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The sensitivity coefficients for dynamic systems described by interconnected difference ordinary equations and equations with the distributed memory

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Abstract. The variational method of calculation of sensitivity coefficients connecting first variation of quality functional (the Bolts's problem) with variable and constant parameters for multivariate non-linear dynamic systems described by interconnected difference ordinary equations and equations with distributed memory on phase coordinates and variable parameters is developed. Sensitivity coefficients are components of sensitivity functional and they are before variations of variable and constant parameters. The base of calculation of sensitivity coefficients is the decision of object equations in the forward direction of discrete time and corresponding difference conjugate equations for Lagrange's multipliers in the opposite direction of discrete time.

Keywords: variational method; sensitivity coefficient; difference equation; conjugate equation; Lagrange's multiplier

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Коэффициенты чувствительности для динамических систем, описываемых взаимосвязанными разностными обыкновенными уравнениями и уравнениями с распределенной памятью

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Аннотация. Вариационный метод применен для расчета коэффициентов чувствительности, которые связывают первую вариацию функционалов качества работы систем (функционала Больца) с вариациями переменных и постоянных параметров, для многомерных нелинейных динамических систем, описываемых взаимосвязанными разностными обыкновенными уравнениями и уравнениями с распределенной памятью по фазовым координатам и переменным параметрам.

Ключевые слова: вариационный метод; коэффициент чувствительности; разностное уравнение; функционал качества работы системы; сопряженное уравнение; множитель Лагранжа

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The sensitivity coefficients (SC) are components of vector gradient from quality functional according to parameters. The problem of calculation of SC for dynamic systems is principal in the analysis and syntheses of control laws, identification, optimization, stability [1–16]. The first-order sensitivity characteristics are mostly used. Later on we shall examine only SC of the first-order.

Consider two vectorial outputs $x(t)$ and $y(t)$ of dynamic object model: interconnected ordinary difference equations and difference equations with distributed memory on phase coordinates under discrete time $t \in [0,1,...,N+1]$ implicitly depending on vector α parameters and $I(\alpha)$ is functional constructed on $x(t)$, $y(t)$, α under $t \in [0,1,...,N+1]$:

$$I(\alpha) = \sum_{t=0}^{N+1} f_0(x(t), y(t), \alpha, t) .$$

SC with respect to constant α parameters are called a gradient from $I(\alpha)$ on α vector: $(dI(\alpha)/d\alpha)^T \equiv \nabla_\alpha I(\alpha)$. SC are a coefficients of single-line relationship between the first variation $\delta_\alpha I(\alpha)$ of functional $I(\alpha)$ and the variations $\delta\alpha$ of constant α parameters:

$$\delta_\alpha I(\alpha) = \frac{dI(\alpha)}{d\alpha} \delta\alpha \equiv \sum_{j=1}^{m} \frac{\partial I(\alpha)}{\partial \alpha_j} \delta\alpha_j .$$

The direct method of SC calculation (by means of the differentiation of quality functional with respect to constant parameters) inevitably requires a solution of cumbersome sensitivity equations to sensitivity functions: $W_\alpha^x(t) = dx(t)/d\alpha$, $W_\alpha^y(t) = dy(t)/d\alpha$. For instance, for functional $I(\alpha)$ we have following SC

$$\frac{dI(\alpha)}{d\alpha} = \sum_{t=0}^{N+1} \left[\frac{\partial f_0(x(t), y(t), \alpha, t)}{\partial x(t)} W_\alpha^x(t) + \frac{\partial f_0(x(t), y(t), \alpha, t)}{\partial y(t)} W_\alpha^y(t)) + \frac{\partial f_0(x(t), y(t), \alpha, t)}{\partial \alpha} \right].$$

For variable parameters such method essentially becomes complicated and practically is not applicable.

At a choice of good initial approach of parameters at identification of objects and also at consecutive calculation of control actions on object often apply a gradient algorithm. It appears that for calculation of components of a gradient from an optimized functional to required variables and constant parameters, it is convenient to apply the conjugate equations (in relation to the dynamic equations of object).

