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Reidemeister torsion of link complements in a 3-torus

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Abstract. We prove that the Reidemeister torsion and the twisted Alexander polynomial of the complement of a link in a three-dimensional torus are the same.
Keywords: knots, links, three-dimensional torus, twisted Alexander polynomial, Reidemeister torsion, *CW*-complex, Fox calculus

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Научная статья

Кручение Рейдемейстера для дополнения зацепления в трехмерном торе

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Аннотация. Классическая теория узлов, изучая задачи вложений окружности в трехмерную сферу, была расширена до более широкой теории. Например, теорию виртуальных узлов можно рассматривать как теорию узлов на утолщенных замкнутых ориентированных поверхностях. Теория узлов в других трехмерных многообразиях, таких как проективное и линзовое пространство, воплотилась в жизнь в последнее десятилетие. Автор исследовал диаграммный подход к изучению узлов в трехмерном торе. В работе предложен алгоритм вычисления скрученных полиномов Александера узлов и зацеплений в трехмерном торе. Доказано, что кручение Рейдемейстера дополнения к зацеплению и его скрученный полином Александера равны. Связь между полиномом Александера узла и инвариантом кручения Рейдемейстера, Франца и де Рама для дополнения узла была впервые замечена Милнором. Как следствие этого соотношения Милнор дал еще одно доказательство симметрии полинома Александера. Милнор применил этот результат к теории узлов, рассматривая случай классического узла в трехмерной сфере, т.е. дополнение узла имеет гомологию окружности. Оказывается, существуют аналогичные отношения между кручением Рейдемейстера и скрученным полиномом Александера для случая дополнения узла в других пространствах, отличных от трехмерной сферы, когда первая группа гомологии содержит также кручение. Технология получения явных отношений была создана Милнором, используя теорию простых гомотопий для *CW*-комплексов и свободное дифференциальное исчисление Фокса. Они допускают клеточную структуру *CW* для узла, связанную с данным представлением фундаментальной группы, так что граничные операторы получаются посредством свободных производных Фокса. Таким образом, показано, что этот метод имеет эффект также для случая узлов и зацеплений в трехмерном торе.

Ключевые слова: узлы, зацепления, трехмерный тор, скрученный полином Александера, кручение Рейдемейстера, *CW*-комплекс, исчисление Фокса

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1. Introduction

Recently, the classical knot theory has been extended to a wider theory, such as the virtual knot theory that can be considered as the theory of knots in thickened closed oriented surfaces. The theory of knots in other 3-manifolds, such as the projective space and the lens space, had come to life in the last decade. In a forthcoming paper [1] the author investigated the diagrammatic approach to the study of knots in a three-dimensional torus. In that work, I establish an algorithm for computing twisted Alexander polynomials of knots and links in a three-dimensional torus. In this paper, I prove that the Reidemeister torsion of the link complement and its twisted Alexander polynomial are equal.

The relation between the Alexander polynomial of a knot and the torsion invariant of Reidemeister, Franz, and de Rham for knot complement was first noticed by Milnor (see [2]). As a consequence of the relation, Milnor gave another proof for symmetry of the Alexander polynomial [3]. Milnor applied the result to the knot theory, considering the case of classical knot i.e., the knot complement has the homology of the circle. It turns out that there are similar relations between Reidemeister torsion and twisted Alexander polynomial for the case of knot complement in other spaces, rather than three-dimensional sphere when the homology group contains also torsion (see [4, 5]). The technology to get explicit relations as Milnor had created making use of the simple homotopy theory for CW-complexes and Fox free differential calculus. Those ensure a CW structure for the knot complement associated with a presentation of the fundamental group, so that the boundary maps are obtained by free derivatives. Thus, in Section 5 I show that the method works out fine also for the case of knots and links in a three-dimensional torus.

1. The first homology group of link complement in T^3

A link L with n components in a three-dimensional torus T^3 is an embedding of a disjoint union of n circles S^1 into three-dimensional torus. If $n = 1$, the link is called knot. Two links are considered equivalent if they are ambient isotopic, that is, if there exists a continuous deformation of T^3 which takes one link to the other.

