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WEIGHT DISTRIBUTION OF LOW-DENSITY PERIODIC RANDOM ERRORS AND THEIR CORRECTING CODES WITH ERROR DECODING PROBABILITY

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We present weight distribution of low-density periodic random errors in the space of all q-ary n-tuples along with the average Hamming weight of the error set. We also provide necessary and sufficient conditions for the existence of linear codes correcting such error pattern. Examples of such codes are given. Finally, probability of decoding error of such codes over a binary symmetric channel is derived.

Keywords: *parity-check matrix, syndromes, periodic random error, error decoding probability.*

ВЕСОВОЕ РАСПРЕДЕЛЕНИЕ ПЕРИОДИЧЕСКИХ СЛУЧАЙНЫХ ОШИБОК МАЛОЙ ПЛОТНОСТИ И ИХ КОРРЕКТИРУЮЩИЕ КОДЫ С ВЕРОЯТНОСТЬЮ ОШИБКИ ДЕКОДИРОВАНИЯ

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Изучается весовое распределение периодических случайных ошибок малой плотности в пространстве всех q-арных n-кортежей и среднее число таких ошибок на блок длины n. Приведены необходимые и достаточные условия существования и примеры линейных кодов, исправляющих такие ошибки. Вычислена вероятность ошибки декодирования таких кодов для двоичного симметричного канала связи.

Ключевые слова: *матрица проверки чётности, синдромы, периодическая случайная ошибка, вероятность ошибки декодирования.*

1. Introduction

Periodic random error is one type of errors which occurs in electronic control unit like power lines, inverters, car electric, compact disc, CD ROM. This was observed by N. Lange in 1994 [1]. This error pattern behaves in such a way that any b consecutive components are disturbed after a gap of some fixed positions repeatedly. Linear codes capable of detecting and correcting such errors along with their Hamming weight distribution and decoding error probability are studied in [2, 3]. A periodic random error can be defined as follows.

Definition 1. An s-periodic random error of length b is an n-tuple whose nonzero components are confined to distinct sets of b consecutive positions such that the sets are separated by s positions.

In 1963, Wyner [4] observed that for low intensity disturbances, only a few components within a burst [5] get disturbed, and he introduced the concept of low-density burst.

We extend this idea to periodic random errors whose intensity is low and define low-density periodic random error as below.

Definition 2. Low-density periodic random error is an s-periodic random error of length b such that each sets of b consecutive components separated by s zeros contains at most w, $w \leqslant b$, nonzero components.

Let $\xi_{(s,b|w),n,q}$ denote the set of all low-density periodic random errors in the space of n-tuples over GF(q). For example, the following vectors are some members of $\xi_{(3,2|1),10,2}$:

$$\begin{gathered}0000001000, 0000010000, 0100000000, 0100001000, 0100010000,\\ 1000000000, 1000001000, 1000010000, 0000000100, 0010000000,\\ 0010000100, 0010001000, 0100000100, 0000000010, 0001000000,\\ 0001000010, 0001000100, 0010000010, 0000000001, 0000100000.\end{gathered}$$

In this paper, we study the Hamming weight distribution of the vectors of $\xi_{(s,b|w),n,q}$. Then we study the existence of linear codes correcting the errors from the set $\xi_{(s,b|w),n,q}$. We denote a linear code that corrects low-density periodic random errors from the set $\xi_{(s,b|w),n,q}$ by LDP$_{(s,b|w),n,q}$RC-code. We further study the probability of decoding error for the errors set $\xi_{(s,b|w),n,q}$ over a binary symmetric channel. Throughout the paper, we consider $n = \lambda(b+s)+l$, where $0 \leqslant l < s+b$ and $\lambda \in \mathbb{N}$.

The organization of remaining part of the paper is as follows. Section 2 gives the Hamming weight distribution of the vectors of $\xi_{(s,b|w),n,q}$ along with examples. Average Hamming weight of the vectors of $\xi_{(s,b|w),n,q}$ is derived. In Section 3, we obtain necessary and sufficient conditions for existence of a LDP$_{(s,b|w),n,q}$RC-code followed by three examples. Finally, we provide the probability of decoding error for the errors of $\xi_{(s,b|w),n,q}$ over a binary symmetric channel.

2. Hamming weight distribution of vectors of $\xi_{(s,b|w),n,q}$

In this section, we give the Hamming weight distribution of vectors of $\xi_{(s,b|w),n,q}$ and average Hamming weight of a vector from the set $\xi_{(s,b|w),n,q}$.

Here $n = \lambda(b+s)+l$, where $0 \leqslant l < s+b$, then the maximum Hamming weight $w_{\max}$ of a vector of $\xi_{(s,b|w),n,q}$ is given by

$$w_{\max} = \begin{cases} w\lambda, & \text{when} \quad l = 0,\\ w\lambda + \min\{l, w\}, & \text{when} \quad 1 \leqslant l < b,\\ w\left(\lambda+1\right), & \text{when} \quad b \leqslant l < s+b. \end{cases}$$

We first state the following lemma from [6].

Lemma 1 [6]. If λ_i denotes the number of sets of non-zero positions and m_i the maximum number of nonzero positions in a vector of $\xi_{(s,b|w),n,q}$ that starts from i^{th} position, then

$$\lambda_i = \left\lceil \frac{n-i+1}{s+b} \right\rceil \text{ and } m_i = \left\lfloor \frac{n-i+1}{s+b} \right\rfloor b + \gamma\big((n-i+1) \bmod (b+s)\big),$$

where $\lfloor x \rfloor$ means the greatest integer $\leqslant x$, $\lceil x \rceil$ means the smallest integer $\geqslant x$, and

$$\gamma(r) = \begin{cases} r, & \text{if} \quad 0 \leqslant r \leqslant b,\\ b, & \text{if} \quad b < r < b+s. \end{cases}$$

Next, we give the following Lemma to derive weight distribution of vectors of $\xi_{(s,b|w),n,q}$.