Variational method [6] makes possible to simplify the process of determination of conjugate equations and formulas of account of SC. On the basis of this method it is an extension of quality functional by means of inclusion into it dynamic equations of object by means of Lagrange's multipliers and obtaining the first variation of extended functional on phase coordinates of object and on interesting parameters. Dynamic equations for Lagrange's multipliers are obtained due to set equal to a zero (in the first variation of extended functional) the functions before the variations of phase coordinates. Given simplification first variation of extended functional brings at presence in the right part only parameter variations, i.e. it is got the sensitivity functional on concerning parameters.

In difference from other papers devoted to calculation of SC in given paper the generalized difference models are used: interconnected ordinary difference equations and difference equations with distributed memory on phase coordinates and variable parameters. Besides variables and constant parameters enter into the right parts of difference equations of dynamic object, in an indicator of quality of system work and initial values of phase coordinates depend on constant parameters. At the right part of the equations of object model there are also phase coordinates and variable parameters during the previous moments of time. Such discrete equations are similar numerical decisions of integro-differential Volterra's equations.

It is proved that both methods to calculation of SC (with use of Lagrange's functions or with use of sensitivity functions) yield the same result, but the first method it is essential more simple in the computing relation.

1. Problem statement

We suppose that the dynamic system is described by non-linear interconnected difference equations

$$x(t+1) = f(x(t), y(t), \tilde{\alpha}(t), \alpha, t), \quad t = 0, 1, 2, \dots, N, \quad x(0) = x_0(\alpha). \quad (1)$$

$$y(t+1) = \sum_{s=0}^{t} K(t, x(s), y(s), \tilde{\alpha}(s), \alpha, s), \quad t = 0, 1, 2, \dots, N, \quad y(0) = y_0(\alpha).$$

Here: x, y are a vector-columns of phase coordinates; $\tilde{\alpha}(t), \alpha$ are a vector-columns of interesting variable and constant parameters; $f(), K(\cdot)$ are known continuously differentiated limited vector-functions.

The quality of functioning of system it is characterised of functional

$$I(\tilde{\alpha}, \overline{\alpha}) = \sum_{t=0}^{N} f_0(x(t), y(t), \tilde{\alpha}(t), \alpha, t) + f_0(x(N+1), y(N+1), \tilde{\alpha}(N+1), \alpha, N+1), \quad (2)$$

depending on $\tilde{\alpha}(t)$ and α. The conditions for function $f_0(\cdot)$ are the same as for $f(\cdot), K(\cdot)$. With use of a functional (2) the optimization problem (in the theory of optimal control) are named as the Bolts's problem.

With the purpose of simplification of appropriate deductions with preservation of a generality in all transformations (1), (2) there are two vectors of parameters $\tilde{\alpha}(t), \alpha$. If in the equations (1), (2) parameters are different then it is possible formally to unit them in two vectors $\tilde{\alpha}(t), \alpha$, to use obtained outcomes and then to make appropriate simplifications, taking into account a structure of a vectors $\tilde{\alpha}(t), \alpha$.

It is shown also that the variation method allows to receive SC in relation to variable and constant parameters:

$$\delta I(\tilde{\alpha}, \alpha) = \sum_{t=0}^{N+1} \frac{\partial I(\tilde{\alpha}, \alpha)}{\partial \tilde{\alpha}(t)} \delta \tilde{\alpha}(t) + \frac{\partial I(\tilde{\alpha}, \alpha)}{\partial \alpha} \delta \alpha. \quad (3)$$

$$\nabla_{\tilde{\alpha}(t)} I(\tilde{\alpha}, \alpha) = \begin{pmatrix} \dfrac{\partial I(\tilde{\alpha}, \alpha)}{\partial \tilde{\alpha}_1(t)} & \cdots & \dfrac{\partial I(\tilde{\alpha}, \alpha)}{\partial \tilde{\alpha}_{m_1}(t)} \end{pmatrix}^{\mathrm{T}}, \quad t = 0, 1, 2, \dots, N, N+1,$$

$$\nabla_{\alpha} I(\tilde{\alpha}, \alpha) = \begin{pmatrix} \dfrac{\partial I(\tilde{\alpha}, \alpha)}{\partial \alpha_1} & \cdots & \dfrac{\partial I(\tilde{\alpha}, \alpha)}{\partial \alpha_{m_2}} \end{pmatrix}^{\mathrm{T}}.$$