A diagram of link in T^3 is a regular plane graph represented on a square (see [1]), which has nodes of 4-valent (with extra structure representing the crossing in the link) and 2-valent nodes (vertices with poles). It is said that two such diagrams are equivalent if there is a sequence of generalized Reidemeister moves (see Fig. 1) and vertex moves indicated in Fig. 2 taking one diagram to the other. These moves are performed locally on the regular plane graph (with extra structure) that constitutes the link diagram.

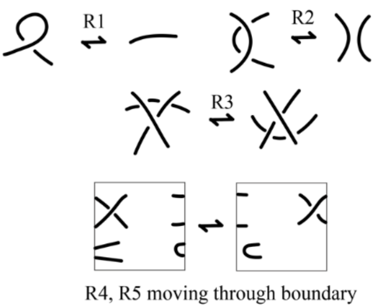

Fig. 1. Generalized Reidemeister moves

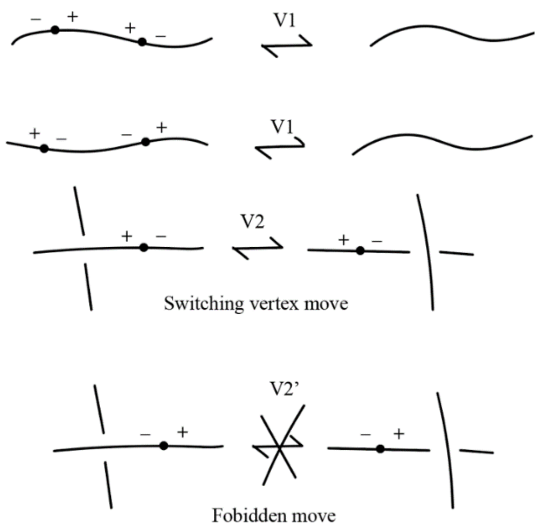

Fig. 2. Vertex moves

I proposed an algorithm to get a presentation of the fundamental group of link complement in [1] from a diagram of a link as defined above. Having a diagram of a knot K in a three-dimensional torus, we can easily define its homology class $[K] \in \mathbb{Z}^3$ of K. Also having a presentation of the fundamental group of link complement, the abelianization of the fundamental group $\pi_1(T^3 \setminus L)/[\pi_1(T^3 \setminus L), \pi_1(T^3 \setminus L)]$ is its first homology group $H_1(T^3 \setminus L)$. Thus, I recall the following theorem from [1] about the first homology group of link complement in T^3.

Theorem 1 [1]. Let L be a link in 3-torus T^3, with components $L_1, ..., L_\omega$. For each $\iota = 1, ..., \omega$, let $(\delta_\iota, \sigma_\iota, \xi_\iota) = [L_\iota] \in \mathbb{Z}^3 = H_1(T^3)$. Then

$$H_1(T^3 \setminus L) \cong \begin{cases} \mathbb{Z}^3 \oplus \mathbb{Z}_\rho, & \text{if } \omega = 1 \\ \mathbb{Z}^3 \oplus \mathbb{Z}_\kappa \oplus \mathbb{Z}_\lambda, & \text{if } \omega = 2 \\ \mathbb{Z}^\omega \oplus \mathbb{Z}_\zeta \oplus \mathbb{Z}_\eta \oplus \mathbb{Z}_\theta, & \text{if } \omega \geq 3. \end{cases}$$

where $\rho = \gcd(\delta_1, \sigma_1, \xi_1)$; κ and λ are the invariant factor of the matrix M_1; ζ, η and θ are the invariant factor of the matrix M_2.

$$M_1 = \begin{pmatrix} \delta_1 & \delta_2 \\ \sigma_1 & \sigma_2 \\ \xi_1 & \xi_2 \end{pmatrix}; \qquad M_2 = \begin{pmatrix} \delta_1 & \delta_2 & \dots & \delta_\omega \\ \sigma_1 & \sigma_2 & \dots & \sigma_\omega \\ \xi_1 & \xi_2 & \dots & \xi_\omega \end{pmatrix}.$$

Now we see that the first homology group might contain torsion part. We say that a link $L \in T^3$ is nontorsion if torsion part $\mathrm{Tors}(H_1(T^3 \setminus L))$ is zero, otherwise we say that L is torsion. A local link or affine link is a link that can be isotoped so that it is contained inside a 3-ball in T^3. A local link is clearly nontorsion.

