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Numerical method for restoring the initial condition for the wave equation

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Abstract. The inverse problem of restoring the initial condition for the time derivative for the one-dimensional wave equation is considered. As an additional condition, the solution of the wave equation at a finite time is given. First, the discretization of the derivative with respect to the spatial variable is carried out and the initial problem is reduced to a differential-difference problem with respect to functions depending on the time variable. To solve the resulting differential-difference problem, a special representation is proposed, with the help of which the problem splits into two independent differential-difference problems. As a result, an explicit formula is obtained for determining the approximate value of the desired function for each discrete value of a spatial variable. The finite difference method is used for the numerical solution of the obtained differential-difference problems. The presented results of numerical experiments conducted for model problems demonstrate the effectiveness of the proposed computational algorithm.
Keywords: wave equation, inverse problem, recovery of the initial condition, differential-difference problem

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Научная статья

Численный метод восстановления начального условия для волнового уравнения

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Аннотация. Рассматривается обратная задача восстановления начального условия для производной по времени для одномерного волнового уравнения. В качестве дополнительного условия задается решение волнового уравнения в конечный момент времени. Сначала проводится дискретизация производной по пространственной переменной, и исходная задача сводится к дифференциально-разностной задаче относительно функций, зависящих от временной переменной. Для решения полученной дифференциально-разностной задачи предлагается специальное представление, с помощью которого задача распадается на две независимые дифференциально-разностные задачи. В результате получена явная формула для определения приближенного значения искомой функции при каждом дискретном значении пространственной переменной. Для численного решения полученных дифференциально-разностных задач используется метод конечных разностей. Представленные результаты численных экспериментов, проведенных для модельных задач, демонстрируют эффективность предложенного вычислительного алгоритма.

Ключевые слова: волновое уравнение, обратная задача, восстановление начального условия, дифференциально-разностная задача

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Introduction

It is known that inverse problems for wave equations occur in mathematical modeling of many physical processes in geophysics, seismics, electrodynamics, thermophysics, medicine, and many other fields of science and technology [1–5]. In these inverse problems, in addition to solving the wave equation, it is necessary to determine either the right-hand sides, or coefficients, or initial conditions. It should be noted that a large number of publications have been devoted to the study of the correctness, existence, and unambiguous solvability of coefficient inverse problems and inverse problems for determining the right parts of wave equations [6–12]. At the same time, much less work has been devoted to the inverse problem of restoring the initial conditions for wave equations. In a number of papers [13–16], Dirichlet-type problems for the wave equation are presented as an inverse problem of restoring the initial condition and gradient iterative methods are proposed for the numerical solution of such problems.

In this paper, a non-iterative computational algorithm is proposed for the numerical solution of the inverse problem of restoring the initial condition for the time derivative for a one-dimensional wave equation.

1. Problem statement and solution method

Let a one-dimensional wave equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial}{\partial x}\left(k(x)\frac{\partial u(x,t)}{\partial x}\right) + f(x,t), \quad 0 < x < 1, \quad 0 < t \le T, \tag{1}$$

be considered with the initial conditions

$$u(x,0) = \varphi(x), \tag{2}$$

$$\frac{\partial u(x,0)}{\partial t} = v(x), \tag{3}$$

and boundary conditions

$$u(0,t) = q(t), \tag{4}$$

$$u(1,t) = p(t). \tag{5}$$

It is known that the direct problem for equation (1) consists in determining a function $u(x,t)$ from equation (1) with a given coefficient $k(x)$, the right side $f(x,t)$ and conditions (2)–(5).

Suppose that in addition to the function $u(x,t)$, the function $v(x)$ is also unknown and the restoration of this function is required. In this case, as an additional condition, the solution of equation (1) is given at a finite time

$$u(x,T) = \psi(x), \tag{6}$$

where $\psi(x)$ is the given function.

Thus, the task is to determine the functions $u(x,t)$ and $v(x)$ satisfying equation (1) and conditions (2)–(6). The problem belongs to the class of inverse problems associated with the restoration of initial conditions for partial differential equations.

First, we transform the assigned task to a semi-discrete task. To this end, we introduce a uniform difference grid in the domain $[0 \le x \le 1]$ of a variable x

$$\overline{\omega_x} = \{x_i = i\Delta x,\ i = 0,1,2,...,n\}$$

with a step $\Delta x = \frac{1}{n}$.