By obtaining of results the obvious designations:

$$f(t) \equiv f(x(t), y(t), \tilde{\alpha}(t), \alpha, t), \quad t = 0, 1, 2, \dots, N, \quad (4)$$

$$K(t, s) \equiv K(t, x(s), y(s), \tilde{\alpha}(s), \alpha, s), \quad t = 0, 1, \dots, N, \quad s = 0, 1, \dots, t,$$

$$f_0(t) \equiv f_0(x(t), y(t), \tilde{\alpha}(t), \alpha, t), \quad t = 0, 1, 2, \dots, N+1.$$

are used.

The index t in functions $f_0(x(t), y(t), \tilde{\alpha}(t), \alpha, t)$, indexes t, s in functions $K(t, x(s), y(s), \tilde{\alpha}(s), \alpha, s)$ and t in functions $f_0(x(t), y(t), \tilde{\alpha}(t), \alpha, t)$ also reflects not only obvious dependence on step number, but also that the kind of functions from a step to a step can change.

Let's receive the conjugate equations for calculation of Lagrange's multipliers and on the basis of them formulas for SC calculation.

2. Conjugate equations

The dynamic equations (1) (written down in the form of restrictions of equalities type) by means of Lagrange's multipliers $\lambda_x(t), \lambda_y(t)$ are added to an initial indicator of optimality $I(\tilde{\alpha}, \alpha)$. The size of the extended indicator of optimality always coincides with size initial functional on which judge an optimality of work of system. SC for both functionals coincide also – section 4 of given paper see. Then for the received extended indicator we write down the first variation and the dynamic equations for $\lambda_x(t), \lambda_y(t)$ turn out from an additional conditions of equality to zero of the functions which are before variations of phase coordinates $\delta x(N+1), ..., \delta x(1), \delta x(0),\quad \delta y(N+1), ..., \delta y(1), \delta y(0)$. Coefficients before variations of parameters $\delta\tilde{\alpha}(t), \delta\alpha$ in the first variation of the extended indicator I represent required SC (3).

Complement a quality functional (2) by restrictions-equalities (1) by means of Lagrange's multipliers $\lambda_x(t),\ \lambda_y(t),\ t = 0, 1, 2, \dots, N+1$ (column vectors) and get the extended functional:

$$I = I(\tilde{\alpha},\alpha) + \sum_{t=0}^{N} \lambda_x^{\mathrm{T}}(t+1)\left[-x(t+1) + f(t)\right] + \lambda_x^{\mathrm{T}}(0)[-x(0) + x_0(\alpha)] + \tag{5}$$

$$+\sum_{t=0}^{N} \lambda_y^{\mathrm{T}}(t+1)\left[-y(t+1) + \sum_{s=0}^{t} K(t,s)\right] + \lambda_y^{\mathrm{T}}(0)\left[-y(0) + y_0(\alpha)\right] = \sum_{t=0}^{N} f_0(t) + f_0(N+1) -$$

$$-\lambda_x^{\mathrm{T}}(N+1)x(N+1) + \sum_{t=0}^{N}\left[-\lambda_x^{\mathrm{T}}(t)x(t) + \lambda_x^{\mathrm{T}}(t+1)\, f_x(t)\right] + \lambda_x^{\mathrm{T}}(0)x_0(\alpha) -$$

$$-\lambda_y^{\mathrm{T}}(N+1)y(N+1) + \sum_{t=0}^{N}\left[-\lambda_y^{\mathrm{T}}(t)y(t)\right] + \sum_{t=0}^{N}\sum_{s=0}^{t} \lambda_y^{\mathrm{T}}(t+1)K(t,s) + \lambda_y^{\mathrm{T}}(0)y_0(\alpha)\,.$$

Functional (5) is equal to $I(\tilde{\alpha},\alpha)$ when (1) is fulfilled.

