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UNIQUE LIST COLORABILITY OF THE GRAPH $K_2^n + K_r$

L. X. Hung

Hanoi University for Natural Resources and Environment, Hanoi, Vietnam

E-mail: lxhung@hunre.edu.vn

Given a list $L(v)$ for each vertex v, we say that the graph G is L-colorable if there is a proper vertex coloring of G, where each vertex v takes its color from $L(v)$. The graph is uniquely k-list colorable if there is a list assignment L such that $|L(v)| = k$ for every vertex v and the graph has exactly one L-coloring with these lists. If a graph G is not uniquely k-list colorable, we also say that G has property $M(k)$. The least integer k such that G has the property $M(k)$ is called the m-number of G, denoted by $m(G)$. In this paper, we characterize the unique list colorability of the graph $G = K_2^n + K_r$. In particular, we determine the number $m(G)$ of the graph $G = K_2^n + K_r$.

Keywords: *vertex coloring, list coloring, uniquely list colorable graph, complete r-partite graph.*

ОДНОЗНАЧНАЯ СПИСОЧНАЯ РАСКРАШИВАЕМОСТЬ ГРАФА $K_2^n + K_r$

Л. Хунг

Ханойский университет природных ресурсов и окружающей среды, г. Ханой, Вьетнам

Имея список $L(v)$ для каждой вершины v, мы говорим, что граф L-раскрашиваем, если существует правильная раскраска его вершин, в которой каждая вершина v окрашена цветом из $L(v)$. Граф является однозначно k-раскрашиваемым, если существует список L, такой, что $|L(v)| = k$ для каждой вершины v и граф имеют ровно одну L-раскраску. Если граф G не является однозначно k-раскрашиваемым, то G обладает свойством $M(k)$. Наименьшее целое число k, такое, что G обладает свойством $M(k)$, называется m-числом графа G и обозначается $m(G)$. В работе охарактеризована однозначность списочной раскрашиваемости графа $G = K_2^n + K_r$, в частности определено значение $m(G)$ этого графа.

Ключевые слова: *раскраска вершин графа, раскраска списком, однозначно раскрашиваемый граф, полный r-долевой граф.*

1. Introduction

All graphs considered in this paper are finite undirected graphs without loops or multiple edges. If G is a graph, then $V(G)$ and $E(G)$ (or V and E in short) will denote its vertex-set and its edge-set, respectively. The set of all neighbours of a subset $S \subseteq V(G)$ is denoted by $N_G(S)$ (or $N(S)$ in short). The subgraph of G induced by $W \subseteq V(G)$ is denoted by $G[W]$. The empty graphs (independent sets) and complete graphs of order n are denoted by O_n and K_n, respectively. Unless otherwise indicated, our graph-theoretic terminology will follow [1].

A graph $G = (V, E)$ is called *r-partite graph* if V admits a partition $V = V_1 \cup V_2 \cup \ldots \cup V_r$ such that the subgraphs of G induced by V_i, $i = 1, \ldots, r$, are empty. An r-partite graph in which every two vertices from different partition classes are adjacent is called complete r-partite graph and is denoted by $K_{|V_1|,|V_2|,\dots,|V_r|}$. The complete r-partite graph $K_{|V_1|,|V_2|,\dots,|V_r|}$ with $|V_1| = |V_2| = \ldots = |V_r| = s$ is denoted by K_s^r.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs such that $V_1 \cap V_2 = \varnothing$. Their *union* $G = G_1 \cup G_2$ has, as expected, $V(G) = V_1 \cup V_2$ and $E(G) = E_1 \cup E_2$. Their *join* is denoted by $G_1 + G_2$ and consists of $G_1 \cup G_2$ and all edges joining V_1 with V_2.

The *complement* $\overline{G} = (\overline{V}, \overline{E})$ of $G = (V, E)$ is the graph with $\overline{V} = V$ and for every $u, v \in V$, $uv \in \overline{E}$ if and only if $uv \notin E$.

Let $G = (V, E)$ be a graph and λ is a positive integer.

A λ-*coloring* of G is a mapping $f : V(G) \to \{1, 2, \dots, \lambda\}$ such that $f(u) \neq f(v)$ for any adjacent vertices $u, v \in V(G)$. The smallest positive integer λ such that G has a λ-coloring is called the *chromatic number* of G and is denoted by $\chi(G)$. We say that a graph G is n-*chromatic* if $n = \chi(G)$.