The differential expression $\frac{\partial}{\partial x}(k(x)\frac{\partial u(x,t)}{\partial x})$ in equation (7) for $x = x_i$, $i = 1,2,...,n-1$ is approximated by the "central" difference

$$\frac{\partial}{\partial x}(k(x)\frac{\partial u(x,t)}{\partial x})\Big|_{x=x_i} \approx$$

$$\approx \frac{1}{\Delta x}\left[k(x_i + \frac{\Delta x}{2})\frac{u(x_{i+1},t) - u(x_i,t)}{\Delta x} - k(x_i - \frac{\Delta x}{2})\frac{u(x_i,t) - u(x_{i-1},t)}{\Delta x}\right].$$

Denoting $u_i(t) \approx u(x_i,t)$, $k_{i\pm1/2} = k\left(x_i \pm \frac{\Delta x}{2}\right)$, equation (1) and conditions (2)–(5) are written as the following system of ordinary differential equations

$$\frac{d^2u_i(t)}{dt^2} = \frac{k_{i+1/2}}{\Delta x^2}u_{i+1}(t) - \frac{2k_i}{\Delta x^2}u_i(t) + \frac{k_{i-1/2}}{\Delta x^2}u_{i-1}(t) + f_i(t),\ 0 < t \le T,\ i = \overline{1,n-1}, \tag{7}$$

$$u_i(0) = \varphi_i,\ \ i = \overline{0,n}, \tag{8}$$

$$\frac{du_i(0)}{dt} = v_i,\ \ i = \overline{0,n}, \tag{9}$$

$$u_0(t) = q(t), \tag{10}$$

$$u_n(t) = p(t), \tag{11}$$

$$u_i(T) = \psi_i,\ \ i = \overline{0,n}, \tag{12}$$

where $k_i = (k_{i+1/2} + k_{i-1/2})/2$, $v_i \approx v(x_i)$, $\varphi_i = \varphi(x_i)$, $\psi_i = \psi(x_i)$, $f_i(t) = f(x_i,t)$.
In the resulting differential-difference problem, the approximate values of the desired functions $v(x)$ in the nodes of the difference grid $\overline{\omega_x}$, i.e. v_i and the functions $u_i(t)$,

$i=1,\ 2,\ ...,\ n-1$, act as unknown. For the decomposition of the differential-difference problem (7)–(12) into mutually independent subtasks, each of which can be solved independently, its solution for each fixed value $i=0,1,2,...,n,$ is represented as [17, 18]

$$u_i(t)=w_i(t)+v_i\,\theta_i(t),\quad i=0,1,2,...,n, \tag{13}$$

where $w_i(t),\ \theta_i(t)$ are unknown functions. Substituting the representation $u_i(t)$ into equation (7), we obtain

$$\frac{d^2w_i(t)}{dt^2}+v_i\frac{d^2\theta_i(t)}{dt^2}=\frac{k_{i+1/2}}{\Delta x^2}w_{i+1}(t)+v_{i+1}\frac{k_{i+1/2}}{\Delta x^2}\theta_{i+1}(t)-\frac{2k_i}{\Delta x^2}w_i(t)-v_i\frac{2k_i}{\Delta x^2}\theta_i(t)+$$
$$+\frac{k_{i-1/2}}{\Delta x^2}w_{i-1}(t)+v_{i-1}\frac{k_{i-1/2}}{\Delta x^2}\theta_{i-1}(t)+f_i(t)\,.$$

Replacing v_{i-1} and v_{i+1} with v_i, the latter relation is represented as

$$\left[\frac{d^2w_i(t)}{dt^2}-\frac{k_{i+1/2}}{\Delta x^2}w_{i+1}(t)+\frac{2k_i}{\Delta x^2}w_i(t)-\frac{k_{i-1/2}}{\Delta x^2}w_{i-1}(t)-f_i(t)\right]+$$
$$+v_i\left[\frac{d^2\theta_i(t)}{dt^2}-\frac{k_{i+1/2}}{\Delta x^2}\theta_{i+1}(t)+\frac{2k_i}{\Delta x^2}\theta_i(t)-\frac{k_{i-1/2}}{\Delta x^2}\theta_{i-1}(t)\right]=0\,.$$

Substitution of representation (13) into (8)–(11) yields

$$w_i(0)+v_i\,\theta_i(0)=\varphi_i\,,$$
$$\frac{dw_i(0)}{dt}+v_i\frac{d\theta_i(0)}{dt}=v_i\,,$$
$$w_0(t)+v_0\theta_0(t)=q(t)\,,$$
$$w_n(t)+v_n\theta_n(t)=p(t)\,.$$