2. Twisted Alexander polynomial

Given a presentation of the group of a link, one may calculate its Alexander polynomial using Fox free calculus [6]. We recall the following definition of Alexander polynomials (compare [5, 7–9]). Let

$$P = \langle x_1, \dots, x_n \mid r_1, \dots, r_m \rangle$$

be a presentation of a group G and denote by $H = G/[G,G]$ its abelianization. Let $F = \langle x_1, \dots, x_n \rangle$ be the corresponding free group. We apply the chain of maps

$$\mathbb{Z}F \xrightarrow{\frac{\partial}{\partial x}} \mathbb{Z}F \xrightarrow{\gamma} \mathbb{Z}G \xrightarrow{\alpha} \mathbb{Z}H,$$

where $\frac{\partial}{\partial x}$ denotes the Fox differential, γ is the quotient map by relations $r_1, \dots, r_m$ and α is the abelianization map. The Alexander–Fox matrix of the presentation P is the matrix $A = [a_{i,j}]$, where $a_{i,j} = \alpha(\gamma(\frac{\partial r_i}{\partial x_j}))$ for $i = 1, \dots, m$ and $j = 1, \dots, n$. For $k = 1, \dots,$ $\min\{m-1, n-1\}$, the k-th elementary ideal $E_k(P)$ is the ideal of $\mathbb{Z}H$, generated by the determinants of all the $(n-k)$ minors of A. The first elementary ideal $E_1(P)$ is the ideal of $\mathbb{Z}H$, generated by the determinants of the all the $(n-1)$ minors of A.

Definition 1. *Let $L \subset S^3$ be a link, and let $E_k(P)$ be the k-th elementary ideal obtained from a presentation P of fundamental group $\pi_1(S^3 \setminus L, *)$. Then the k-th link polynomial $\Delta_k(L)$ is the generator of the smallest principal ideal containing $E_k(P)$. The Alexander polynomial of L, denoted by $\Delta(L)$, is the first link polynomial of L.*

For a classical link L in S^3, the abelianization of $\pi_1(S^3 \setminus L, *)$ is the free abelian group, whose generators correspond to the components of L. For a link in a 3-torus T^3, the abelianization of its link group may also contain torsion, as we know by Theorem 1. In this case, the Alexander polynomial is not defined, we need the notion of twisted Alexander polynomials. Thus, we recall the definition of twisted Alexander polynomials.

Let G be a group with a finite presentation P and abelianization $H = G/[G,G]$ and denote $K = H/\mathrm{Tors}(H)$. Then every representation $\phi: \mathrm{Tors}(H) \to \mathbb{C}^* = \mathbb{C}\setminus\{0\}$ determines a twisted Alexander polynomial $\Delta^\phi(P)$ as follows. Choosing a splitting $H = Tors(H)\times K$, ϕ induces a ring homomorphism $\phi: \mathbb{Z}[H] \to \mathbb{C}[K]$ sending $(f, g) \in Tors(H)\times K$ to $\phi(f)g$. The ring homomorphism is called twisted homomorphism. Thus, we apply the chain of maps

$$\mathbb{Z}[F] \xrightarrow{\frac{\partial}{\partial x}} \mathbb{Z}[F] \xrightarrow{\gamma} \mathbb{Z}[G] \xrightarrow{\alpha} \mathbb{Z}[H] \xrightarrow{\phi} \mathbb{C}[K]$$

and obtain the ϕ-twisted Alexander matrix $A^\phi = \left[\phi(\alpha(\gamma(\frac{\partial r_i}{\partial x_j})))\right]$. The twisted Alexander polynomial is then defined by $\Delta^\phi(P) = \gcd(\phi(E_1(P)))$.

Definition 2. *Let $L \subset T^3$ be a link in the three-dimensional torus T^3. For any presentation P of the link group $\pi_1(T^3 \setminus L, *)$, we may define the following.*

The Alexander polynomial of L, denoted by $\Delta(L)$, is the generator of the smallest principal ideal containing $E_1(P)$.

For any homomorphism $\phi: Tors(H_1(T^3\setminus L)) \to \mathbb{C}^*$, the ϕ-twisted Alexander polynomial of L is $\Delta^\phi(L) = gcd(\phi(E_1(P)))$.