Let $(L(v))_{v\in V}$ be a family of sets. We call a coloring f of G with $f(v) \in L(v)$ for all $v \in V$ a *list coloring from the lists* $L(v)$. We will refer to such a coloring as an L-coloring. The graph G is called λ-*list-colorable*, or λ-*choosable*, if for every family $(L(v))_{v\in V}$ with $|L(v)| = \lambda$ for all v, there is a coloring of G from the lists $L(v)$. The smallest positive integer λ such that G is λ-choosable is called the *list-chromatic number*, or *choice number* of G and is denoted by $ch(G)$. The idea of list colorings of graphs was given independently by V. G. Vizing [2], P. Erdös, A. L. Rubin, and H. Taylor [3].

Let G be a graph with n vertices and suppose that for each vertex v in G, there exists a list of k colors $L(v)$, such that there exists a unique L-coloring for G, then G is called a *uniquely k-list colorable graph* or a UkLC graph in short. If a graph G is not uniquely k-list colorable, we also say that G has property $M(k)$. So G has the property $M(k)$ if and only if for any collection of lists assigned to its vertices, each of size k, either there is no list coloring for G or there exist at least two list colorings. The smallest positive integer k such that G has the property $M(k)$ is called the *m-number* of G, denoted by $m(G)$. The idea of uniquely colorable graph was introduced in [4, 5].

For example, one can easily see that the graph $G = K_{1,1,2}$ is U2LC and it has the property M(3), i.e., $m(G) = 3$. Indeed, let $V(G) = \{u_1, u_2, v_1, v_2\}$, $E(G) = \{u_1v_1, u_1v_2, u_2v_1, u_2v_2, v_1v_2\}$. We assign the following lists for the vertices: $L(u_1) = \{1, 3\}$, $L(u_2) = \{2, 3\}$, $L(v_1) = \{1, 3\}$, and $L(v_2) = \{2, 3\}$. Then, a unique coloring f of G exists from the assigned lists: $f(u_1) = f(u_2) = 3$, $f(v_1) = 1$, $f(v_2) = 2$. Thus, G is U2LC. If $G = K_{1,1,2}$ is U3LC, then there exists lists for the vertices $L(u_1) = \{a_{11}, a_{12}, a_{13}\}$, $L(u_2) = \{a_{21}, a_{22}, a_{23}\}$, $L(v_1) = \{b_{11}, b_{12}, b_{13}\}$, and $L(v_2) = \{b_{21}, b_{22}, b_{23}\}$ such that there exists a unique coloring f of G, we may assume that $f(u_1) = a_{11}$, $f(u_2) = a_{21}$, $f(v_1) = b_{11}$, $f(v_2) = b_{21}$. If there exists $x \in \{a_{12}, a_{13}\}$ such that $x \notin \{b_{11}, b_{21}\}$, then there is a coloring g of G with $g(u_1) = x$, $g(u_2) = a_{21}$, $g(v_1) = b_{11}$, and $g(v_2) = b_{21}$, it follows that $g \neq f$, a contradiction. So $\{a_{12}, a_{13}\} = \{b_{11}, b_{21}\}$. Similarly, we can show that $\{a_{22}, a_{23}\} = \{b_{11}, b_{21}\}$. If $a_{11} \in \{b_{12}, b_{13}\}$, then there is a coloring g of G with $g(u_1) = b_{11}$, $g(u_2) = b_{11}$, $g(v_1) = a_{11}$, and $g(v_2) = b_{21}$, it follows that $g \neq f$, a contradiction. So $a_{11} \notin \{b_{12}, b_{13}\}$. Similarly, we can show that $a_{21} \notin \{b_{12}, b_{13}\}$. Let $y \in \{b_{12}, b_{13}\} \setminus \{b_{21}\}$. Then there is a coloring g of G with $g(u_1) = a_{11}$, $g(u_2) = a_{21}$, $g(v_1) = y$, and $g(v_2) = b_{21}$, it follows that $g \neq f$, a contradiction. Thus, G is not U3LC.

The list coloring model can be used in the channel assignment. The fixed channel allocation scheme leads to low channel utilization across the whole channel. It requires a more effective channel assignment and management policy, which allows unused parts of channel to become available temporarily for other usages so that the scarcity of the channel can be largely mitigated [6]. It is a discrete optimization problem. A model for channel availability, observed by the secondary users, is introduced in [6]. The research of list coloring consists of two parts: choosability and unique list colorability. In [7], we characterized list-chromatic number of the graph $G = K_2^n + O_r$, we have proved that $ch(G) = n + 1$ if $1 \leqslant r \leqslant 2$. In [8], we characterized list-chromatic number and characterized unique list colorability of the graph $G = K_2^n + K_r$, we have proved that $ch(G) = n + r$, G is U3LC if and only if $2n + r \geqslant 7$ and $n \geqslant 2$.