We consider equality [16]

$$\sum_{t=0}^{N}\sum_{s=0}^{t} \lambda_y^{\mathrm{T}}(t+1)K(t,s) = \sum_{0}^{t=N}\sum_{t}^{s=N} \lambda_y^{\mathrm{T}}(s+1)K(s,t)\,.$$

The proof of correctness of equality is realized by an mathematical induction method.

Extended functional now becomes:

$$I = f_0(N+1) + \sum_{0}^{t=N} f_0(t) - \tag{6}$$

$$-\lambda_x^{\mathrm{T}}(N+1)x(N+1) + \sum_{0}^{t=N}\left[-\lambda_x^{\mathrm{T}}(t)x(t) + \lambda_x^{\mathrm{T}}(t+1)\, f(t)\right] + \lambda_x^{\mathrm{T}}(0)x_0(\alpha) -$$

$$-\lambda_y^{\mathrm{T}}(N+1)y(N+1) + \sum_{0}^{t=N}\left[-\lambda_y^{\mathrm{T}}(t)y(t) + \sum_{t}^{s=N} \lambda_y^{\mathrm{T}}(s+1)K(s,t)\right] + \lambda_y^{\mathrm{T}}(0)y_0(\alpha)\,.$$

We calculate the first variation of extended functional, caused by a variation of phase coordinates, and also a variation of variables and constant parameters:

$$\delta I = \sum_{t=0}^{N+1} \frac{\partial I}{\partial x(t)}\delta x(t) + \sum_{t=0}^{N+1} \frac{\partial I}{\partial y(t)}\delta y(t) + \sum_{t=0}^{N+1} \frac{\partial I}{\partial \tilde{\alpha}(t)}\delta\tilde{\alpha}(t) + \frac{\partial I}{\partial \alpha}\delta\alpha. \tag{7}$$

The factors standing in the formula (7) before variations of phase coordinates look like:

$$\frac{\partial I}{\partial x(N+1)} = -\lambda_x^{\mathrm{T}}(N+1) + \frac{\partial f_0(N+1)}{\partial x(N+1)}, \tag{8}$$

$$\frac{\partial I}{\partial x(t)} = -\lambda_x^{\mathrm{T}}(t) + \lambda_x^{\mathrm{T}}(t+1)\frac{\partial f(t)}{\partial x(t)} + \sum_{t}^{s=N} \lambda_y^{\mathrm{T}}(s+1)\frac{\partial K(s,t)}{\partial x(t)} + \frac{\partial f_0(t)}{\partial x(t)},\ t = N, N-1, \dots, 1, 0,$$

$$\frac{\partial I}{\partial y(N+1)} = -\lambda_y^{\mathrm{T}}(N+1) + \frac{\partial f_0(N+1)}{\partial y(N+1)},$$

$$\frac{\partial I}{\partial y(t)} = -\lambda_y^{\mathrm{T}}(t) + \lambda_x^{\mathrm{T}}(t+1)\frac{\partial f(t)}{\partial y(t)} + \sum_{t}^{s=N} \lambda_y^{\mathrm{T}}(s+1)\frac{\partial K(s,t)}{\partial y(t)} + \frac{\partial f_0(t)}{\partial y(t)},\ t = N, N-1, \dots, 1, 0.$$

We equate values (8) to zero and receive the conjugate equations for Lagrange's multipliers:

$$\lambda_x^{\mathrm{T}}(N+1) = \frac{\partial f_0(N+1)}{\partial x(N+1)},\ \lambda_y^{\mathrm{T}}(N+1) = \frac{\partial f_0(N+1)}{\partial y(N+1)}, \tag{9}$$

$$\lambda_x^{\mathrm{T}}(t) = \lambda_x^{\mathrm{T}}(t+1)\frac{\partial f(t)}{\partial x(t)} + \sum_{t}^{s=N}\lambda_y^{\mathrm{T}}(s+1)\frac{\partial K(s,t)}{\partial x(t)} + \frac{\partial f_0(t)}{\partial x(t)},\ t = N, N-1, \dots, 1, 0\,;$$

$$\lambda_y^{\mathrm{T}}(t) = \lambda_x^{\mathrm{T}}(t+1)\frac{\partial f(t)}{\partial y(t)} + \sum_{t}^{s=N}\lambda_y^{\mathrm{T}}(s+1)\frac{\partial K(s,t)}{\partial y(t)} + \frac{\partial f_0(t)}{\partial y(t)},\ t = N, N-1, \dots, 1, 0\,.$$