Lemma 2. Let p_i be the number of common nonzero positions of the errors of $\xi_{(s,b|w),n,q}$ that starts from the i^{th} $(i = s+2, \ldots, s+b)$ position with an error vector of $\xi_{(s,b|w),n,q}$ that starts from the 1^{st} position. Then p_i is given by

(1) when $l = 0$ and $b-1 \leqslant l \leqslant s+b$: $\quad p_i = (i-s-1)\beta_i$, and

(2) when $1 \leqslant l < b-1$: $\quad p_i = \begin{cases} (i-s-1)\beta_{i-1} & \text{for} \quad i = s+2, \\ (i-s-1)\beta_{i-1} + l & \text{for} \quad i = s+3, s+4, \ldots, s+b, \end{cases}$

where $\beta_i = \left\lceil \dfrac{n-i-b+1}{s+b} \right\rceil$.

Proof.

Case 1: $l = 0$ and $b-1 \leqslant l \leqslant s+b$.

The common nonzero positions of the error pattern of $\xi_{(s,b|w),n,q}$ that starts from the $(s+2)^{\text{th}}$ position with the error pattern that starts from the 1^{st} position are $s+b+1$, $2(s+b)+1, \ldots, \beta_{s+1}(s+b)+1$, where $\beta_{s+1} = \left\lceil \dfrac{n-(s+1)-b+1}{s+b} \right\rceil$. This number of common nonzero position is given by β_{s+1}.

The common nonzero positions of the error pattern of $\xi_{(s,b|w),n,q}$ that starts from the $(s+3)^{\text{th}}$ position with the error pattern that starts from the 1^{st} position are $s+b+1$, $s+b+2$, $2(s+b)+1$, $2(s+b)+2, \ldots, \beta_{s+2}(s+b)+1$, $\beta_{s+2}(s+b)+2$, where $\beta_{s+3} = \left\lceil \dfrac{n-(s+3)-b+1}{s+b} \right\rceil$. This number of common nonzero position is given by $2\beta_{s+2}$.

Continuing this, the common nonzero positions of the error pattern of $\xi_{(s,b|w),n,q}$ that starts from the $(s+b)^{\text{th}}$ position with the error pattern that starts from the 1^{st} position are $s+b+1$, $s+b+2, \ldots, s+b+(b-1)$, $2(s+b)+1$, $2(s+b)+2, \ldots, 2(s+b)+(b-1), \ldots, \beta_{s+b-1}(s+b)+1$, $\beta_{s+b-1}(s+b)+2, \ldots, \beta_{s+b-1}(s+b)+(b-1)$, where $\beta_{s+b-1} = \left\lceil \dfrac{n-(s+b-1)-b+1}{s+b} \right\rceil$. This number of common nonzero position is given by $(b-1)\beta_{s+b-1}$.

Thus $p_i = (i-s-1)\beta_{i-1}$ for $i = s+2, s+3, \ldots, s+b$, where $\beta_i = \left\lceil \dfrac{n-i-b+1}{s+b} \right\rceil$.

Case 2: $1 \leqslant l < b-1$.

The common nonzero positions of the error pattern of $\xi_{(s,b|w),n,q}$ that starts from the $(s+2)^{\text{th}}$ position with the error pattern that starts from the 1^{st} position are $s+b+1$, $2(s+b)+1, \ldots, \beta_{s+1}(s+b)+1$, where $\beta_{s+1} = \left\lceil \dfrac{n-(s+2)-b+1}{s+b} \right\rceil$. This number of common nonzero position is given by β_{s+1}.

If the error pattern starts from the $(s+3)^{\text{th}}$ position, the common nonzero positions with the error pattern that starts from the 1^{st} position, excluding the last set of nonzero positions, are $s+b+1$, $s+b+2$, $2(s+b)+1$, $2(s+b)+2, \ldots, \beta_{s+2}(s+b)+1$, $\beta_{s+2}(s+b)+2$, where $\beta_{s+2} = \left\lceil \dfrac{n-(s+2)-b+1}{s+b} \right\rceil$.

The common positions with the last set are $\lambda_{s+3}(s+b)+1$, $\lambda_{s+3}(s+b)+2, \ldots, \lambda_{s+3}(s+b)+l$ (λ_i are given by Lemma 1), whose number is l. Therefore, the total number of common nonzero positions is given by $2\beta_{s+2}+l$.

Continuing this, if the error pattern starts from the $(s+b)^{\text{th}}$ position, the common nonzero positions with the error pattern that starts from the 1^{st} position, excluding the last set of nonzero positions, are $s+b+1$, $s+b+2, \ldots, s+b+(b-1)$, $2(s+b)+1$, $2(s+b)+2$,

$\ldots,\ 2(s+b)+(b-1),\ \ldots,\ \beta_{s+b-1}(s+b)+1,\ \beta_{s+b-1}(s+b)+2,\ \ldots,\ \beta_{s+b-1}(s+b)+(b-1)$, where $\beta_{s+b-1} = \left\lceil \dfrac{n-(s+b-1)-b+1}{s+b} \right\rceil$.

The last set has common positions $\lambda_{s+b}(s+b)+1,\ \lambda_{s+b}(s+b)+2,\ \ldots,\ \lambda_{s+b}(s+b)+l$, whose number is l. This number of common nonzero position is given by $(b-1)\beta_{s+b-1}+l$. Thus $p_i = \begin{cases} (i-s-1)\beta_{i-1} & \text{for} \quad i = s+2, \\ (i-s-1)\beta_{i-1}+l & \text{for} \quad i = s+3, s+4, \ldots, s+b. \end{cases}$

Lemma 2 is proven. ■

Theorem 1. Let $R^n_{s,b|w}(j)$ be the total number of vectors of $\xi_{(s,b|w),n,q}$, whose Hamming weight is j. Then:

For $j = 1$:

$$R^n_{s,b|w}(1) = \sum_{i=1}^{s+1}\left[\binom{m_i}{1} - \binom{k_{i-1}}{1}\right](q-1) + \sum_{i=s+2}^{s+b}\left[\binom{m_i}{1} - \binom{k_{i-1}}{1} - \binom{\beta_{i-1}}{1}\right](q-1).$$