From the obtained relations, it is possible to obtain differential-difference problems for determining auxiliary functions $w_i(t),\ \ \theta_i(t)$, $i=0,1,2,...,n$

$$\frac{d^2w_i(t)}{dt^2}-\frac{k_{i+1/2}}{\Delta x^2}w_{i+1}(t)+\frac{2k_i}{\Delta x^2}w_i(t)-\frac{k_{i-1/2}}{\Delta x^2}w_{i-1}(t)-f_i(t)=0\,,\quad i=\overline{1,n-1}\,, \tag{14}$$

$$w_i(0)=\varphi_i\,, \tag{15}$$

$$\frac{dw_i(0)}{dt}=0\,, \tag{16}$$

$$w_0(t)=q(t)\,, \tag{17}$$

$$w_n(t)=p(t)\,. \tag{18}$$

$$\frac{d^2\theta_i(t)}{dt^2}-\frac{k_{i+1/2}}{\Delta x^2}\theta_{i+1}(t)+\frac{2k_i}{\Delta x^2}\theta_i(t)-\frac{k_{i-1/2}}{\Delta x^2}\theta_{i-1}(t)=0\,,\quad i=\overline{1,n-1}\,, \tag{19}$$

$$\theta_i(0)=0\,, \tag{20}$$

$$\frac{d\theta_i(0)}{dt}=1\,, \tag{21}$$

$$\theta_0(t)=0\,, \tag{22}$$

$$\theta_n(t)=0\,. \tag{23}$$

And substituting representation (13) into (12), we have

$$w_i(T)+v_i\,\theta_i(T)=\psi_i\,.$$

From here we get a formula for determining the value of the desired function $v(x)$ for each fixed value $x=x_i$

$$v_i=\frac{\psi_i-w_i(T)}{\theta_i(T)},\quad i=\overline{1,n-1}. \tag{24}$$

Thus, the computational algorithm for the numerical solution of the differential-difference problem (7)–(12), by definition $u_i(t)$, v_i, $i=1,2,\ldots,n-1$, consists of the following:

– the solutions of two independent differential-difference problems (14)–(18) and (19)–(23) with respect to auxiliary functions $w_i(t)$, $\theta_i(t)$, $i=0,1,2,\ldots,n$, are determined on the segment $[0,T]$;

– according to formula (24), approximate values of the desired function $v(x)$ are determined for $x=x_i$, i.e. v_i, $i=1,2,\ldots,n-1$;

– the formula (13) determines the values of the functions $u_i(t)$, $i=0,1,2,\ldots,n$, on the segment $[0,T]$.

It should be noted that the approximate values of the desired function $v(x)$ at the boundary points $x_0=0$ and $x_n=1$ cannot be determined by formula (24) due to the fulfillment of conditions (22) and (23). Therefore, the values of the desired function $v(x)$ at the boundary points can be determined by interpolation.

It should be noted that the applicability of the proposed computational algorithm is associated with the fulfillment of the condition

$$\theta_i(T)\neq 0,\quad i=\overline{1,n-1}\,.$$

For an equation with a constant coefficient, it is possible to find out in advance the fulfillment of this condition. To do this, it is enough to write a differential approximation of the differential-difference problem (19)–(23) for the case $k(x)=k_0=const$

$$\frac{\partial^2\theta(x,t)}{\partial t^2}=k_0\,\frac{\partial^2\theta(x,t)}{\partial x^2},\quad 0<x<1,\ \ 0<t\le 0.1\,,$$

$$\theta(x,0)=0\,,\quad \frac{\partial\theta(x,0)}{\partial t}=a(x)\equiv 1\,,$$

$$\theta(0,t)=0\,,\quad \theta(1,t)=0\,.$$

The exact solution of this problem is determined by the explicit formula

$$\theta(x,t)=\sum_{r=1}^{\infty}\frac{2}{\pi r\sqrt{k_0}}\left[\int_0^1 a(\xi)\sin\pi r\xi\,d\xi\right]\sin\pi r\sqrt{k_0}\,t\,\sin\pi r x\,.$$

It follows that when $T\neq 1/\sqrt{k_0}$ the condition $\theta_i(T)\neq 0$, $i=\overline{1,n-1}$ is satisfied. However, for an equation with variable coefficients, due to the complexity of constructing an analytical solution, the condition $\theta_i(T)\neq 0$, $i=\overline{1,n-1}$ can be fulfilled using a numerical experiment.