These equations are decided in the opposite direction changes of an independent integer variable t.

3. Sensitivity coefficients

SC in the equation (7) for variables and constant parameters look like:

$$\frac{\partial I}{\partial\tilde{\alpha}(N+1)} = \frac{\partial f_0(N+1)}{\partial\tilde{\alpha}(N+1)}, \tag{10}$$

$$\frac{\partial I}{\partial\tilde{\alpha}(t)} = \frac{\partial f_0(t)}{\partial\tilde{\alpha}(t)} + \lambda_x^{\mathrm{T}}(t+1)\frac{\partial f(t)}{\partial\tilde{\alpha}(t)} + \sum_{t}^{s=N}\lambda_y^{\mathrm{T}}(s+1)\frac{\partial K(s,t)}{\partial\tilde{\alpha}(t)},\ t = N, N-1, \dots, 1, 0,$$

$$\frac{\partial I}{\partial\alpha} = \frac{\partial f_0(N+1)}{\partial\alpha} +$$

$$+\sum_{0}^{t=N}\left[\frac{\partial f_0(t)}{\partial\alpha} + \lambda_x^{\mathrm{T}}(t+1)\frac{\partial f(t)}{\partial\alpha} + \sum_{t}^{s=N}\lambda_y^{\mathrm{T}}(s+1)\frac{\partial K(s,t)}{\partial\alpha}\right] + \lambda_x^{\mathrm{T}}(0)\frac{dx_0(\alpha)}{d\alpha} + \lambda_y^{\mathrm{T}}(0)\frac{dy_0(\alpha)}{d\alpha}.$$

This result is more common in relation to appropriate results of monograph [13] and paper [16].

4. Equivalence of sensitivity coefficient for initial (2) and extended (5) functionals

We take extended functional (5):

$$I = I(\tilde{\alpha},\alpha) + \sum_{t=0}^{N}\lambda_x^{\mathrm{T}}(t+1)\left[-x(t+1) + f(t)\right] + \lambda_x^{\mathrm{T}}(0)[-x(0) + x_0(\alpha)] +$$

$$+\sum_{t=0}^{N}\lambda_y^{\mathrm{T}}(t+1)\left[-y(t+1) + \sum_{s=0}^{t}K(t,s)\right] + \lambda_y^{\mathrm{T}}(0)\left[-y(0) + y_0(\alpha)\right].$$

In brackets (before $\lambda_x^{\mathrm{T}}(\cdot), \lambda_y^{\mathrm{T}}(\cdot)$) there are the dynamic equations of the object which have been written down in the form of the equality equations. Hence, values of functions in brackets are always equal to zero.

Let's calculate derivatives from both parts of the previous equations in the beginning on constant α parameters:

$$\frac{\partial I}{\partial\alpha} = \frac{\partial I(\tilde{\alpha},\alpha)}{\partial\alpha} + \sum_{t=0}^{N}\lambda_x^{\mathrm{T}}(t+1)\left[-W_\alpha^x(t+1) + \frac{\partial f(t)}{\partial x(t)}W_\alpha^x(t) + \frac{\partial f(t)}{\partial y(t)}W_\alpha^y(t) + \frac{\partial f(t)}{\partial\alpha}\right] + \tag{11}$$

$$+\lambda_x^{\mathrm{T}}(0)\left[-W_\alpha^x(0) + \frac{dx_0(\alpha)}{d\alpha}\right] +$$

$$+\sum_{t=0}^{N}\lambda_y^{\mathrm{T}}(t+1)\left[-W_\alpha^y(t+1) + \sum_{s=0}^{t}\left(\frac{\partial K(t,s)}{\partial x(s)}W_\alpha^x(s) + \frac{\partial K(t,s)}{\partial y(s)}W_\alpha^y(s) + \frac{\partial K(t,s)}{\partial\alpha}\right)\right] +$$

$$+\lambda_y^{\mathrm{T}}(0)\left[-W_\alpha^y(0) + \frac{dy_0(\alpha)}{d\alpha}\right].$$