For $2 \leqslant j \leqslant w$:

$$R^n_{s,b|w}(j) = \sum_{i=1}^{s+1}\left[\binom{m_i}{j} - \binom{k_{i-1}}{j}\right](q-1)^j + \sum_{i=s+2}^{s+b}\left[\binom{m_i}{j} - \binom{k_{i-1}}{j} - \binom{p_i}{j} + \binom{p_{i-1}}{j}\right](q-1)^j.$$

For $w+1 \leqslant j \leqslant w_{\max}-1$:

$$R^n_{s,b|w}(j) = \sum_{i=1}^{s+1}\left[\binom{m_i}{j} - \binom{k_{i-1}}{j} - \binom{(b-w)\beta_{i-1}}{1}\binom{m_i-b}{j-w-1}\right](q-1)^j +$$
$$+ \sum_{i=s+2}^{s+b}\left[\binom{m_i}{j} - \binom{k_{i-1}}{j} - \binom{p_i}{j} + \binom{p_{i-1}}{j} - \binom{(b-w)\beta_{i-1}}{1}\binom{m_i-b}{j-w-1}\right](q-1)^j.$$

For $j = w_{\max}$:

$$R^n_{s,b|w}(w_{\max}) = \begin{cases} \left[(s+1)b^{\lambda} + b^{\lambda-1} - s - 1\right](q-1)^{w_{\max}}, & \text{when} \quad l = 0, \\ b^{\lambda}(q-1)^{w_{\max}}, & \text{when} \quad 1 \leqslant l < b, \\ \left[(l-b+1)b^{\lambda+1} + b^{\lambda} - l + b - 1\right](q-1)^{w_{\max}}, & \text{when} \quad b \leqslant l < s+b, \end{cases}$$

where $p_{s+1} = 1$, $k_0 = 0$, $k_i = m_{i+1} - \beta_i$, β_i, m_i, and p_i are given by Lemmas 1, 2.

Proof.

C a s e 1 : $j = 1$.

The number of error patterns of weight 1 that start from the i^{th} positions, where $i = 1, 2, \ldots, s+1$, is given by $\binom{m_i}{1}(q-1)$. But in the calculation $\binom{m_{i+1}}{1}(q-1)$, number of already counted nonzero components in $\binom{m_i}{1}(q-1)$ is $k_i = m_{i+1} - \beta_i$ for $i = 1, 2, 3, \ldots, s+1$, where $\beta_i = \left\lceil \dfrac{n-i-b+1}{s+b} \right\rceil$ represents the total number of complete sets of b consecutive positions in which the nonzero elements of the error pattern start from the i^{th} position. Therefore, the total number of the errors having weight 1 is $\sum_{i=1}^{s+1}\left[\binom{m_i}{1} - \binom{k_{i-1}}{1}\right](q-1)^1$ with $k_0 = 0$.

For error patterns whose starting position is $i = s+2, \dots, s+b$, all the weight 1 vectors are already present in the first position (i.e., the b^{th} positions of each set of nonzero positions except the last set which may be less than b components). The number of these nonzero components is given by β_{i-1}. Thus, $\binom{\beta_{i-1}}{1}$, number of weight 1, need to be subtracted from each starting position of the error pattern. So, the number of weight 1 in these positions is given by the quantity $\binom{m_i}{1} - \binom{k_{i-1}}{1} - \binom{\beta_{i-1}}{1}$. We have

$$R^n_{s,b|w}(1) = \sum_{i=1}^{s+1} \left[\binom{m_i}{1} - \binom{k_{i-1}}{1}\right](q-1) + \sum_{i=s+2}^{s+b} \left[\binom{m_i}{1} - \binom{k_{i-1}}{1} - \binom{\beta_{i-1}}{1}\right](q-1).$$

Case 2: $2 \leqslant j \leqslant w$.

As above, the total number of the errors having weight j that start from the i^{th} positions, where $i = 1, 2, \dots, s+1$, is the quantity $\sum_{i=1}^{s+1} \left[\binom{m_i}{j} - \binom{k_{i-1}}{j}\right](q-1)^j$ with $k_0 = 0$.

But, for error patterns that start from positions $i = s+2, \dots, s+b$, there are some more common vectors with the already counted error vectors that starts from the 1^{st} position.

By Lemma 2, p_i denotes the number of common nonzero components that start from the i^{th} ($i = s+2, \dots, s+b$) position which are already present in the errors that start from the first position; $\binom{p_i}{j}(q-1)^j$ gives the number of vectors of weight j in the i^{th} ($i = s+2, \dots, s+b$) positions which are already counted in the error pattern that start from the first position. This includes some vectors which are deleted by the term $\binom{k_{i-1}}{j}(q-1)^j$, thus the term $\binom{p_{i-1}}{j}(q-1)^j$ is added to include such already deleted error vectors, here $p_{s+1} = 1$. So the exact number of common vectors that need to be excluded is $\left[\binom{p_i}{j} - \binom{p_{i-1}}{j}\right](q-1)^j$. Therefore, we have

$$R^n_{s,b|w}(j) = \sum_{i=1}^{s+1} \left[\binom{m_i}{j} - \binom{k_{i-1}}{j}\right](q-1)^j + \sum_{i=s+2}^{s+b} \left[\binom{m_i}{j} - \binom{k_{i-1}}{j} - \binom{p_i}{j} + \binom{p_{i-1}}{j}\right](q-1)^j.$$

Case 3: $w+1 \leqslant j \leqslant w_{\max} - 1$.

In this case, we can also similarly calculate the total number of all error vectors having weight j and starting from the i^{th} position, where $i = 1, 2, \dots, s+1$, after deleting the common vectors as the quantity

$$\sum_{i=1}^{s+1} \left[\binom{m_i}{j} - \binom{k_{i-1}}{j} - \binom{(b-w)\beta_{i-1}}{1}\binom{m_i - b}{j-w-1}\right](q-1)^j \quad \text{with } k_0 = 0.$$

Again, for error vectors having weight j starting from $(s+2)^{\text{th}}$ to $(s+b)^{\text{th}}$ positions, there are some more common vectors which we have calculated, as in the previous case: $\left[\binom{p_i}{j} - \binom{p_{i-1}}{j}\right](q-1)^j$. Therefore,

$$R^n_{s,b|w}(j) = \sum_{i=1}^{s+1} \left[\binom{m_i}{j} - \binom{k_{i-1}}{j} - \binom{(b-w)\beta_{i-1}}{1}\binom{m_i - b}{j-w-1}\right](q-1)^j +$$
$$+ \sum_{i=s+2}^{s+b} \left[\binom{m_i}{j} - \binom{k_{i-1}}{j} - \binom{(b-w)\beta_{i-1}}{1}\binom{m_i - b}{j-w-1} - \binom{p_i}{j} + \binom{p_{i-1}}{j}\right](q-1)^j.$$

C a s e 4 : $j = w_{\max}$.

