Original article
UDC 512.57, 512.579 MSC: 20M05, 08B20
doi: 10.17223/19988621/87/4

Multi-groups

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Abstract. In the present paper we define homogeneous algebraic systems. Particular cases of these systems are semigroup (monoid, group) systems. These algebraic systems were studied by J. Loday, A. Zhuchok, T. Pirashvili, and N. Koreshkov. Quandle systems were introduced and studied by V. Bardakov, D. Fedoseev, and V. Turaev.
We construct some group systems on the set of square matrices over a field $\Bbbk$. Also, we define rack systems on the set $V \times G$, where V is a vector space of dimension n over $\Bbbk$ and G is a subgroup of $GL_n(\Bbbk)$. Finally, we find the connection between skew braces and dimonoids.
Keywords: algebraic system, homogeneous algebraic system, groupoid, semigroup, monoid, group, semigroup system, quandle system, dimonoid, skew brace, multi-group, multi-quandle

Acknowledgments: This work was supported by the Ministry of Science and Higher Education of Russia (agreement no. 075-02-2023-943).

For citation: Kozlovskaya, T.A. (2024) Multi-groups. *Vestnik Tomskogo gosudarstvennogo universiteta. Matematika i mekhanika – Tomsk State University Journal of Mathematics and Mechanics*. 87. pp. 34–43. doi: 10.17223/19988621/87/4

Научная статья

Мульти-группы

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Аннотация. Определяются однородные алгебраические системы. Примерами таких систем являются полугрупповые, моноидальные и групповые системы. Они изучались в работах Ж. Лодея, А. Жучок, Т. Пирашвили и Н. Корешкова. Квандловые системы были введены и изучались в работах В. Бардакова, Д. Федосеева и В. Тураева.
В статье строятся некоторые групповые системы на множестве квадратных матриц над полем $\Bbbk$. Определяются рэковые системы на множестве $V \times G$ где V – векторное пространство размерности n над $\Bbbk$, G – подгруппа $GL_n(\Bbbk)$. В заключение найдена связь между косыми брейсами и димоноидами.

Ключевые слова: алгебраическая система, однородная алгебраическая система, группоид, полугруппа, моноид, группа, полугрупповая система, квандловая система, димоноид, косой брейс, мульти-группа, мульти-квандл

Благодарности: Работа выполнена при поддержке Министерства науки и высшего образования РФ (соглашение № 075-02-2023-943).

Для цитирования: Козловская Т.А. Мульти-группы // Вестник Томского государственного университета. Математика и механика. 2024. № 87. С. 34–43. doi: 10.17223/19988621/87/4

1. Introduction

In the theory of algebraic systems there exist algebraic systems with a set of one type algebraic operations. Let us give some examples of these algebraic systems.

A brace (skew brace) is a set with two group operations, which satisfy some axioms ([1, 2]). A generalization of skew braces was suggested in the paper by Bardakov–Neshchadim–Yadav [3], where brace systems were introduced as a set with a family of group operations connected by some axioms.

Dimonoids were introduced by J.L. Loday [4] in his construction of a universal enveloping algebra for the Leibniz algebra. A dimonoid is a set with two semigroup operations which are connected by a set of axioms. The construction of a free dimonoid generated by a given set was presented in [4] and applied to the study of free dialgebras and cohomology of dialgebras. Structural properties of free dimonoids have been investigated by A.V. Zhuchok in [5]. In [6], a construction of a free product of arbitrary dimonoids was presented. It generalizes the free dimonoid and describes its structure. Dimonoids are examples of duplexes which were introduced by T. Pirashvili in [7]. A duplex is an algebraic system with two associative binary operations (without added connections between these operations). T. Pirashvili constructed a free duplex generated by a given set via planar trees and proved that the set of all permutations forms a free duplex on an explicitly described set of generators.

In [8], N. Koreshkov introduced n-tuple semigroup as an algebraic system

$$\mathcal{S} = (S, *_i, i \in I)$$

such that $(S, *_i)$ is a semigroup for any $i \in I$ and with the following axiom which connects these operations,

$$(a *_i b) *_j c = a *_i (b *_j c),\ a, b, c \in S,\ i, j \in I.$$

The free n-tuple semigroup of an arbitrary rank was first constructed in [9].