We know from Theorem 1 that the torsion subgroup of $H_1(T^3 \setminus L))$ is the group $\mathbb{Z}_\zeta \oplus \mathbb{Z}_\eta \oplus \mathbb{Z}_\theta$ in general. So the image of the group homomorphism $\phi: \mathrm{Tors}(H_1(T^3\setminus L)) \to \mathbb{C}^*$ is contained in the cyclic group, generated by Ω, the d-root of unity, where d is $\mathrm{lcm}(\zeta, \eta, \theta)$. The ϕ-twisted Alexander polynomial $\Delta^\phi(L) \in \mathbb{Z}[\Omega][K]$ is defined up to multiplication by a unit.

3. Reidemeister torsion of cell complex

In this section I recall the definition of Reidemeister torsion following [4] (for further references see Turaev [9, 10], Milnor [11]).

Let $\mathbb{F}$ be a field, V be a k-dimensional vector space over $\mathbb{F}$. Suppose that $b = (b_1, b_2, ..., b_k)$ and $c = (c_1, c_2, ..., c_k)$ are two bases of V then there is a non-singular $k\times k$ matrix (a_{ij}) such that $b_j = \sum_{i=1}^{k} a_{ij} c_i$. We write $[b/c] = \det(a_{ij}) \in \mathbb{F}^*$. Two bases b and c are said to have the same orientation if $[b/c] > 0$, and to be equivalent if $[b/c] = 1$.

Let $0 \to C \xrightarrow{\alpha} D \xrightarrow{\beta} E \to 0$ be a short exact sequence of vector spaces. Let $c = (c_1, c_2, ..., c_k)$ be a basis for C and $e = (e_1, e_2, ..., e_l)$ be a basis for E. Since the map β is surjective we can lift e_i to a vector $\tilde{e}_i$ in D. Then $ce = (c_1, ..., c_k, \tilde{e}_1, ..., \tilde{e}_l)$ is a basis for D and its equivalent class depends not on the choice of $\tilde{e}_i$ but only on the equivalence classes of c and e.

The finite chain complex $(C,\partial) = (0 \to C_m \xrightarrow{\partial_m} C_{m-1} \xrightarrow{\partial_{m-1}} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1}$ $\xrightarrow{\partial_1} C_0 \to 0)$ of finite-dimensional vector spaces over $\mathbb{F}$ is called acyclic if it is exact. The chain is called based if for each C_i a basis is chosen.

Assume that (C,∂) is acyclic and based with basis c. Choose a basis b_i for $B_i = \operatorname{Im} \partial_{i+1} = \ker \partial_i$. From the short exact sequence $0 \to B_i \to C_i \to B_{i-1} \to 0$ we get a basis $b_i b_{i-1}$ for C_i.

Definition 3. *The torsion of the acyclic and based chain complex C is defined to be* $\tau(C) = \prod_{i=0}^{m} [b_i b_{i-1} / c_i]^{(-1)^{i+1}} \in \mathbb{F}$. *If* C *is not acyclic, then* $\tau(C)$ *is defined to be 0.*

The torsion $\tau(C)$ depends on c but does not depend on the choice of b_i's. If a basis $c_{i'}$ is used instead of c_i, then the torsion is multiplied with $[c_i / c_{i'}]^{(-1)^{i+1}}$.

Let X be a finite connected CW-complex and let $\pi = \pi_1(X)$. The universal cover $\tilde{X}$ of X has a canonical CW-complex structure obtained by lifting the cells of X. If $\{e_i^k, 1 \le i \le n_k\}$ is an ordered set of oriented k-cells of X and $\tilde{e}_i^k$ is any lift of e_i^k, then the ordered set $\{\tilde{e}_i^k, 1 \le i \le n_k\}$ is a basis of the $\mathbb{Z}[\pi]$-module $C_i(\tilde{X})$.