In this case, the number of error vectors with weight j is calculated, and different formulas are found for $l = 0$, $1 \leqslant l < b$, and $b \leqslant l < s + b$:

$$R^n_{s,b|w}(w_{\max}) = \begin{cases} \left[(s+1)b^\lambda + b^{\lambda-1} - s - 1\right](q-1)^{w_{\max}}, & \text{when} \quad l = 0, \\ b^\lambda (q-1)^{w_{\max}}, & \text{when} \quad 1 \leqslant l < b, \\ \left[(l-b+1)b^{\lambda+1} + b^\lambda - l + b - 1\right](q-1)^{w_{\max}}, & \text{when} \quad b \leqslant l < s + b. \end{cases}$$

Theorem 1 is proven. ■

Remark 1. The values of m_i and p_i in [3] are given by

$$m_i = \begin{cases} b\lambda & \text{for} \quad 1 \leqslant i \leqslant s+1, \\ b\lambda + s - i + 1 & \text{for} \quad s+2 \leqslant i \leqslant s+b, \end{cases} \qquad \text{if} \quad l = 0,$$

$$m_i = \begin{cases} b\lambda + l - i + 1 & \text{for} \quad 1 \leqslant i \leqslant l, \\ b\lambda & \text{for} \quad l+1 \leqslant i \leqslant s+l+1, \\ b\lambda + s + l - i + 1 & \text{for} \quad s+l+2 \leqslant i \leqslant s+b, \end{cases} \qquad \text{if} \quad 1 \leqslant l < b,$$

$$m_i = \begin{cases} b(\lambda+1) & \text{for} \quad 1 \leqslant i \leqslant l-b+1, \\ b(\lambda+1) + (l-b-i+1) & \text{for} \quad l-b+1 < i \leqslant l, \\ b\lambda & \text{for} \quad 1+l \leqslant i \leqslant s+b, \end{cases} \qquad \text{if} \quad b \leqslant l < s+b,$$

$$p_i = i\beta_{i+s}, \qquad \text{if} \quad l = 0 \;\; \text{or} \;\; b \leqslant l < s+b,$$

$$p_i = \begin{cases} i\beta_{i+s} & \text{for} \quad 1 \leqslant i \leqslant l, \\ i(p_1 - 1) + l & \text{for} \quad l+1 \leqslant i \leqslant b-1, \end{cases} \qquad \text{if} \quad 1 \leqslant l < b.$$

In this paper, we consider the simplified form for m_i and p_i in Lemma 1 and Theorem 1, and for $b = w$ we get

$$w_{\max} = \begin{cases} b\lambda, & \text{when} \quad l = 0, \\ b\lambda + l, & \text{when} \quad 1 \leqslant l < b, \\ b\,(\lambda+1), & \text{when} \quad b \leqslant l < s+b, \end{cases} \qquad \text{and} \qquad \binom{(b-w)\beta_{i-1}}{1}\binom{m_i - b}{j - w - 1} = 0.$$

Then Theorem 1 coincides with Lemma 3.1 [3].

Example 1. Considering $q = 3$, $n = 11$, $s = 3$, $b = 2$, and $w = 1$ in Theorem 1, we have $\lambda = 2$, $l = 11 \bmod 5 = 1$, $m_1 = 5$, $m_2 = \cdots = m_5 = 4$, $p_{s+1} = 1$, $p_{s+2} = 2$, $\beta_0 = \beta_1 = \cdots = \beta_4 = 2$. Then

$$R^{11}_{3,2|1}(1) = \left[\binom{5}{1} - \binom{0}{1}\right](3-1)^1 + \left[\binom{4}{1} - \binom{2}{1}\right](3-1)^1 +$$
$$+ \left[\binom{4}{1} - \binom{2}{1}\right](3-1)^1 + \left[\binom{4}{1} - \binom{2}{1}\right](3-1)^1 + \left[\binom{4}{1} - \binom{2}{1} - \binom{2}{1}\right](3-1)^1 =$$
$$= 5 \cdot 2 + 2 \cdot 3 \cdot 2 + 0 = 22;$$

$$R^{11}_{3,2|2}(2) = \left[\binom{5}{2} - \binom{0}{2} - \binom{(3-2)\cdot 2}{1}\binom{3}{2-1-1}\right](3-1)^2 +$$
$$+ \left[\binom{4}{2} - \binom{2}{2} - \binom{(3-2)\cdot 2}{1}\binom{2}{2-1-1}\right](3-1)^2 +$$

$$+\left[\binom{4}{2}-\binom{2}{2}-\binom{(3-2)\cdot 2}{1}\binom{2}{2-1-1}\right](3-1)^2+$$
$$+\left[\binom{4}{2}-\binom{2}{2}-\binom{(3-2)\cdot 2}{1}\binom{2}{2-1-1}\right](3-1)^2+$$
$$+\left[\binom{4}{2}-\binom{2}{2}-\binom{2}{2}+\binom{1}{2}-\binom{(3-2)\cdot 2}{1}\binom{2}{2-1-1}\right](3-1)^2=76;$$
$$R^{11}_{3,2|3}(3)=2^2(3-1)^3=32.$$

Here the maximum weight is $w_{\max}=3$. This example can be verified by using Example 4 in the next section.