In the present paper we define homogeneous algebraic systems (see Definition 2.1). Particular cases of these systems are semigroup (monoid, group) system $\mathcal{G} = (G, *_i, i \in I)$, where $(G, *_i)$ is a semigroup (monoid, group) for any $i \in I$. An example of a semigroup system with two operations is a duplex. We call $\mathcal{G}$ a multi-semigroup (multi-monoid, multi-group) if the operations are connected by the following condition

$$(a *_i b) *_j c = a *_i (b *_j c),\ a, b, c \in G,\ i, j \in I.$$

An example of a multi-semigroup with n operations is an n-tuple semigroup [8].

V.G. Bardakov and D.A. Fedoseev [10] considered quandle systems $\mathcal{Q}=(Q,*_i,i\in I)$, where $(Q,*_i)$ is a quandle for any $i\in I$, and defined a multiplication $*_i *_j$ of the operations $*_i$ and $*_j$ by the rule

$$p(*_i *_j)q=(p*_i q)*_j q, \quad p,q\in Q.$$

In the general case, the algebraic system $(Q,*_i *_j)$ is not a quandle, but if the operations satisfy the axioms

$$(x*_i y)*_j z=(x*_j z)*_i (y*_j z),\ (x*_j y)*_i z=(x*_i z)*_j (y*_i z),\ x,y,z\in Q,$$

then $(Q,*_i *_j)$ and $(Q,*_j *_i)$ are quandles. V. Turaev called quandle systems that satisfy the last axioms for all $i,j\in I$ multi-quandles and gave them a topological interpretation (see [11]).

In 1971, V.M. Buchstaber and S.P. Novikov [12] introduced a notion of *n*-valued group in which the product of each pair of elements is an *n*-multi-set, the set of *n* elements with multiplicities. An appropriate survey on n-valued groups and its applications can be found in [13].

If we have a group system $\mathcal{G}=(G,*_i,i\in I)$, where $|I|=n$, we can define *n*-valued multiplication

$$a*b=[a*_1 b,a*_2 b,\ldots,a*_n b],\ a,b\in G,$$

and study the algebraic system $(G,*)$. In [14], connections between group systems and *n*-valued groups were investigated. It was proved that if all groups $(G,*_i)$ have a common unit and $(G,*)$ is an *n*-valued group, then $*_i=*_j$ for all $1\le i,j\le n$.

In the present paper we study connections between skew braces and dimonoids and define a semigroup systems on the set of square matrices. We investigate semigroup systems on the set of matrices $M_n(\Bbbk)$ and give an answer on a question from [14]. Also, we construct some rack systems and multi-racks on the set $V\times G$, where V is a vector space of dimension *n* over a field $\Bbbk$, G is a subgroup of $GL_n(\Bbbk)$.

The paper is organized as follows.

In Section 2 we introduce homogeneous algebraic systems which include the algebraic systems from the introduction.

In Section 3 we construct some group systems on the set of square matrices over a field and give an answer on a question from [14].

In Section 4, rack systems on the set $V\times G$, $G\le GL_n(\Bbbk)$ are defined.

In Section 5, the connection between skew braces and dimonoids is established.

2. Homogeneous algebraic systems

In this section we introduce homogeneous algebraic systems.

Definition 2.1. Let $\mathcal{A}=(A,f_i,i\in I)$ be an algebraic system with a set of algebraic operations f_i of arity n_i. It is said to be an *m*-homogeneous *I*-system if all arities n_i are equal to *m*. In particular, if $|I|=n$, we will say about an *m*-homogeneous *n*-system. If $m=2$, we will say instead 2-homogeneous *n*-system on groupoid *n*-system or simply on a groupoid system.

A typical example is a ring $(K,+,\cdot)$ that is a groupoid 2-system. Other examples of 2-homogeneous *I*-systems are a semigroup (monoid, group) system $\mathcal{G}=(G,*_i,i\in I)$, where $(G,*_i)$ is a semigroup (monoid, group) for all $i\in I$. An example of a semigroup system with two operations is a duplex. We call a system $\mathcal{G}$ a multi-semigroup (multi-monoid, multi-group) if the operations are connected by the following condition

$$(a*_i b)*_j c = a*_i (b*_j c),\ a,b,c\in G,\ i,j\in I.$$

An example of a multi-semigroup with *n* operations is an *n*-tuple semigroup (see [8]).

Let us give other examples of semigroup systems.