If $\mathbb{Z}[\pi] \xrightarrow{\phi} \mathbb{F}$ is a ring homomorphism then by the change of rings construction $\mathbb{F} \otimes C_*(\tilde{X})$ is a chain complex of finite dimensional vector spaces over $\mathbb{F}$. If this chain complex is acyclic then its torsion $\tau(\mathbb{F} \otimes C_*(\tilde{X})) \in \mathbb{F}^*$ is defined. However, $\tau(\mathbb{F} \otimes C_*(\tilde{X}))$ depends on the chosen of basis for $C_*(\tilde{X})$, that is on the choices of lifting cells $\{\tilde{e}_i^k, 1 \le i \le n_k\}$. If we fix a choice of a set of lifting cells as a basis for the $\mathbb{Z}[\pi]$-module $C_i(\tilde{X})$ but change the order of the cells in the basis then $\tau(\mathbb{F} \otimes C_*(\tilde{X}))$ is multiplied with ± 1. If we change the orientations of the cells, then torsion is also multiplied with ± 1. If we choose a different lifting cell for e_i^k —by an action $h.\tilde{e}_i^k$ of a covering transformation $h \in \pi$—then torsion is multiplied with $\phi(h)^{\pm 1}$.

Definition 4. *The Reidemeister torsion* $\tau^{\phi}(X)$ *of the CW-complex* X *is defined to be the image of* $\tau(\mathbb{F} \otimes C_*(\tilde{X}))$ *under the quotient map* $\mathbb{F} \to \mathbb{F} / \pm\phi(\pi)$.

It is well known that torsion is a simple homotopy invariant and a topological invariant of compact connected CW-complexes. And for every topological manifold of dimension 3 admits a piecewise linear structure or in other words admits a triangulation. Such a piecewise linear structure is unique in the sense that every homeomorphism h between two piecewise linear manifolds is isotopic to a piecewise linear homeomorphism. In terms of triangulations, the triangulations can be subdivided so that there is an isomorphism of the subdivided triangulations isotopic to h. Thus, torsion is well-defined for our cases.

Remark. We see that for defining the twisted Alexander polynomial we need a representation $\phi : \operatorname{Tors}(H) \to \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ of the torsion part $\operatorname{Tors}(H)$ into $\mathbb{C}^*$ as described in the section 3. The representation induces the twisted homomorphism $\mathbb{Z}[H]\xrightarrow{\phi}\mathbb{C}[K]$, that we also denote by ϕ. If $\mathbb{Q}(K)$ denotes the field of quotient of $\mathbb{C}[K]$. Then by composing with the projection into the quotient, twisted homomorphism ϕ determines a ring homomorphism from $\mathbb{Z}[H]$ to the field $\mathbb{Q}(K)$ that we still denote by ϕ. Thus with a representation $\phi:\mathrm{Tors}(H)\to\mathbb{C}^*$ we can define both a twisted Alexander polynomial Δ^{ϕ} and a torsion τ^{ϕ}.

4. Reidemeister torsion of link complements in a 3-torus

Let L be a link in a three-dimensional torus T^3. The Euler characteristic of a 3-torus T^3 is $\chi(T^3)=0$. Removing a tubular neighborhood $N(L)$ of the link L from the 3-torus T^3, we obtain a compact 3-manifold X with a boundary that is the link complement in the 3-torus. In terms of the Euler characteristic, we have $0=\chi(T^3)=\chi(X\cup N(L))=\chi(X)+\chi(N(L))\ -\chi(X\cap N(L))$, that implies $\chi(X)=0$.

The complement X, then by pushing in one free face at a time, we can collapse X down to a 2-dimensional subcomplex Y, so X is simple homotopic to Y (see Whitehead [12]). The 2-cell complex Y is of Euler characteristic zero. We can ensure that Y has a cellular structure, containing only one 0-cell σ^0; n 1-cells $\sigma_1^1,...,\sigma_n^1$, m 2-cells $\sigma_1^2,...,\sigma_m^2$, where $m=n-1$.

The boundary maps are $\partial_1=0$ and $\partial_2(\sigma_i^2)=r_i$, where r_i is a word in σ_j^1, giving a presentation of fundamental group as $\pi=\langle x_1,x_2,...,x_n\mid r_1,r_2,...,r_m\rangle$. This presentation is not necessarily the same as the one, given in another paper (see [1]).

