	41	10	1	741	349	353
7	1	351	160	164	10	2	1173	548	556
7	2	495	221	229	10	3	1365	631	643
7	3	495	214	226	10	4	1365	622	638
7	4	399	163	179	10	5	1221	545	565
7	5	255	92	112	10	6	981	424	448
7	6	111	25	49	10	7	693	283	311
8	1	465	215	219	10	8	405	146	178
8	2	689	314	322	10	9	165	37	73

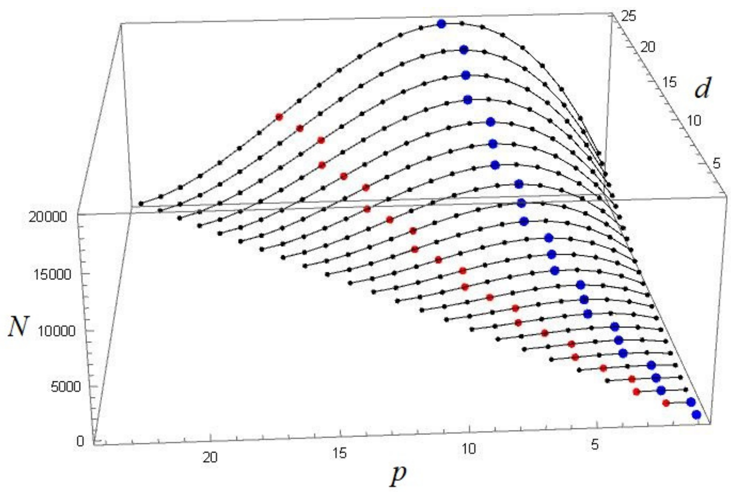

Fig. 5. Number of vertices $N = N(d, p)$ for $C(N; 1, s_2, s_3) \in \Psi$

7. Conclusion

The graphs of three series of the circulants obtained in this paper have not only general topological properties, but also general communicative properties that relate to the organization of routing for these graphs. We do not review in detail the existing routing algorithms for degree 6 circulant graphs. This is a topic for a separate paper. Let's just say that few routing algorithms are known for these graphs and they are not efficient for use in NoCs, therefore analytical routing algorithms that can be developed for graphs of the series obtained are of interest. It should also be noted that the proof of Lemma 2 implies the existence of a general analytical method for calculating the distance function in all graphs of Φ. This property in turn allowed us to develop a general analytical routing algorithm for all families of Φ. The routing algorithm is described in [18, 30] and implemented in NoC using the example of one of the possible families of Φ as a NoC topology.

Thus, as we could demonstrate on the example of the series Φ of circulans, a general structure of graphs of each found series makes it possible to develop a general analytical method for calculating the distance function and then to obtain the shortest-path vectors for a routing algorithm for all graphs included in the series, changing only the value of the parameter $p(d)$, which defines the form of a considered family of circulants.

The second interesting question is whether there are other series of circulant graphs analytically described and given by two parameters d and $p(d)$ that have the same structural and communication properties? Or is it a common property in the space of analytical descriptions of circulant graphs? Are there other designs of circulant series that contain the largest possible graphs for a given degree and a given diameter? Generalizing this problem, we can say that the question is to discover new regularities in obtaining series of circulant graphs that establish relationships between the analytical connection of the order and generators of a graph and the geometry of the resulting series of graphs.

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REFERENCES

1. *Monakhova E. A.* A survey on undirected circulant graphs. Discr. Math. Algorithms Appl., 2012, vol. 4, no. 1, 1250002, 30 p.
2. *Bermond J.-C., Comellas F., and Hsu D. F.* Distributed loop computer networks: a survey. J. Parallel Distrib. Comput., 1995, vol. 24, pp. 2–10.
3. *Feria-Puron R., Pérez-Rosés H., and Ryan J.* Searching for Large Circulant Graphs. arXiv:1503.07357v1 [math.CO], Aug. 11, 2018, 31 p.
4. *Hwang F. K.* A survey on multi-loop networks. Theor. Comput. Sci., 2003, vol. 299, pp. 107–121.
5. *Chen S. and Jia X.-D.* Undirected loop networks. Networks, 1993, no. 23(4), pp. 257–260.
6. *Stojmenović I.* Multiplicative circulant networks. Topological properties and communication algorithms. Discr. Appl. Math., 1997, no. 77, pp. 281–305.
7. *Thomson A. and Zhou S.* Gossiping and routing in undirected triple-loop networks. Networks, 2010, no. 55(4), pp. 341–349.
8. *Wong C. K. and Coppersmith D.* A combinatorial problem related to multimodule memory organizations. J. Assoc. Comput. Mach., 1974, no. 21(3), pp. 392–402.
9. *Monakhova E.* Optimal triple loop networks with given transmission delay: Topological design and routing. Proc. INOC'2003, Evry/Paris, France, 2003, pp. 410–415.
10. *Lewis R. R.* The degree-diameter problem for circulant graphs of degree 8 and 9. Electron. J. Combin., 2014, no. 21(4), article no. P4.50.
11. *Lewis R. R.* The degree-diameter problem for circulant graphs of degrees 10 and 11. Discrete Math., 2018, no. 341, pp. 2553–2566.
12. *Dalfó C., Fiol M. A., Lopéz N., and Ryan J.* An improved Moore bound and some new optimal families of mixed Abelian Cayley graphs. Discr. Math., 2020, vol. 343, no. 10.
13. *Miller M. and Siran J.* Moore graphs and beyond: A survey of the degree/diameter problem. Electron. J. Combin., Dyn. Surv., 2005, no. DS14, 61 p.
14. *Pérez-Rosés H.* Algebraic and computer-based methods in the undirected degree/diameter problem – A brief survey. Electron. J. Graph Theory Appl., 2014, no. 2(2), pp. 166–190.