Skew braces (see [1, 2]). A triple $(G,\cdot,\circ)$, where $(G,\cdot)$ and $(G,\circ)$ are groups, is said to be a *skew (left) brace* if

$$g_1\circ(g_2\cdot g_3)=(g_1\circ g_2)\cdot g_1^{-1}\cdot(g_1\circ g_3)$$

for all $g_1,g_2,g_3\in G$, where g_1^{-1} denotes the inverse of g_1 in $(G,\cdot)$. We call $(G,\cdot)$ the *additive group* and $(G,\circ)$ the *multiplicative group* of the skew left brace $(G,\cdot,\circ)$. A skew left brace $(G,\cdot,\circ)$ is said to be a *(left) brace* if $(G,\cdot)$ is an abelian group. In this case we will use the notation $+$ instead $\cdot$ in additive group. We see that a skew left brace is an example of group system with 2 operations.

Dimonoids (see [4, 15]). A dimonoid is a set X together with two binary operations $\vdash$ and $\dashv$ satisfying the following axioms:

$$\begin{cases} x\dashv(y\dashv z)\overset{1}{=}(x\dashv y)\dashv z\overset{2}{=}x\dashv(y\vdash z),\\ (x\vdash y)\dashv z\overset{3}{=}x\vdash(y\dashv z),\\ (x\dashv y)\vdash z\overset{4}{=}x\vdash(y\vdash z)\overset{5}{=}(x\vdash y)\vdash z \end{cases}$$

for all $x,y,z\in X$. Observe that relations 1 and 5 are the “associativity”' of the products $\vdash$ and $\dashv$ respectively.

The typical examples of dimonoid are the following.

a) Let M be a monoid. Put $D=M\times M$ and define the products by

$$(m,n)\dashv(m',n'):=(m,nm'n'),$$
$$(m,n)\vdash(m',n'):=(mnm',n').$$

Then $\mathcal{D}=(D,\dashv,\vdash)$ is a dimonoid.

b) Let *G* be a group and *X* be a *G*-set. The following formulas define a dimonoid structure on $X\times G$:

$$(x,g)\dashv(y,h):=(x,gh),$$
$$(x,g)\vdash(y,h):=(g\cdot x,gh).$$

We see that a dimonoid is an example of a group system with 2 operations.

3. Group systems and multi-groups

Let $M_n(\Bbbk)$ be a set of $n\times n$ matrices over a field $\Bbbk$. The next multiplication was defined in [14]:

$$A*_{s,t,M_1,M_2} B = sAM_1B + tAM_2B,\ \ s,t \in \Bbbk,\ M_1, M_2 \in M_n(\Bbbk),$$

and the following was formulated:

Question 3.1. What can we say on this multiplication? What algebraic systems one can construct using these multiplications? Is there z connection of these multiplications with non-standard matrix multiplications that were studied in [18]?

Let us find conditions under which $(M_n(\Bbbk), *_{s,t,M_1,M_2})$ is a semigroup. It is need to check axiom of associativity,

$$(A*B)*C = (sAM_1B + tAM_2B)*C =$$
$$= s(sAM_1B + tAM_2B)M_1C + t(sAM_1B + tAM_2B)M_2C.$$

On the other side,

$$A*(B*C) = A*(sBM_1C + tBM_2C) =$$
$$= sAM_1(sBM_1C + tBM_2C) + tAM_2(sBM_1C + tBM_2C).$$

We have a system

$$\begin{cases} AM_1BM_1C = AM_1BM_1C; \\ AM_2BM_1C + AM_1BM_2C = AM_1BM_2C + AM_2BM_1C; \\ AM_2BM_2C = AM_2BM_2C. \end{cases}$$

It is easy to see that $(A*B)*C = A*(B*C)$.

Lemma 3.2. The multiplication $*_{s,t,M_1,M_2}$ is associative.

Corollary 3.3. The algebraic system $(M_n(\Bbbk), *_{s,t,M_1,M_2}, s,t \in \Bbbk, M_1, M_2 \in M_n(\Bbbk))$ is a semigroup system.