Example 2. Considering $q=2$, $n=12$, $s=3$, $b=2$, and $w=1$ in Theorem 1, we have $\lambda=2$, $l=12 \bmod 5=2$, $m_1=6$, $m_2=5$, $m_3=\cdots=m_5=4$, $p_{s+1}=1$, $p_{s+2}=2$, $\beta_0=\beta_1=\cdots=\beta_4=2$. Then

$$R^{11}_{3,2|1}(1)=\left[\binom{6}{1}-\binom{0}{1}\right]+\left[\binom{5}{1}-\binom{3}{1}\right]+\left[\binom{4}{1}-\binom{2}{1}\right]+\left[\binom{4}{1}-\binom{2}{1}\right]+$$
$$+\left[\binom{4}{1}-\binom{2}{1}-\binom{2}{1}\right]=6+2\cdot 3+0=12;$$
$$R^{11}_{3,2|2}(2)=\left[\binom{6}{2}-\binom{0}{2}-\binom{(3-2)\cdot 3}{1}\binom{4}{2-1-1}\right]+$$
$$+\left[\binom{5}{2}-\binom{3}{2}-\binom{(3-2)\cdot 2}{1}\binom{3}{2-1-1}\right]+\left[\binom{4}{2}-\binom{2}{2}-\binom{(3-2)\cdot 2}{1}\binom{2}{2-1-1}\right]+$$
$$+\left[\binom{4}{2}-\binom{2}{2}-\binom{(3-2)\cdot 2}{1}\binom{2}{2-1-1}\right]+$$
$$+\left[\binom{4}{2}-\binom{2}{2}-\binom{2}{2}+\binom{1}{2}-\binom{(3-2)\cdot 2}{1}\binom{2}{2-1-1}\right]=12+5+3\cdot 2+2=25;$$
$$R^{11}_{3,2|3}(3)=(2-2+1)\,2^3+2^2-2+2-1=11.$$

Here the maximum weight is $w_{\max}=3$.

Example 3. Taking $q=2$, $n=14$, $s=4$, $b=3$, and $w=2$ in Theorem 1, we have $\lambda=2$, $l=14 \bmod 7=0$, $m_1=m_2=\cdots=m_5=6$, $m_6=5$, $m_7=4$, $p_{s+1}=1$, $p_{s+2}=1$, $p_{s+3}=2$, $\beta_0=\beta_1=\cdots=\beta_4=2$, $\beta_5=\beta_6=1$. Then

$$R^{14}_{4,3|1}(1)=\left[\binom{6}{1}-\binom{0}{1}\right]+\left[\binom{6}{1}-\binom{4}{1}\right]+\left[\binom{6}{1}-\binom{4}{1}\right]+\left[\binom{6}{1}-\binom{4}{1}\right]+$$
$$+\left[\binom{6}{1}-\binom{4}{1}\right]+\left[\binom{6}{1}-\binom{4}{1}\right]+$$
$$+\left[\binom{5}{1}-\binom{3}{1}-\binom{2}{1}\right]+\left[\binom{4}{1}-\binom{2}{1}-\binom{1}{1}\right]=6+4\cdot 2+0=14;$$
$$R^{14}_{4,3|2}(2)=\left[\binom{6}{2}-\binom{0}{2}\right]+\left[\binom{6}{2}-\binom{4}{2}\right]+$$
$$+\left[\binom{6}{2}-\binom{4}{2}\right]+\left[\binom{6}{2}-\binom{4}{2}\right]+\left[\binom{6}{2}-\binom{4}{2}\right]+$$
$$+\left[\binom{5}{2}-\binom{4}{2}-\binom{1}{2}+\binom{1}{2}\right]+\left[\binom{4}{2}-\binom{3}{2}-\binom{2}{2}+\binom{1}{2}\right]=15+9\cdot 4+4+2=57;$$

$$R^{14}_{4,3|3}(3) = \left[\binom{6}{3} - \binom{0}{3} - \binom{(3-2)\cdot 2}{1}\binom{4}{3-2-1}\right] +$$
$$+\left[\binom{6}{3}-\binom{4}{3}-\binom{(3-2)\cdot 2}{1}\binom{4}{3-2-1}\right]+\left[\binom{6}{3}-\binom{4}{3}-\binom{(3-2)\cdot 2}{1}\binom{4}{3-2-1}\right]+$$
$$+\left[\binom{6}{3} - \binom{4}{3} - \binom{(3-2)\cdot 2}{1}\binom{4}{3-2-1}\right]+$$
$$+\left[\binom{5}{3} - \binom{4}{3} - \binom{1}{3} + \binom{1}{3} - \binom{(3-2)\cdot 1}{1}\binom{2}{3-2-1}\right]+$$
$$+\left[\binom{4}{3} - \binom{3}{3} - \binom{2}{3} + \binom{1}{3} - \binom{(3-2)\cdot 1}{1}\binom{1}{3-2-1}\right]=$$
$$= (20-2) + (20-4-2)\cdot 4 + (10-4-1) + (4-1-1) = 81;$$
$$R^{14}_{4,3|4}(4) = (4+1)\cdot 3^2 + 3^{2-1} - 4 - 1 = 43.$$

Here the maximum weight is $w_{\max} = 4$.

Theorem 2. The average weight of a vector of the set $\xi_{(s,b|w),n,q}$ is

$$\sum_{j=1}^{w_{\max}} jR^n_{s,b|w}(j) \Big/ \sum_{j=1}^{w_{\max}} R^n_{s,b|w}(j),$$

where $R^n_{s,b|w}(j)$ is given by Theorem 1.

Proof. By Theorem 1, the number of vectors of $\xi_{(s,b|w),n,q}$ having Hamming weight j is $R^n_{s,b|w}(j)$, and the total weight of all vectors of $\xi_{(s,b|w),n,q}$ is given by $\sum\limits_{j=1}^{w_{\max}} jR^n_{s,b|w}(j)$. The ratio gives the required average weight. ■

3. Existence of LDP$_{(s,b|w),n,q}$RC-codes

In this section, we obtain necessary and sufficient conditions for the existence of q-ary LDP$_{(s,b|w),n,q}$RC-codes. We also derive an upper bound on the number of codewords for such a code. We also construct examples based on the results.