In this paper, we continue to characterize the unique list colorability of the graph $G = K_2^n + K_r$. In particular, we determine the number $m(G)$ of the graph $G = K_2^n + K_r$.

2. Preliminaries

We need the following Lemmas 1–10 to prove our results.

Lemma 1 [5]. Each UkLC graph is also a U$(k-1)$LC graph.

Lemma 2 [5]. The graph G is UkLC if and only if $k < m(G)$.

Lemma 3 [5]. A connected graph G has the property $M(2)$ if and only if every block of G is either a cycle, a complete graph, or a complete bipartite graph.

Lemma 4 [9]. For every graph G we have $m(G) \leqslant |E(\overline{G})| + 2$.

Lemma 5 [9]. Every UkLC graph has at least $3k - 2$ vertices.

Lemma 6. With $G = K_2^n + K_r$, we have $m(G) \leqslant n + 2$.

Proof. It is clear that $|E(\overline{G})| = n$. By Lemma 4, $m(G) \leqslant n + 2$. ∎

Lemma 7.

(i) If $n = 1$ and $r = 1$, then $G = K_2^n + K_r$ has the property $M(2)$.

(ii) If $n = 1$ and $r \geqslant 2$, then $m(K_2^n + K_r) = 3$.

Proof.

(i) If $n = 1$ and $r = 1$, then $G = K_2^n + K_r$ is a complete bipartite graph, then by Lemma 3, G has the property $M(2)$.

(ii) By Lemma 3, $G = K_2^n + K_r$ is U2LC. It is not difficult to see that $|E(\overline{G})| = 1$. By Lemma 4, $m(K_2^n + K_r) \leqslant 3$. Thus, $m(K_2^n + K_r) = 3$ if $n = 1$ and $r \geqslant 2$. ∎

Lemma 8. $m(K_2^2 + K_r) = 3$ for every $1 \leqslant r \leqslant 2$.

Proof. By Lemma 3, $G = K_2^2 + K_r$ is U2LC. Suppose that G is U3LC. By Lemma 5, $|V(G)| \geqslant 7$, a contradiction. So $m(G) = 3$. ∎

Lemma 9 [8]. $G = K_2^2 + K_r$ is U3LC for every $r \geqslant 3$.

Lemma 10. $m(K_2^2 + K_r) = 4$ if and only if $r \geqslant 3$.

Proof. Suppose that $m(K_2^2+K_r) = 4$. If $1 \leqslant r \leqslant 2$, then by Lemma 8, $m(K_2^2+K_r) = 3$, a contradiction.

Suppose that $r \geqslant 3$. By Lemma 9, $G = K_2^2 + K_r$ is U3LC. So $m(K_2^2 + K_r) \geqslant 4$. By Lemma 6, $m(K_2^2 + K_r) \leqslant 4$. Thus, $m(K_2^2 + K_r) = 4$. ∎

3. Main Results

Theorem 1. Let $K_2^{n-1} + K_{r-1}$ is UkLC for every $n, r \geqslant 2$. Then

(i) $K_2^n + K_{r-1}$ is UkLC and

$$m(K_2^{n-1} + K_{r-1}) \leqslant m(K_2^n + K_{r-1}) \leqslant m(K_2^{n-1} + K_{r-1}) + 2;$$

(ii) $K_2^{n-1} + K_r$ is UkLC and

$$m(K_2^{n-1} + K_{r-1}) \leqslant m(K_2^{n-1} + K_r) \leqslant m(K_2^{n-1} + K_{r-1}) + 1.$$

Proof.

(i) We prove $G = K_2^n + K_{r-1}$ is UkLC by induction on n. If $n = 2$, then by Lemma 7 to Lemma 10, we deduce what to prove. So let $n > 2$ and assume the assertion for smaller values of n.