Let us check, is this semigroup system a multi-semigroup. Let we have two different multiplications: $* = *_{s,t,M_1,M_2}, \circ = \circ_{p,q,N_1,N_2}$ and check the axiom

$$(A*B)\circ C = A*(B\circ C).$$

The left hand side:

$$(A*B)\circ C = (sAM_1B + tAM_2B)\circ C =$$
$$= p(sAM_1B + tAM_2B)N_1C + q(sAM_1B + tAM_2B)N_2C.$$

The right hand side:

$$A*(B\circ C) = A*(pBN_1C + qBN_2C) =$$
$$= sAM_1(pBN_1C + qBN_2C) + tAM_2(pBN_1C + qBN_2C).$$

We get a system

$$\begin{cases} AM_1BN_1C = AM_1BN_1C; \\ AM_2BN_1C = AM_2BN_1C; \\ AM_1BN_2C = AM_1BN_2C; \\ AM_2BN_2C = AM_2BN_2C. \end{cases}.$$

Since this system is true for all matrices, we obtain

Proposition 3.4. The semigroup system $(M_n(\Bbbk), *_{s,t,M_1,M_2}, s,t \in \Bbbk, M_1, M_2 \in M_n(\Bbbk))$ is a multi-semigroup.

Let us find the unit element:

$$A * X = sAM_1X + tAM_2X = A.$$

It means that $t = 0, s = 1, M_1X = E \Rightarrow X = M_1^{-1}$.

Hence, $A * X = AM_1X$, $E^{(*)} = M_1^{-1}$. On the other side, $X * A = XM_1A = A$.

Lemma 3.5. We have the unit element only for multiplication $A * B = AMB$, $\det M \neq 0$, $E^{(*)} = M^{-1}$.

The inverse element $A * Y = E^{(*)} \Leftrightarrow AMY = M^{-1}$. Hence, $Y = M^{-1}A^{-1}M^{-1}$.

Theorem 3.6. 1) Let $M \in M_n(\Bbbk)$, $\det M \neq 0$. Then $(GL_n(\Bbbk), *_M)$ is a group with the product $A *_M B = AMB$, with unit element $E^{(*)} = M^{-1}$ and inverse $\bar{A}^{(*)} = M^{-1}A^{-1}M^{-1}$.

2) The algebraic system $(GL_n(\Bbbk), *_M, M \in GL_n(\Bbbk))$ is a group system.

4. Rack systems

Some examples of quandle systems and multi-quandles can be found in [10, 11]. In this section we give some other examples. At first, recall basic definitions.

Definition 4.1. ([16, 17]).

A *quandle* is a non-empty set Q with a binary operation $(x, y) \mapsto x * y$ satisfying the following axioms:

(Q1) $x * x = x$ for all $x \in Q$,

(Q2) for any $x, y \in Q$ there exists a unique $z \in Q$ such that $x = z * y$,

(Q3) $(x * y) * z = (x * z) * (y * z)$ for all $x, y, z \in Q$.

An algebraic system satisfying only (Q2) and (Q3) is called a *rack*. Many interesting examples of quandles come from groups.

Example 4.2.

1. If G is a group, m is an integer, then the binary operation $a *_m b = b^{-m}ab^m$ turns G into the quandle $\mathrm{Conj}_m(G)$ called the m*-conjugation quandle* on G. If $m = 1$, this quandle is called a *conjugation quandle* and is denoted as $\mathrm{Conj}(G)$.

2. A group G with the binary operation $a * b = ba^{-1}b$ turns the set G into the quandle $\mathrm{Core}(G)$ called the *core quandle* of G. In particular, if $G = \mathbb{Z}_n$, the cyclic group of order n, then it is called the *dihedral quandle* and denoted by R_n.

3. Let G be a group and $\varphi \in \mathrm{Aut}(G)$. Then the set G with the binary operation $a *_\varphi b = \varphi(ab^{-1})b$ forms a quandle $\mathrm{Alex}(G, \varphi)$ referred as the *generalized Alexander quandle* of G with respect to φ.

From the last example, it follows that if $Q = GL_n(\Bbbk)$, $\varphi \in \mathrm{Aut}(GL_n(\Bbbk))$, then we can define a quandle system $(Q, *_\varphi, \varphi \in \mathrm{Aut}(GL_n(\Bbbk)))$.

In the present section we study the following question: what rack (quandle) systems can be defined on $V \times G$, where V is a vector space of dimension n over a field $\Bbbk$, G is a subgroup of $GL_n(\Bbbk)$? On the set $Q = V \times G$, we can define the operation

$$(a,A)\circ(b,B)=(Ab,\varphi(AB^{-1})B),\;\; a,b\in V,\;A,B\in G,\;\varphi\in Aut(G).$$

In this case

$$(a,A)\circ(a,A)=(Aa,\varphi(E)A)=(Aa,A).$$

It means that $A=E$; hence, $G=\{E\}$ is the trivial group.