Theorem 3. Every (n,k) LDP$_{(s,b|w),n,q}$RC-code satisfies

$$n-k \geqslant \log_q\left[1+\sum_{j=1}^{w_{\max}} R^n_{s,b|w}(j)\right], \text{ where } R^n_{s,b|w}(j) \text{ is given by Theorem 1.}$$

Proof. By Theorem 1, the number of error vectors of $\xi_{(s,b|w),n,q}$ including the zero vector is $1 + |\xi_{(s,b|w),n,q}| = 1 + \sum\limits_{j=1}^{w_{\max}} R^n_{s,b|w}(j)$. As the maximum available coset is q^{n-k} and LDP$_{(s,b|w),n,q}$RC-code corrects all such errors, we have

$$q^{n-k} \geqslant 1+\sum_{j=1}^{w_{\max}} R^n_{s,b|w}(j) \implies n-k \geqslant \log_q\left[1+\sum_{j=1}^{w_{\max}} R^n_{s,b|w}(j)\right].$$

Theorem 3 is proven. ■

Remark 2. The maximum number of codewords of an (n,k) LDP$_{(s,b|w),n,q}$RC-code is

$$M \leqslant \frac{q^n}{1+\sum\limits_{j=1}^{w_{\max}} R^n_{s,b|w}(j)}.$$

In the following theorem, we apply the well known technique used in Varshamov — Gilbert — Sacks bound (see [7] and [8, Theorem 4.7]).

Theorem 4. For existence of an (n,k) $\mathrm{LDP}_{(s,b|w),n,q}$RC-code, the following condition is sufficient:

$$q^{n-k} > \sum_{j=0}^{w-1}\binom{b-1}{j}(q-1)^j\left(\sum_{j=0}^{w}\binom{b-1}{j}(q-1)^j\right)^{\lambda-1}\sum_{j=0}^{\min\{w,g\}}\binom{g}{j}(q-1)^j\left(1+\sum_{j=1}^{w_{\max}}R^{n-b}_{s,b|w}(j)\right), \tag{1}$$

where $g=\gamma(l)$ and $R^{n-b}_{s,b|w}(j)$ is given by Theorem 1. Here $\sum\limits_{j=0}^{\min\{w,g\}}\binom{g}{j}(q-1)^j = 1$ for $g=0$.

Proof. The proof is done by constructing an appropriate $(n-k)\times n$ parity-check matrix H of the code. Suppose that the first $n-1$ columns $h_1, h_2, h_3, \ldots, h_{n-1}$ are added suitably to H. Then any (nonzero) column h_n is added to H provided that it is not a linear combination of at most $w-1$ columns among the immediately preceding $b-1$ columns together with at most w columns from each set of previous b consecutive columns which are at gap of s columns (the last set may contain less than b columns), along with a linear combination of at most w columns from each set of b consecutive columns which are at gap of s columns confined to the first $n-b$ columns (the last set may contain less than b columns). This can be written as

$$\begin{aligned} h_n \neq {} & \left(\sum_{i=1}^{b-1} a_{i1}h_{n-i} + \sum_{i=0}^{b-1} b_{i1}h_{n-(s+b)-i} + \sum_{i=0}^{b-1} b_{i2}h_{n-2(s+b)-i} + \ldots + \sum_{i=0}^{g-1} b_{i\lambda}h_{n-\lambda(s+b)-i}\right) + \\ & + \left(\sum_{i=0}^{b-1} \alpha_{i1}h_{j'-i} + \sum_{i=0}^{b-1} \beta_{i1}h_{j'-(s+b)-i} + \sum_{i=0}^{b-1} \beta_{i2}h_{j'-2(s+b)-i} + \ldots + \sum_{i=0}^{g'-1} \beta_{i\lambda'}h_{j'-\lambda'(s+b)-i}\right), \end{aligned} \tag{2}$$

where $a_{ij}, b_{ij}, \alpha_{ij}, \beta_{ij} \in \mathrm{GF}(q)$ such that the number of nonzero a_{ij} is at most $w-1$, and that of b_{ij}, α_{ij}, β_{ij} is at most w; $j' \leqslant n-b$; $g = \gamma\big(n \bmod (s+b)\big) = \gamma(l)$, $g' = \gamma\big((n-b-j'+1) \bmod (s+b)\big)$, and $\lambda' = \left\lfloor \dfrac{n-b}{s+b} \right\rfloor$.

The number of coefficients a_{i1} is $\sum\limits_{j=0}^{w-1}\binom{b-1}{j}(q-1)^j$.

The number of coefficients b_{ij} is $\left(\sum\limits_{j=0}^{w}\binom{b-1}{j}(q-1)^j\right)^{\lambda-1}\sum\limits_{j=0}^{\min\{w,g\}}\binom{g}{j}(q-1)^j$. So the number of all possible linear combinations in the first bracket of the right-hand side (2) is

$$\sum_{j=0}^{w-1}\binom{b-1}{j}(q-1)^j\left(\sum_{j=0}^{w}\binom{b-1}{j}(q-1)^j\right)^{\lambda-1}\sum_{j=0}^{\min\{w,g\}}\binom{g}{j}(q-1)^j.$$

The second bracket in (2) gives the total number of low-density periodic random error in a vector of length $n-b$. This is given by Theorem 3 as $1+\sum\limits_{j=1}^{w_{\max}}R^{n-b}_{s,b|w}(j)$. Therefore, the total number of all the possible linear combinations of the right-hand side (2) is

$$\sum_{j=0}^{w-1}\binom{b-1}{j}(q-1)^j\left(\sum_{j=0}^{w}\binom{b-1}{j}(q-1)^j\right)^{\lambda-1}\sum_{j=0}^{\min\{w,g\}}\binom{g}{j}(q-1)^j\left(1+\sum_{j=1}^{w_{\max}}R^{n-b}_{s,b|w}(j)\right). \tag{3}$$

Since we have at most q^{n-k} columns, so taking q^{n-k} greater than or equal to the term computed in (3) gives the sufficient condition for the existence of the required code. ■

In the following examples, λ', p'_i, and β'_j represent the values of λ, p_i, and β_j respectively, when n is replaced by $n-b$.