Let $V(G) = V_1 \cup V_2 \cup V_3 \cup \ldots \cup V_{n+r-1}$ is a partition of $V(G)$ such that $|V_1| = |V_2| = \ldots = |V_n| = 2$, $|V_{n+1}| = |V_{n+2}| = \ldots = |V_{n+r-1}| = 1$ and for every $i = 1, 2, \ldots, n$ the subgraph of G induced by V_i is an independent set. Set $V_i = \{u_{i1}, u_{i2}\}$ for every $i = 1, \ldots, n$ and $G' = G - V_n$. By the induction hypothesis, for each vertex v in G', there exists a list of k colors $L'(v)$, such that there exists a unique f' for G'. We assign the following lists for the vertices of G:

$$L(u_{n1}) = L(u_{n2}) = \{f'(u_{11}), f'(u_{21}), \ldots, f'(u_{(k-1)1}), l\},$$

with $l \notin f'(G')$, $L(v) = L'(v)$ if $v \in V(G')$. A unique coloring f of G exists from the assigned lists: $f(u_{n1}) = f(u_{n2}) = l$, $f(v) = f'(v)$ if $v \in V(G')$.

Thus, $G = K_2^n + K_{r-1}$ is UkLC. It follows that $m(K_2^{n-1} + K_{r-1}) \leqslant m(K_2^n + K_{r-1})$.

Put $m(K_2^{n-1} + K_{r-1}) = t$. For suppose on the contrary that graph $G = K_2^n + K_{r-1}$ satisfies $m(G) = h > t + 2$. So there exists a list of $h - 1$ colors $L(v)$ for each vertex $v \in V(G)$, such that there exists a unique L-coloring f for G. We consider separately two cases.

Case 1: $|f(V_n)| = 1$.

In this case, $f(u_{n1}) = f(u_{n2}) = a$. We assign the following lists $L'(v)$ for the vertices v of G':

(a) If $a \in L(v)$, then $L'(v) = L(v) \setminus \{a\}$;

(b) If $a \notin L(v)$, then $L'(v) = L(v) \setminus \{b\}$, where $b \in L(v)$ and $b \neq f(v)$.

It is clear that $|L'(v)| = h - 2 \geqslant t + 1$ for every $v \in V(G')$. Since G' has the property $M(t)$, by Lemma 1, G' has the property $M(t+1)$, so G' has the property $M(h-2)$. It follows that with lists $L'(v)$, there exist at least two list colorings for the vertices v of G'. So it is not difficult to see that with lists $L(v)$, there exist at least two list colorings for the vertices v of G, a contradiction.

Case 2: $|f(V_n)| = 2$.

In this case, $f(u_{11}) = a, f(u_{12}) = b, a \neq b$. We assign the following lists $L'(v)$ for the vertices v of G':

(a) If $a, b \in L(v)$, then $L'(v) = L(v) \setminus \{a, b\}$;

(b) If $a \in L(v), b \notin L(v)$, then $L'(v) = L(v) \setminus \{a, c\}$, where $c \in L(v)$ and $c \neq f(v)$;

(c) If $a \notin L(v), b \in L(v)$, then $L'(v) = L(v) \setminus \{b, c\}$, where $c \in L(v)$ and $c \neq f(v)$;

(d) If $a, b \notin L(v)$, then $L'(v) = L(v) \setminus \{c, d\}$, where $c, d \in L(v), c \neq d$ and $c, d \neq f(v)$.

It is clear that $|L'(v)| = h - 3 \geqslant t$ for every $v \in V(G')$. Since G' has the property $M(t)$, by Lemma 1, G' has the property $M(h-3)$. It follows that with lists $L'(v)$, there exist at least two list colorings for the vertices v of G'. So it is not difficult to see that with lists $L(v)$, there exist at least two list colorings for the vertices v of G, a contradiction.

Thus, $m(K_2^{n-1} + K_{r-1}) \leqslant m(K_2^n + K_{r-1}) \leqslant m(K_2^{n-1} + K_{r-1}) + 2$.

(ii) We prove $G = K_2^{n-1} + K_r$ is UkLC by induction on r. For $r = 2$, it is not difficult we deduce what to prove. So let $r > 2$ and assume the assertion for smaller values of r. Let $V(G) = V_1 \cup V_2 \cup V_3 \cup \ldots \cup V_{n+r-1}$ is a partition of $V(G)$ such that $|V_1| = |V_2| = \ldots = |V_{n-1}| = 2$, $|V_n| = |V_{n+1}| = \ldots = |V_{n+r-1}| = 1$ and for every $i = 1, 2, \ldots, n-1$ the subgraph of G induced by V_i is an independent set. Set $V_i = \{v_i\}$ for every $i = n, n+1, \ldots, n+r-1$ and $G' = G - V_n$. By the induction hypothesis, for each vertex v in G', there exists a list of k colors $L'(v)$, such that there exists a unique f' for G'.