Here $W_\alpha^x(t) = dx(t)/d\alpha$, $W_\alpha^y(t) = dy(t)/d\alpha$ there are the sensitivity functions.

Before $\lambda_x^{\mathrm{T}}(\cdot), \lambda_y^{\mathrm{T}}(\cdot)$ now there are sensitivity equations for a matrix of sensitivity functions. These equations are written down as in the form of the equality equations. Values of functions in brackets also are always equal to zero.

Hence, SC rather on α parameters both for initial functional and for its extended variant have identical values. That the sensitivity equations have the specified appearance, it is necessary for equations (1) to impose a condition of differentiability of $f(t)$ and $K(t,s)$ on phase coordinates and on considered parameters on the right member of equations of movement of dynamic object (1). On α parameters should be differentiated initial functions $x_0(\alpha)$ and $y_0(\alpha)$.

We receive the same result and for SC in relation to variable parameters. The sensitivity equations for each fixed value of argument of variable $\tilde{\alpha}(j),\ j = 0,1,..., N+1$ parameters have more difficult form. They demand special consideration. Important that these sensitivity equations objectively exist.

5. Example

We assume that the discrete model of object (1) is set in the form of two vector difference models are not connected with each other on phase coordinates:

$$x(t+1) = f(x(t), \tilde{\alpha}(t), \alpha, t),\ \ t = 0, 1, 2, \ldots, N,\ \ x(0) = x_0(\alpha), \tag{12}$$

$$y(t+1) = \sum_{s=0}^{t} K(t, y(s), \tilde{\alpha}(s), \alpha, s),\ \ t = 0, 1, 2, \ldots, N,\ \ y(0) = y_0(\alpha).$$

However, both models are connected with each other of functional (2) and they can contain the general variables and constant parameters. The first model is similar to the numerical decision of the vector ordinary differential equation, and the second model has memory on phase coordinates and on variable parameters. This model is similar to the decision of the vector integrated equation of Volterra's type.

The conjugate equations (9) for Lagrange's multipliers become more simple:

$$\lambda_x^{\mathrm{T}}(N+1) = \frac{\partial f_0(N+1)}{\partial x(N+1)};\ \lambda_x^{\mathrm{T}}(t) = \lambda_x^{\mathrm{T}}(t+1)\frac{\partial f(t)}{\partial x(t)} + \frac{\partial f_0(t)}{\partial x(t)},\ \ t = N, N-1, \ldots, 1, 0;$$

$$\lambda_y^{\mathrm{T}}(N+1) = \frac{\partial f_0(N+1)}{\partial y(N+1)};\ \lambda_y^{\mathrm{T}}(t) = \sum_{t}^{s=N} \lambda_y^{\mathrm{T}}(s+1)\frac{\partial K(s,t)}{\partial y(t)} + \frac{\partial f_0(t)}{\partial y(t)},\ t = N, N-1, \ldots, 1, 0.$$

The equations (10) for SC do not change.

Conclusion

Variational method allowed to receive effective algorithms of SC calculation for multivariate non-linear dynamic systems described by interconnected difference ordinary equations and difference equations with distributed memory on phase coordinates and variable parameters. Variables and constant parameters are present at object model and at a quality functional for systems.

In a basis of calculation of SC is the decision of the difference equations of object model in a forward direction of time and obtained difference equations for Lagrange's multipliers in the opposite direction of time. It is proved that both methods to calculation of SC (with use of Lagrange's functions or with use of sensitivity functions) yield the same result, but the first method it is essential more simple in the computing relation.