Example 4. Consider $n = 11$, $s = 3$, $b = 2$, $w = 1$, and $q = 3$ in Theorem 4, then $\lambda = 2$, $l = 11 \bmod 5 = 1$, $\lambda' = 1$, $p'_{s+1} = 1$, $p'_{s+2} = 1$, $\beta'_0 = \beta'_1 = \beta'_2 = 2$, $\beta'_3 = \beta'_4 = 1$. Putting these values in the inequality (1), we get

$$3^{n-k} > \sum_{j=0}^{0}\binom{1}{j}(3-1)^j\left(\sum_{j=0}^{1}\binom{2-1}{j}(3-1)^j\right)^1\sum_{j=0}^{\min\{w=1,g=1\}}\binom{1}{j}(3-1)^j\left(1+\sum_{j=1}^{3}R_{3,2|1}^{11-2}(j)\right) =$$
$$= 1\cdot 3\cdot 3\,(1+66)\ \big[\text{Using Theorem 1 and Example 1}\big] = 603.$$

This implies $n-k \geqslant 6$. Thus, we can construct a parity check matrix H of order 6×11, which generates the $(11,5)$ ternary $\mathrm{LDP}_{(3,2|1),11,3}$RC-code:

$$H = \begin{bmatrix} 1 & 0 & 2 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 2 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 2 \end{bmatrix}.$$

It can be verified from the Error Pattern-Syndromes Table 1 that the syndromes of all the errors are nonzero and distinct, showing that the code is a $(11,5)$ ternary $\mathrm{LDP}_{(3,2|1),11,3}$RC-code.

Table 1

Error Pattern-Syndrome

Error Patterns	Syndromes	Error Patterns	Syndromes
00 000 00 000 1	001002	20 000 02 000 2	210000
00 000 00 000 2	002001	10 000 10 000 1	221000
00 000 01 000 0	002000	10 000 10 000 2	222000
00 000 02 000 0	001000	10 000 20 000 1	021000
00 000 10 000 0	100000	10 000 20 000 2	022002
00 000 20 000 0	200000	20 000 10 000 1	011001
00 000 01 000 1	000002	20 000 10 000 2	012000
00 000 01 000 2	001001	20 000 20 000 1	111001
00 000 02 000 1	002002	20 000 20 000 2	112000
00 000 02 000 2	000001	0 00 000 01 000	000100
00 000 10 000 1	101002	0 00 000 02 000	000200
00 000 10 000 2	102001	0 01 000 00 000	200010
00 000 20 000 1	201002	0 02 000 00 000	100020
00 000 20 000 2	202001	0 01 000 01 000	200110
01 000 00 000 0	010100	0 01 000 02 000	200210
02 000 00 000 0	020200	0 02 000 10 000	100120
01 000 00 000 1	011102	0 02 000 20 000	100220
01 000 00 000 2	012101	0 01 000 10 000	202010
02 000 00 000 1	021202	0 01 000 20 000	201010
02 000 00 000 2	022201	0 02 000 10 000	102010
01 000 01 000 0	012100	0 02 000 20 000	101020
01 000 02 000 0	011100	0 10 000 01 000	010200
02 000 01 000 0	022200	0 10 000 02 000	010000

End of Table 1

Error Patterns	Syndroms	Error Patterns	Syndromes
02 000 02 000 0	021200	0 20 000 10 000	020000
01 000 10 000 0	110100	0 20 000 20 000	020100
01 000 20 000 0	210100	00 00 000 01 00	111111
02 000 10 000 0	120200	00 00 000 02 00	222222
02 000 20 000 0	220200	00 01 000 00 00	100111
01 000 01 000 1	010102	00 02 000 00 00	200222
01 000 01 000 2	011101	00 01 000 01 00	211222
01 000 02 000 1	012102	00 01 000 02 00	022000
01 000 02 000 2	010101	00 02 000 01 00	011000
02 000 01 000 1	020202	00 02 000 02 00	122111
02 000 01 000 2	021201	00 01 000 10 00	100211
02 000 02 000 1	022202	00 01 000 20 00	100011
02 000 02 000 2	020201	00 02 000 10 00	200022
01 000 10 000 1	111102	00 02 000 20 00	200122
01 000 10 000 2	112101	00 10 000 01 00	011121
01 000 20 000 1	211102	00 10 000 02 00	122202
01 000 20 000 2	212101	00 20 000 01 00	211101
02 000 10 000 1	121202	00 20 000 02 00	022212
02 000 10 000 2	122201	000 00 000 01 0	000010
02 000 20 000 1	221202	000 00 000 02 0	000020
02 000 20 000 2	222201	000 01 000 00 0	000200
10 000 00 000 0	120001	000 02 000 00 0	000100
20 000 00 000 0	210002	000 01 000 01 0	000210
10 000 00 000 1	121000	000 01 000 02 0	000220
10 000 00 000 2	122002	000 02 000 01 0	000110
20 000 00 000 1	211001	000 02 000 02 0	000120
20 000 00 000 2	212000	000 01 000 10 0	112011
10 000 01 000 0	122001	000 01 000 20 0	222122
10 000 02 000 0	121001	000 02 000 10 0	111211
20 000 01 000 0	212002	000 02 000 20 0	222022
20 000 02 000 0	211002	000 10 000 01 0	100121
10 000 10 000 0	220001	000 10 000 02 0	100101
10 000 20 000 0	020001	000 20 000 01 0	200202
20 000 10 000 0	010002	000 20 000 02 0	200212
20 000 20 000 0	110002	0000 01 000 10	100010
10 000 01 000 1	120000	0000 01 000 20	100020
10 000 01 000 2	121002	0000 02 000 10	200010
10 000 02 000 1	122000	0000 02 000 20	200020
10 000 02 000 2	120002	0000 10 000 01	001202
20 000 01 000 1	210001	0000 10 000 02	002201
20 000 01 000 2	211000	0000 20 000 01	001102
20 000 02 000 1	212001	0000 20 000 02	002101