The second quandle axiom:

$$(u,X)\circ(a,A)=(b,B)\Leftrightarrow(u,X)\circ(a,A)=(Xa,\varphi(XA^{-1})A).$$

Hence,

$$Xa=b,\;X=\varphi^{-1}(BA^{-1})A.$$

It means that such element (u,X) exists but it is not unique.

Let us check the third quandle axiom:

$$((a,A)\circ(b,B))\circ(c,C)=((a,A)\circ(c,C))\circ((b,B)\circ(c,C)).$$

The left-hand side:

$$((a,A)\circ(b,B))\circ(c,C)=(Ab,\varphi(AB^{-1})B)\circ(c,C)=(\varphi(AB^{-1})Bc,\varphi(\varphi(AB^{-1})BC^{-1})C).$$

The right-hand side:

$$((a,A)\circ(c,C))\circ((b,B)\circ(c,C))=(Ac,\varphi(AC^{-1})C)\circ(Bc,\varphi(BC^{-1})C)=$$
$$=(\varphi(AC^{-1})CBc,\varphi(\varphi(AC^{-1})C).$$

We have the system

$$\begin{cases}\varphi(AB^{-1})Bc=\varphi(AC^{-1})CBc,\\ \varphi(\varphi(AB^{-1})BC^{-1})C=\varphi(\varphi(AC^{-1})C).\end{cases}$$

Since $A*B=\varphi(AB^{-1})B$ satisfies the quandle operation, the second equation is true. Consider the first equation of the system. It is equivalent to the equality

$$\varphi(AB^{-1}CA^{-1})=C,$$

which must be true for arbitrary $A,B,C\in G$. Evidently, this is true for the trivial group.

Let us define the operation (see [19])

$$(a,A)\circ(b,B)=(Ab,ABA^{-1}),\;\; a,b\in V,\;A,B\in G,$$

and check the left self-distributivity,

$$(a,A)\circ((b,B)\circ(c,C))=((a,A)\circ(b,B))\circ((a,A)\circ(c,C)).$$

Since

$$(a,A)\circ((b,B)\circ(c,C))=(a,A)\circ(Bc,BCB^{-1})=(ABc,ABCB^{-1}A^{-1})$$

and

$$((a,A)\circ(b,B))\circ((a,A)\circ(c,C))=(Ab,ABA^{-1})\circ(Ac,ACA^{-1})=(ABc,ABCB^{-1}A^{-1}),$$

the left self-distributivity holds.

Let us take $n\in\mathbb{Z}$ and define more general operation,

$$(a,A)\circ_n(b,B)=(A^nb,A^nBA^{-n}),\;\; a,b\in V,\;A,B\in G.$$

Check the left self-distributivity,

$$(a,A)\circ_n((b,B))\circ_n((c,C))=(a,A)\circ_n(B^nc,B^nCB^{-n})=(A^nB^nc,A^nB^nCB^{-n}A^{-n}),$$

$$((a,A)\circ_n(b,B))\circ_n((a,A)\circ_n(c,C))=(A^nb,A^nBA^{-n})\circ_n(A^nc,A^nCA^{-n})=$$
$$=(A^nB^nc,A^nB^nCB^{-n}A^{-n}).$$

Hence, the operation $(a,A)\circ_n(b,B)=(A^n b, A^n BA^{-n})$ is left self-distributive.

Let us check the left divisibility axiom: $(a,A)\circ_n(u,X)=(b,B)$. We have

$$(a,A)\circ_n(u,X)=(A^n u, A^n XA^{-n}).$$

Hence,

$$\begin{cases} u = A^{-n}b, \\ X = A^{-n}BA^n. \end{cases}$$

Since this system has a unique solution, the left divisibility holds.

Summarizing the previous calculations, we get

Theorem 4.3. Let $Q=(V,G)$, where V is a vector space of dimension n over a field $\Bbbk$, G be a subgroup of $GL_n(\Bbbk)$. Then the algebraic system $(Q,*_n, n\in\mathbb{Z})$, where

$$(a,A)\circ_n(b,B)=(A^n b, A^n BA^{-n}),\ \ a,b\in V,\ A,B\in G,$$

satisfies the following axioms:

1) left self-distributivity,

$(a,A)\circ_n((b,B)\circ_n(c,C))=((a,A)\circ_n(b,B))\circ_n((a,A)\circ_n(c,C)),\ a,b,c\in V,\ A,B,C\in G.$

2) left divisibility,

for any $(a,A),(b,B)\in Q$ there is unique $(u,X)\in Q$ such that

$$(a,A)\circ_n(u,X)=(b,B).$$

From this theorem follows

Corollary 4.4. The algebraic system $(Q,*_n^{op}, n\in\mathbb{Z})$, where the opposite operations are defined by the rules

$$(a,A)*_n^{op}(b,B)=(b,B)*_n(a,A)$$

is a rack system.