Variation method of calculation of SC allows to generalize it on dynamic systems described by more general nonlinear difference equations and characterized more general nonlinear functionals.

Results of present paper are applicable at design of high-precision systems and devices.

This paper continues research in [13, 16].

References

1. Ostrovsky, G.M. & Volin, Yu.M. (1967) *Metody optimizatsii khimicheskikh reaktorov* [Methods of Optimization of Chemical Reactors]. Moscow: Khimiya.
2. Rosenvasser, E.N. & Yusupov, R.M. (1969) *Chuvstvitel'nost' sistem avtomaticheskogo upravleniya* [Sensitivity of Automatic Control Systems]. Leningrad: Energiya.
3. Krutyko, P.D. (1969) Reshenie zadachi identifikatsii metodom teorii chuvstvitel'nosti [The decision of a identification problem by a sensitivity theory method]. *Izvestiya Akademii nauk SSSR. Tekhnicheskaya kibernetika*. 6. pp. 146–153.
4. Petrov, B.N. & Krutyko, P.D. (1970) Primenenie teorii chuvstvitel'nosti v zadachakh avtomaticheskogo upravleniya [Application of the sensitivity theory in automatic control problems]. *Izvestiya Akademii nauk SSSR. Tekhnicheskaya kibernetika*. 2. pp. 202–212.
5. Gorodetskiy, V.I., Zakharin, F.M., Rosenvasser, E.N. & Yusupov, R.M. (1971) *Metody teorii chuvstvitel'nosti v avtomaticheskom upravlenii* [Methods of Sensitivity Theory in Automatic Control]. Leningrad: Energiya.
6. Bryson, A.E. & Ho, Ju-Chi. (1972) *Applied Theory of Optimal Control*. Moscow: Mir.
7. Speedy, C.B., Brown, R.F. & Goodwin, G.C. (1973) *Control Theory: Identification and Optimal Control*. Oliver & Boyd.
8. Gekher, K. (1973) *Teoriya chuvstvitel'nosti i dopuski elektronnykh skhem* [Theory of Sensitivity and Tolerances of Electronic Circuits]. Moscow: Sovetskoe radio.
9. Rouban, A.I. (1975) *Identifikatsiya nelineynykh dinamicheskikh ob"ektov na osnove algoritma chuvstvitel'nosti* [Nonlinear dynamic objects identification on the base of sensitivity algorithm]. Tomsk: Tomsk State University.
10. Rosenvasser, E.N. & Yusupov, R.M. (1977) *Problemy kibernetiki. Teoriya chuvstvitel'nosti i ee primenenie* [Cybernetics problems. Sensitivity theory and its application]. Vol. 23. Moscow: Svyaz'.
11. Voronov, A.A. (1979) *Ustoychivost', upravlyaemost', nablyudaemost'* [Stability, Controllability, Observability]. Moscow: Nauka.
12. Rosenvasser, E.N. & Yusupov, R.M. (1981) *Chuvstvitel'nost' sistem upravleniya* [Sensitivity of Control Systems]. Moscow: Nauka.
13. Rouban, A.I. (1982) *Identifikatsiya i chuvstvitel'nost' slozhnykh sistem* [Identification and Sensitivity of Complex Systems]. Tomsk: Tomsk State University.
14. Haug, E.J., Choi, K.K. & Komkov, V. (1988) *Design sensitivity analysis of structural systems*. Moscow: Mir.
15. Afanasiev, V.N., Kolmanovskiy, V.B. & Nosov, V.R. (1998) *Matematicheskaya teoriya proektirovaniya sistem upravleniya* [The Mathematical Theory of Designing Control Systems]. Moscow: Vysshaya shkola.
16. Rouban, A.I. (2021) The sensitivity coefficients for dynamic systems described by difference equations with the distributed memory on phase coordinates and variable parameters. *Vestnik Tomskogo gosudarstvennogo universiteta. Upravlenie vychislitelnaya tekhnika i informatika – Tomsk State University Journal of Control and Computer Science*. 57. pp. 95–100. DOI: 10.17223/19988605/57/10

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