For the sake of completeness, I carry out derivation of some formulas from a paper by Huynh and Le [4] related to the Reidemeister torsion and Alexander–Fox matrix for a presentation of a group, that is the fundamental group of a manifold. Let $\tilde{Y}$ be the maximal abelian cover of Y. The cellular complexes of $\tilde{Y}$ is considered as modules over integral group ring $\mathbb{Z}(H)$, where H is the first homology group of Y. We have a chain complex of $\mathbb{Z}(H)$-modules

$$C_2(\tilde{Y})\xrightarrow{\partial_2}C_1(\tilde{Y})\xrightarrow{\partial_1}C_0(\tilde{Y})\to 0.$$

The boundary maps are obtained by Fox's free differential calculus (Compare Fox [13. P. 547] and [14], Milnor [2. P. 146]): $\partial_1(\tilde{\sigma}_i^1)=\mathrm{pr}(x_i-1)\tilde{\sigma}^0$ and $\partial_2(\tilde{\sigma}_i^2)=\sum_{j=1}^{n}\mathrm{pr}(\frac{\partial r_i}{\partial x_j})\tilde{\sigma}_j^1$,

where the tilde sign denotes a lift of the cell to $\tilde{Y}$. The natural projection pr is the composition of the maps α,γ in the chain $\mathbb{Z}[F]\xrightarrow{\gamma}\mathbb{Z}[G]\xrightarrow{\alpha}\mathbb{Z}[H]$ as defined in section 3 for the case group G is the fundamental group π.

Fix a splitting of H as a product $H=K\times\mathrm{Tors}(H)$ of the free part $K=H/\mathrm{Tors}(H)$ and the torsion part $Tors(H)$. Denote the quotient field $Q(\mathbb{C}[K])$ of $\mathbb{C}[K]$ by $\mathbb{Q}(K)$. Using the homomorphism $\phi:\mathbb{Z}[H]\to\mathbb{C}[K]\xrightarrow{\text{embedding}}\mathbb{Q}(K)$, construct the tensor $\mathbb{Q}(K)\otimes_{\mathbb{Z}[H]} C_i(\tilde{Y})$, considered as a vector space over $\mathbb{Q}(K)$. We have a chain complex of vector spaces over $\mathbb{Q}(K)$:

$$C = (\mathbb{Q}(K)\otimes_{\mathbb{Z}[H],\phi} C_2(\tilde{Y}) \xrightarrow{\partial_2} \mathbb{Q}(K)\otimes_{\mathbb{Z}[H],\phi} C_1(\tilde{Y}) \xrightarrow{\partial_1} \mathbb{Q}(K)\otimes_{\mathbb{Z}[H],\phi} C_0(\tilde{Y}) \to 0).$$

The boundary maps are $[\partial_1]_i = \phi(x_i)-1$, and $[\partial_2]_{i,j} = \phi(\frac{\partial r_j}{\partial x_i})$, $1 \le u < n$, $1 \le j \le n-1$.

Let $A = [\partial_2]^t$. Denote the columns of A by u_i, $1 \le i \le n$, and denote the $(n-1)\times(n-1)$ matrix obtained from A by omitting the column u_i by A_i. Since C is a chain, we have $0 = \partial_1(\partial_2(\tilde{\sigma}_i^2)) = (\sum_{j=1}^{n} \phi\left(\frac{\partial r_i}{\partial x_j}\right)(\phi(x_j)-1))\tilde{\sigma}^0$, hence $\sum_{j=1}^{n} \phi\left(\frac{\partial r_i}{\partial x_j}\right)(\phi(x_j)-1) = 0$. That means $\sum_{j=1}^{n}(\phi(x_j)-1)u_j = 0$. For any $i > j$ we have

$$\begin{aligned}(\phi(x_j)-1)\det A_i &= \det[u_1,\dots,u_{j-1},(\phi(x_j)-1)u_j,u_{j+1},\dots,u_i,\dots,u_n]\\ &= \det[u_1,\dots,u_{j-1},-\sum_{k\neq j}(\phi(x_k)-1)u_k,u_{j+1},\dots,u_i,\dots,u_n]\\ &= (-1)^{i-j+1}(\phi(x_i)-1)\det A_j.\end{aligned}$$

Thus, for any *i* and *j*,

$$(\phi(x_i)-1)\det A_j = \pm(\phi(x_j)-1)\det A_i. \tag{1}$$

Because H has at least three free generators (Theorem 1), the image $\phi(\pi)$ cannot be $\{1\}$, thus there is at least one x_i such that $\phi(x_i)\neq 1$. The property $\partial_1(\tilde{\sigma}_i^1) = (\phi(x_i)-1)\tilde{\sigma}^0$ implies $\partial_1\left(\frac{1}{\phi(x_i)-1}\tilde{\sigma}_i^1\right) = \tilde{\sigma}^0$, so ∂_1 is onto. Therefore, the chain C is exact if and only if ∂_2 is injective, which means the rank of its matrix is exactly $n-1$. Thus C is acyclic if and only if A has a nonzero $(n-1)\times(n-1)$ minor.