5. Connection between skew braces and dimonoids

In this section we find some connections between skew braces and dimonoids.

Proposition 5.1. Let $(G,\cdot)$ be a group.

1) If $a\circ b=ab$, then $(G,\cdot,\circ)$ is a skew brace. If $a\vdash b=a\dashv b=ab$, then we get a dimonoid.

2) If $a\circ b=ba$, then $(G,\cdot,\circ)$ is a skew brace. If $a\vdash b=ab$ and $a\dashv b=ba$, then $(G,\dashv,\vdash)$ is not a dimonoid. □

The binary operation $\vdash$ is associative since it corresponds to the product in group G.

Let us check the following axiom:

$$(a\dashv b)\dashv c=a\dashv(b\dashv c).$$

So let us compute both sides of equation:

$$(ba)\dashv c=c(ba),$$

$$a\dashv(cb)=(cb)a.$$

Since they are the same, the operation is associative.

Let us check the following axiom: $a\dashv(b\dashv c)=a\dashv(b\vdash c)$. We have $cba=bca$. Therefore, it must satisfy $bc=cb$ and this group is Abelian group.

It means that if $a \dashv b = a \vdash b = ab,$ then $(G, \vdash, \dashv)$ is a dimonoid. ■

If we a have a skew brace $(G, \cdot, \circ),$ then we can define operations $a \vdash b = ab,$ $a \dashv b = a \circ b$ and formulate the question: is $(G, \vdash, \dashv)$ a dimonoid?

The next example shows that in a general case the answer is negative.

Example 5.2. Let us take the brace $(\mathbb{Z}, +, \circ)$, where $(\mathbb{Z}, +)$ is the infinite cyclic group and $a \circ b = a + (-1)^a b,\ a, b \in \mathbb{Z}$.

Note that

$$a \circ b = a + (-1)^a b = \begin{cases} a + b, & \textit{if } a \textit{ is even}; \\ a - b, & \textit{if } a \textit{ is odd}. \end{cases}$$

Put

$$a \vdash b = a + b, a \dashv b = a \circ b.$$

It is evident that the associativity holds for the binary operations $\vdash$ and $\dashv$. Let us check that

$$a \dashv (b \dashv c) = a \dashv (b \vdash c).$$

We have that

$$a \circ (b \circ c) = a \circ \begin{cases} b + c, & \text{if } b \text{ is even}; \\ b - c, & \text{if } b \text{ is odd}. \end{cases} = \begin{cases} a + b + c & \text{if } a \text{ and } b \text{ are even}; \\ a - b - c & \text{if } a \text{ is odd and } b \text{ is even}; \\ a + b - c & \text{if } a \text{ is even and } b \text{ is odd}; \\ a - b + c & \text{if } a \text{ and } b \text{ are odd}. \end{cases}$$

On the other side we get

$$a \circ (b + c) = \begin{cases} a + b + c, & \text{if } a \text{ is even}; \\ a - b - c, & \text{if } a \text{ is odd}. \end{cases}$$

Let us take $a = 2,\ \ b = 3,\ \ c = 4.$ Then $a \dashv (b \dashv c) = a + b - c = 1.$ On the other side, $a \dashv (b \vdash c) = a + b + c = 9.$

Therefore, the skew brace $(\mathbb{Z}, +, \circ)$ is not a dimonoid.

At the end, we formulate the following questions.

Question 5.3. Under which conditions a skew brace $(G, \cdot, \circ)$ is a dimonoid with respect to the operations $a \vdash b = ab,\ a \dashv b = a \circ b$?

Question 5.4. Let $\mathcal{G} = (G, *_i, i \in I)$ be a semigroup system. Define a product of semigroup operations,

$$g(*_i *_j)h = (g *_i h) *_j h,\ g, h \in G.$$

Find necessary and sufficient conditions under which $(Q, *_i *_j)$ is a semigroup.

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The article was submitted 02.11.2023; accepted for publication 12.02.2024

Статья поступила в редакцию 02.11.2023; принята к публикации 12.02.2024