Example 5. Consider $n = 12$, $s = 3$, $b = 2$, $w = 1$, and $q = 2$ in Theorem 4, then $\lambda = 2$, $l = 12 \bmod 5 = 2$, $\lambda' = 2$, $p'_{s+1} = 1$, $p'_{s+2} = 1$, $\beta'_0 = \beta'_1 = \cdots = \beta'_3 = 2$, $\beta'_4 = 1$. From inequality (1), we get

$$2^{n-k} > \sum_{j=0}^{0} \binom{2-1}{j} (2-1)^j \left(\sum_{j=0}^{1} \binom{2-1}{j} (2-1)^j \right)^1 \sum_{j=0}^{\min\{w=1,g=2\}} \binom{2}{j} (2-1)^j \left(1 + \sum_{j=1}^{3} R^{12-2}_{3,2|1}(j) \right) =$$
$$= 1 \cdot 2^1 \cdot 3\,(1+24) = 150.$$

This implies $n-k \geqslant 8$, which gives rise to a $(12,4)$ binary $\mathrm{LDP}_{(3,2|1),12,2}$RC-code and its parity check matrix H is given by

$$H = \begin{bmatrix} 1&0&0&0&0&0&0&0&0&1&1&1\\ 0&1&0&0&0&0&0&0&0&1&0&1\\ 0&0&1&0&0&0&0&0&1&1&1&0\\ 0&0&0&1&0&0&0&0&1&1&0&1\\ 0&0&0&0&1&0&0&0&0&1&1&0\\ 0&0&0&0&0&1&0&0&1&1&0&0\\ 0&0&0&0&0&0&1&0&0&1&1&1\\ 0&0&0&0&0&0&0&1&0&1&0&1 \end{bmatrix}.$$

It can be verified like the previous one that the syndromes of all the errors are nonzero and distinct, so the code is a $(12,4)$ binary $\mathrm{LDP}_{(3,2|1),12,2}$RC-code.

Example 6. Consider $n=14$, $s=4$, $b=3$, $w=2$, and $q=2$ in Theorem 4, then $\lambda=2$, $l=14 \bmod 7=0$, $\lambda'=1$, $p'_{s+1}=1$, $p'_{s+2}=1$, $p'_{s+3}=2$, $\beta'_0=\beta'_1=\beta'_2=2$, $\beta'_3=\beta'_4=\beta'_5=\beta'_6=1$. Inequality (1) gives

$$2^{n-k} > \sum_{j=0}^{2-1}\binom{3-1}{j}(2-1)^j\left(\sum_{j=0}^{2}\binom{3-1}{j}(2-1)^j\right)^1 \sum_{j=0}^{\min\{w=2,g=0\}}\binom{g}{j}(2-1)^j\left(1+\sum_{j=1}^{4}R^{14-3}_{4,3|2}(j)\right)=$$
$$= 3\cdot 4^1\cdot 1\,(1+135) = 1632.$$

This implies $n-k \geqslant 11$ which leads to a $(14,3)$ binary $\mathrm{LDP}_{(3,2|2),14,2}$RC-code with parity check matrix

$$H = \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0&1&1&1\\ 0&1&0&0&0&0&0&0&0&0&0&0&1&1\\ 0&0&1&0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&1&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&1&0&0&0&0&0&0&0&1&1\\ 0&0&0&0&0&1&0&0&0&0&0&1&1&0\\ 0&0&0&0&0&0&1&0&0&0&0&1&1&0\\ 0&0&0&0&0&0&0&1&0&0&0&0&1&1\\ 0&0&0&0&0&0&0&0&1&0&0&1&1&0\\ 0&0&0&0&0&0&0&0&0&1&0&0&1&1\\ 0&0&0&0&0&0&0&0&0&0&1&0&1&1 \end{bmatrix}.$$

Here we can also verify that all error patterns give nonzero and distinct syndromes.

Finally, we give the probability of decoding error for a $\mathrm{LDP}_{(s,b|w),n,q}$RC-code over a binary symmetric channel.

Theorem 5. Let $\mathrm{PD}_R(E)$ be the probability of decoding error of an (n,k) binary $\mathrm{LDP}_{(s,b|w),n,2}$RC-code on a binary symmetric channel with transition probability ϵ, then

$$\mathrm{PD}_R(E) = 1-\sum_{j=1}^{w_{\max}} R^n_{s,b|w}(j)\,\epsilon^j(1-\epsilon)^{n-j}, \quad \text{where } R^n_{s,b|w}(j) \text{ is given by Theorem 1.}$$

Proof. Since the binary symmetric channel has the transition probability ϵ, the probability of occurring of any one of the error vector of weight j is $\epsilon^j(1-\epsilon)^{n-j}$. So the probability of occurring of any error vector from the set $\xi_{(s,b|w),n,q}$ is $\sum\limits_{j=1}^{w_{\max}} R^n_{s,b|w}(j)\,\epsilon^j(1-\epsilon)^{n-j}$.

Since the code corrects all such error patterns, the probability of a decoding error of the code is $\mathrm{PD}_R(E) = 1 - \sum_{j=1}^{w_{\max}} R^n_{s,b|w}(j)\, \epsilon^j (1-\epsilon)^{n-j}$. ■

Remark 3. For $s = 3$, $b = 2$, and $\epsilon = 0.1$, we determine the probability of decoding error $\mathrm{PD}_R(E)$ of binary $\mathrm{LDP}_{(s,b|w),n,2}$RC-codes of different lengths as follows (Table 2).

Table 2

Values of $\mathbf{PD}_R(E)$

n	λ	l	$\mathrm{PD}_R(E)$
10	2	0	0.19
11	2	1	0.21
12	2	2	0.23
13	2	3	0.29
14	2	4	0.31
15	3	0	0.33

We find that the probability of decoding error of $\mathrm{LDP}_{(s,b|w),n,q}$RC-code increases as the length of the code increases. So a smaller length code is more efficient.

4. Conclusion

This paper derives the weight distribution of low-density periodic random errors. Then necessary and sufficient conditions for the existence of linear codes that correct such errors, along with the error decoding probability of the codes, are presented. It can be interesting to explore some more systematic methods by which we can construct such codes. We can also investigate array code or cyclic code instead of linear code that can deal with such errors.

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