The Reidemeister torsion of C with respect to ϕ is the torsion $\tau^\phi(Y)$ of Y, and since torsion is a simple homotopy invariant, it is also the torsion $\tau^\phi(X)$ of X.

Now if we assume that C is acyclic. Take the standard bases of $\mathbb{Q}(K)\otimes_{\mathbb{Z}[H],\phi} C_*(\tilde{Y})$ given by $\tilde{\sigma}_j^i$ as above. A lift of $c_0 = \{\tilde{\sigma}_0\}$ is $\{\frac{1}{\phi(x_i)-1}\tilde{\sigma}_i^1\}$. Then

$$\tau^\phi(X) = [\left(\sum_{j=1}^{n}\phi\left(\frac{\partial r_1}{\partial x_j}\right)\tilde{\sigma}_j^1,\dots,\sum_{j=1}^{n}\phi\left(\frac{\partial r_{n-1}}{\partial x_j}\right)\tilde{\sigma}_j^1,\frac{1}{\phi(x_i)-1}\tilde{\sigma}_i^1\right)/(\tilde{\sigma}_1^1,\dots,\tilde{\sigma}_n^1)] = \frac{(-1)^{i+n}}{(\phi(x_i)-1)}\det A_j.$$

Thus if $\phi(x_i)\neq 1$ then $\tau^\phi(X) = \pm\det A_i/(\phi(x_i)-1)$. By equation (1), if $\phi(x_j)=1$ then $\det A_j = 0$, hence the following formula is correct for all *i*, whether C is acyclic or not:

$$(\phi(x_i)-1)\tau^\phi(X) = \pm\det A_i \in \mathbb{Q}(K)/\pm K. \tag{2}$$

Remark. Equation (2) derived in the work by Huynh and Le [4] for links in projective space holds for links in lens space [5] and for link complement in any space of Euler characteristic zero. The derivation above is carried over from [4].

Theorem. The Reidemeister torsion and the twisted Alexander polynomial of the complement of a link in 3-torus are the same.

Proof.

The cellular structure of the link complement X in a 3-torus is simple homotopic to a 2-dimensional subcomplex Y of Euler characteristic zero, so the structure admits a presentation for the fundamental group of X with n generators and $m = n-1$ relations. So the Alexander–Fox matrix A associated to such a presentation is a $(n-1)\times n$ matrix. So the twisted Alexander polynomial $\Delta^{\phi}(X)$ is defined to be the greatest common devisor $\gcd(\det A_1, ..., \det A_n)$ of all $(m-1)$-minor A_i of matrix A, obtained by removing the i-th column of A.

By equation (2), we have

$$\Delta^{\phi}(X) = \gcd(\det A_1, ..., \det A_n) = \gcd((\phi(x_1)-1)\tau^{\phi}(X), ..., (\phi(x_n)-1)\tau^{\phi}(X))$$

We will show that $\gcd((\phi(x_1)-1), ..., (\phi(x_n)-1) = 1$ in the case of non-torsion and torsion links.

Case 1: L is a non-torsion knot or link. We have the first homology group of complement (see section 2) is $H_1(T^3 \setminus L) \cong \mathbb{Z}^3 \oplus \mathbb{Z}^{\omega}$, where ω is the number of components. Denote with $t_1, ..., t_{\omega+3}$ the generators of H_1. Then $\phi(x_i) = t_1^{h_i^1} ... t_{\omega+3}^{h_i^{\omega+3}}$ for $i = 1, ..., n$.

Let $g = \gcd((\phi(x_1)-1), ..., (\phi(x_n)-1) \in \mathbb{Z}[t, t^{-1}]$.

For a moment we set $t_2 = ... = t_{\omega+3} = 1$. So g divides each of $(t_1^{h_i^1} - 1)$ for $i = 1, ..., \omega+3$.