We assign the following lists for the vertices of G:

$$L(v_n) = \{t_1, t_2, \ldots, t_{k-1}, t_k\}$$

with $t_1, t_2, \ldots, t_{k-1} \in f'(G')$, $t_k \notin f'(G')$, $L(v) = L'(v)$ if $v \in V(G')$. A unique coloring f of G exists from the assigned lists: $f(v_n) = t_k$, $f(v) = f'(v)$ if $v \in V(G')$.

Thus, $K_2^{n-1} + K_r$ is UkLC. It follows that $m(K_2^{n-1} + K_{r-1}) \leqslant m(K_2^{n-1} + K_r)$.

Put $m(K_2^{n-1} + K_{r-1}) = t$. For suppose on the contrary that graph $G = K_2^{n-1} + K_r$ satisfies $m(G) = h > t + 1$. So there exists a list of $h - 1$ colors $L(v)$ for each vertex $v \in V(G)$, such that there exists a unique L-coloring f for G. Let $f(v_n) = a$. We assign the following lists $L'(v)$ for the vertices v of G':

(a) If $a \in L(v)$, then $L'(v) = L(v) \setminus \{a\}$;

(b) If $a \notin L(v)$, then $L'(v) = L(v) \setminus \{b\}$, where $b \in L(v)$ and $b \neq f(v)$.

It is clear that $|L'(v)| = h - 2 \geqslant t$ for every $v \in V(G')$. Since G' has the property $M(t)$, so G' has the property $M(h-2)$. It follows that with lists $L'(v)$, there exist at least two list colorings for the vertices v of G'. So it is not difficult to see that with lists $L(v)$, there exist at least two list colorings for the vertices v of G, a contradiction.

Thus, $m(K_2^{n-1} + K_{r-1}) \leqslant m(K_2^{n-1} + K_r) \leqslant m(K_2^{n-1} + K_{r-1}) + 1$. ■

Lemma 11 [8]. $G = K_2^3 + K_r$ is U3LC for every $r \geqslant 1$.

Theorem 2.

(i) $m(K_2^3 + K_r) = 5$ if and only if $r \geqslant 4$;

(ii) $m(K_2^3 + K_r) = 4$ if and only if $1 \leqslant r \leqslant 3$.

Proof.

(i) Suppose that $m(K_2^3 + K_r) = 5$. If $1 \leqslant r \leqslant 3$, then by Lemma 11 and Lemma 5, $m(K_2^3 + K_r) = 4$, a contradiction.

Now we suppose that $r \geqslant 4$. First, we prove $G = K_2^3 + K_r$ is U4LC. Let $V(G) = V_1 \cup V_2 \cup V_3 \cup \ldots \cup V_{3+r}$ is a partition of $V(G)$ such that $|V_1| = |V_2| = |V_3| = 2$, $|V_4| = |V_5| = \ldots = |V_{3+r}| = 1$ and for every $i = 1, 2, 3$ the subgraph of G induced by V_i is an independent set. Set $V_i = \{u_{i1}, u_{i2}\}$ for every $i = 1, 2, 3$ and $V_{3+i} = \{v_i\}$ for every $i = 1, 2, \ldots, r$. We assign the following lists for the vertices of $G = K_2^3 + K_r$:

$L(u_{i1}) = L(v_1) = \{1, 2, 3, 4\}$ for every $i = 1, 2, 3$;

$L(u_{i2}) = \{5, 6, 7, i+1\}$ for every $i = 1, 2, 3$;

$L(v_j) = \{2, 3, 4, 3+j\}$ for every $j = 2, 3, \ldots, r$.

A unique coloring f of G exists from the assigned lists:

$f(u_{i1}) = i + 1$ for every $i = 1, 2, 3$;

$f(u_{i2}) = i + 1$ for every $i = 1, 2, 3$;

$f(v_1) = 1$, $f(v_j) = 3 + j$ for every $j = 2, 3, \ldots, r$.

It follows that G is U4LC. So $m(G) \geqslant 5$. By Lemma 6, $m(G) \leqslant 5$. Thus, $m(G) = 5$.