15. The Degree/Diameter Problem For Circulant Graphs. http://combinatoricswiki.org/wiki/The_Degree_Diameter_Problem_for_Circulant_Graphs.
16. *Monakhova E. A., Romanov A. Yu., and Lezhnev E. V.* Shortest path search algorithm in optimal two-dimensional circulant networks: Implementation for networks-on-chip. IEEE Access, 2020, no. 8, pp. 215010–215019.
17. *Romanov A., Amerikanov A., and Lezhnev E.* Analysis of approaches for synthesis of networks-on-chip by using circulant topologies. J. Physics. Conf. Series, 2018, vol. 1050, no. 1, pp. 1–12.
18. *Monakhova E. A., Monakhov O. G., Romanov A. Yu., and Lezhnev E. V.* Analytical routing algorithm for networks-on-chip with the three-dimensional circulant topology. Proc. MWENT 2020, Moscow, Russia, March 11–13, 2020, pp. 1–6.
19. *Romanov A. Yu. and Starykh V. A.* Routing in triple loop circulants: A case of networks-on-chip. Heliyon, 2020, no. 6(7), pp. 1–7.
20. *Ansari A. Q., Ansari M. R., and Khan M. A.* Modified quadrant-based routing alglrithm for 3D torus network-on-chip architecture. Perspect. Sci., 2016, vol. 8, pp. 718–721.
21. *Joseph J. M., Blochwitz C., Garcia-Ortiz A., and Pionteck T.* Area and power savings via asymmetric organization of buffers in 3D-NoCs for heterogeneous 3D-SoCs. Microprocess. Microsyst., 2017, vol. 48, pp. 36–47.
22. *Yebra J. L. A., Fiol M. A., Morillo P., and Alegre I.* The diameter of undirected graphs associated to plane tessellations. Ars Combinatoria, 1985, no. 20B, pp. 159–171.
23. *Martinez C., Stafford E., Beivide R., and Gabidulin E. M.* Modeling hexagonal constellations with Eisenstein — Jacobi graphs. Problems Inform. Transmission, 2008, no. 44(1), pp. 1–11.
24. *Barrière L., Fàbrega J., Simo E., and Zaragozá M.* Fault-tolerant routings in chordal ring networks. Networks, 2000, no. 36(3), pp. 180–190.
25. *Liestman A. L., Opatrny J., and Zaragozá M.* Network properties of double and triple fixed-step graphs. Int. J. Found. Comp. Sci., 1998, no. 9(1), pp. 57–76.
26. *Jha P. K.* A family of efficient six-regular circulants representable as a Kronecker product. Discr. Appl. Math., 2016, no. 203, pp. 72–84.
27. *Parhami B.* A class of odd-radix chordal ring networks. CS'J J. Comput. Sci. Eng., 2006, vol. 4, no. 2–4, pp. 1–9.
28. *Dougherty R. and Faber V.* The degree-diameter problem for several varieties of Cayley Graphs, 1: The Abelian case. SIAM J. Discr. Math., 2004, no. 17(3), pp. 478–519.
29. *Monakhova E. A.* Parametricheskoe zadanie serii semeystv analiticheski opisyvaemykh tsirkulyantnykh setey stepeni shest' [A set of families of analytically described triple loop networks defined by a parameter]. Prikladnaya Diskretnaya Matematika, 2020, no. 49, pp. 108–119. (in Russian)
30. *Monakhova E. A. and Monakhov O. G.* Dinamicheskij algoritm parnoj marshrutizacii dlya analiticheski zadavaemyh semeystv cirkulyantnyh setey stepeni shest' [A dynamic algorithm of two-terminal routing for analytically described families of circulant networks of degree six]. Proc. XIX Inter. Conf. "Problems of Information in Education, Management, Economics and Technology", Penza, Russia, 2019, pp. 30–37. (in Russian)