Observe that (see Lickorish [15]) for any $a, b \in \mathbb{Z}$

$$(t^a - 1) + t^a(t^b - 1) = t^{a+b} - 1$$

and

$$(t^a - 1) - t^{a-b}(t^b - 1) = t^{a-b} - 1.$$

Applying the argument, we conclude that g divides $t_1^{\sum_{i=1}^{n} \alpha_i h_i^1} - 1$ for any $\alpha_i \in \mathbb{Z}$.

Since t_1 is an element of canonical projection of fundamental group $\mathrm{pr}(\pi)$ there is a collection of $\alpha_i \in \mathbb{Z}$ such that $t_1 = \prod_{i=1}^{n} \mathrm{pr}(x_i^{\alpha_i}) = t_1^{\sum_{i=1}^{n} \alpha_i h_i^1}$. Thus g divides $(t_1 - 1)$. Now by letting $t_j = 1$ for $j \neq i, i = 2, .., (\omega+3)$ and repeating the argument we obtain $g = \gcd((\phi(x_1)-1), ..., (\phi(x_n)-1) = \gcd((t_1 - 1), ..., (t_{\omega+3} - 1) = 1$.

Case 2: L is a torsion link. The first homology group of link complement has free part of rank at most $\omega+2$ and the torsion part might be a product of at most three cyclic group $\mathbb{Z}_{\zeta} \oplus \mathbb{Z}_{\eta} \oplus \mathbb{Z}_{\theta}$, where $\zeta, \eta, \theta \in \mathbb{N}$ are the order of respective groups. Without loss of generality, we consider the case when the first homology group has the rank r and torsion part has the structure $\mathbb{Z}_\zeta \oplus \mathbb{Z}_\eta \oplus \mathbb{Z}_\theta$. Now denote with $t_1,...,t_r$ the generators of free part F and u_1,u_2,u_3 are the generator of torsion part $\text{Tors}(H_1)$.

We have projection of x_i is $pr(x_i)=t_1^{h_i^1}...t_r^{h_i^r}u_1^{k_i^1}u_2^{k_i^2}u_3^{k_i^3}$. For some homomorphism $\phi:\text{Tors}(H_1)\to\mathbb{C}^*$ the image $\phi(x_i)$ is defined to be $\phi(x_i)=t_1^{h_i^1}...t_r^{h_i^r}\phi(u_1^{k_i^1}u_2^{k_i^2}u_3^{k_i^3})$.

Setting $t_2=...=t_r=1$, applying the previous reasoning we conclude that $g=\gcd((\phi(x_1)-1),...,(\phi(x_n)-1)$ divides $t_1^{\sum_{i=1}^n \alpha_i h_i^1}\phi(u_1^{\sum_{i=1}^n \alpha_i k_i^1}u_2^{\sum_{i=1}^n \alpha_i k_i^2}u_3^{\sum_{i=1}^n \alpha_i k_i^3})-1$ for any $\alpha_i\in\mathbb{Z}$.

Since t_1 is an element of canonical projection of fundamental group $\text{pr}(\pi)$ there is a collection of $\alpha_i\in\mathbb{Z}$ such that $t_1=\prod_{i=1}^n \text{pr}(x_i^{\alpha_i})=t_1^{\sum_{i=1}^n \alpha_i h_i^1}\phi(u_1^{\sum_{i=1}^n \alpha_i k_i^1}u_2^{\sum_{i=1}^n \alpha_i k_i^2}u_3^{\sum_{i=1}^n \alpha_i k_i^3})$.

So $\sum_{i=1}^n \alpha_i h_i^1=1$ and $\phi(u_1^{\sum_{i=1}^n \alpha_i k_i^1}u_2^{\sum_{i=1}^n \alpha_i k_i^2}u_3^{\sum_{i=1}^n \alpha_i k_i^3})=1$. Thus, analogously we get $\gcd((\phi(x_1)-1),...,(\phi(x_n)-1)=1$ that completes the proof.

Remark. The identifications between Alexander type polynomial and Reidemeister torsion for knot complements in different cases were proved by different people (see Milnor [2], Kitano [16], Kirk and Livingston [17], Turaev [9], Cattabriga [5], Huynh and Le [4]). The proving method of Theorem 2 is technically due to Huynh–Le's work.

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