(ii) Suppose that $m(K_2^3 + K_r) = 4$. If $r \geqslant 4$, then by (i), $m(K_2^3 + K_r) = 5$, a contradiction. Suppose that $1 \leqslant r \leqslant 3$. By Lemma 11 and Lemma 5, $m(K_2^3 + K_r) = 4$. ■

Theorem 3. Let $G = K_2^n + K_r$ be a graph with $n \geqslant 4$ and $r \geqslant 1$. Then

(i) G is UkLC with $k = \lfloor n/2 \rfloor + 1$;

(ii) If $1 \leqslant r < n - 2$, then G has the property $M(n)$;

(iii) If $r \geqslant n - 1$, then G is UnLC;

(iv) If $n - 1 \leqslant r \leqslant n$, then $m(G) = n + 1$;

(v) If $r \geqslant n + 1$, then $m(G) = n + 2$.

Proof. Let $V(G) = V_1 \cup V_2 \cup V_3 \cup \ldots \cup V_{n+r}$ is a partition of $V(G)$ such that $|V_1| = |V_2| = \ldots = |V_n| = 2$, $|V_{n+1}| = |V_{n+2}| = \ldots = |V_{n+r}| = 1$ and for every $i = 1, 2, \ldots, n$ the subgraph of G induced by V_i is an independent set. Set $V_i = \{u_{i1}, u_{i2}\}$ for every $i = 1, \ldots, n$ and $V_{n+i} = \{v_i\}$ for every $i = 1, 2, \ldots, r$.

(i) Put $t = \lfloor n/2 \rfloor$. We assign the following lists for the vertices of G:
$L(u_{i1}) = \{1, 2, \ldots, t+1\}$ for every $i = 1, 2, \ldots, t+1$;
$L(u_{i2}) = \{t+2, t+3, \ldots, 2t+1, i\}$ for every $i = 1, 2, \ldots, t+1$;
$L(u_{(t+1+i)j}) = \{2, 3, \ldots, t+1, t+1+i\}$ for every $i = 1, 2, \ldots, n-t-1$, $j = 1, 2$;
$L(v_i) = \{2, 3, \ldots, t+1, n+i\}$ for every $i = 1, 2, \ldots, r$.
A unique coloring f of G exists from the assigned lists:
$f(u_{i1}) = i$ for every $i = 1, 2, \ldots, t+1$;
$f(u_{i2}) = i$ for every $i = 1, 2, \ldots, t+1$;
$f(u_{(t+1+i)j}) = t+1+i$ for every $i = 1, 2, \ldots, n-t-1$, $j = 1, 2$;
$f(v_i) = n+i$ for every $i = 1, 2, \ldots, r$.

(ii) If $G = K_2^n + K_r$ is UnLC, then by Lemma 5, $|V(G)| \geqslant 3n-2$, a contradiction.

(iii) We assign the following lists for the vertices of G:
$L(u_{i1}) = \{1, 2, \ldots, n\}$ for every $i = 1, 2, \ldots, n$;
$L(u_{i2}) = \{n+1, n+2, \ldots, 2n-1, i\}$ for every $i = 1, 2, \ldots, n$;
$L(v_j) = \{1, 2, \ldots, n-1, n+j\}$ for every $j = 1, 2, \ldots, r$.
A unique coloring f of G exists from the assigned lists:
$f(u_{i1}) = i$ for every $i = 1, 2, \ldots, n$;
$f(u_{i2}) = i$ for every $i = 1, 2, \ldots, n$;
$f(v_j) = n+j$ for every $j = 1, 2, \ldots, r$.
Thus, G is UnLC.

(iv) By (iii), G is UnLC. If G is U$(n+1)$LC, then by Lemma 5, $|V(G)| \geqslant 3n+1$, a contradiction. So $m(G) = n+1$.

(v) We assign the following lists for the vertices of G:
$L(u_{i1}) = L(v_1) = \{1, 2, \ldots, n+1\}$ for every $i = 1, 2, \ldots, n$;
$L(u_{i2}) = \{n+2, n+3, \ldots, 2n+1, i+1\}$ for every $i = 1, 2, \ldots, n$;
$L(v_j) = \{2, 3, \ldots, n+1, n+j\}$ for every $j = 2, 3, \ldots, r$.
A unique coloring f of G exists from the assigned lists:
$f(u_{i1}) = i+1$ for every $i = 1, 2, \ldots, n$;
$f(u_{i2}) = i+1$ for every $i = 1, 2, \ldots, n$;
$f(v_1) = 1$, $f(v_j) = n+j$ for every $j = 2, 3, \ldots, r$.
It follows that G is U$(n+1)$LC. So $m(G) \geqslant n+2$.

By Lemma 6, $m(G) \leqslant n+2$. Thus, $m(G) = n+2$. ■

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