For the numerical solution of problems (14)–(18) and (19)–(23), the finite difference method can be used. We introduce a uniform difference grid with a step Δt on the segment $[0, T]$ in the variable t

$$\bar{\omega}_t = \{t_j = j\Delta t,\ j = 0,1,2,...,m,\ \Delta t = T/m\}.$$

The discrete analogs of problems (14)–(18) and (19)–(23) on the grid $\bar{\omega}_t$ are represented as

$$\frac{w_i^{j+1} - 2w_i^j + w_i^{j-1}}{\Delta t^2} - \frac{k_{i+1/2}}{\Delta x^2} w_{i+1}^{j+1} + \frac{2k_i}{\Delta x^2} w_i^{j+1} - \frac{k_{i-1/2}}{\Delta x^2} w_{i-1}^{j+1} - f_i^{j+1} = 0, \tag{25}$$

$$w_i^0 = \varphi_i, \tag{26}$$

$$\frac{w_i^1 - w_i^0}{\Delta t} = 0, \tag{27}$$

$$w_0^{j+1} = q^{j+1}, \tag{28}$$

$$w_n^{j+1} = p^{j+1}, \tag{29}$$

$$\frac{\theta_i^{j+1} - 2\theta_i^j + \theta_i^{j-1}}{\Delta t^2} - \frac{k_{i+1/2}}{\Delta x^2} \theta_{i+1}^{j+1} + \frac{2k_i}{\Delta x^2} \theta_i^{j+1} - \frac{k_{i-1/2}}{\Delta x^2} \theta_{i-1}^{j+1} = 0, \tag{30}$$

$$\theta_i^0 = 0, \tag{31}$$

$$\frac{\theta_i^1 - \theta_i^0}{\Delta t} = 1, \tag{32}$$

$$\theta_0^{j+1} = 0, \tag{33}$$

$$\theta_n^{j+1} = 0, \tag{34}$$

where $w_i^j \approx w_i(t_j)$, $\theta_i^j \approx \theta_i(t_j)$, $f_i^{j+1} = f_i(t_{j+1})$.

The obtained difference problems (25)–(29) and (30)–(34) for each fixed value $j = 1, 2, \ldots, m-1$ are systems of linear algebraic equations with a tridiagonal matrix, the solutions of which can be found by the Thomas method [17].

2. Numerical examples

To find out the effectiveness of the proposed computational algorithm, numerical experiments were carried out for model problems. Calculations were carried out on a space-time difference grid with steps $\Delta x = 0.05$, $\Delta t = 0.0001$.

Example 1.

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{1}{8\pi^2}\frac{\partial^2 u(x,t)}{\partial x^2} + e^{0.5t}(1.5 + 3\cos 2\pi x),\quad 0 < x < 1,\ 0 < t \le 0.,$$

$$u(x,0) = 2(3 + 2\cos 2\pi x),\quad \frac{\partial u(x,0)}{\partial t} = v(x),$$

$$u(0,t) = 10e^{0.5t},\quad u(1,t) = 10e^{0.5t},$$

$$u(x,0.1) = 2e^{0.05}(3 + 2\cos 2\pi x).$$

This problem has an exact solution

$$u(x,t) = 2e^{0.5t}(3 + 2\cos 2\pi x),\quad v(x) = 3 + 2\cos 2\pi x.$$

Example 2.

$$\frac{\partial^2 u(x,t)}{\partial t^2} = 0.025\frac{\partial^2 u(x,t)}{\partial x^2} + e^{0.5t}(10x - 10x^2 + 2)\,, \quad 0 < x < 1,\ \ 0 < t \le 0.1\,,$$

$$u(x,0) = 40x - 40x^2\,, \quad \frac{\partial u(x,0)}{\partial t} = v(x)\,,$$

$$u(0,t) = 0\,, \quad u(1,t) = 0,$$

$$u(x,0.1) = e^{0.05}(40x - 40x^2).$$

The exact solution to this problem has the form

$$u(x,t) = e^{0.5t}(40x - 40x^2)\,, \quad v(x) = 20x - 20x^2.$$

Example 3.

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial}{\partial x}(xe^{0.2x}\frac{\partial u(x,t)}{\partial x}) + \sin 3t(1 - 45e^{-0.2x})\,, \quad 0 < x < 1,\ \ 0 < t \le 0.1\,,$$

$$u(x,0) = 0\,, \quad \frac{\partial u(x,0)}{\partial t} = v(x)\,,$$

$$u(0,t) = 5\sin 3t\,, \quad u(1,t) = 5e^{-0.2}\sin 3t,$$

$$u(x,0.1) = 5e^{-0.2x}\sin 0.3.$$

The exact solution of the problem has the form

$$u(x,t) = 5e^{-0.2x}\sin 3t\,, \quad v(x) = 15e^{-0.2x}.$$

The results of numerical experiments to determine the approximate values of the desired function $v(x)$ at $x = x_i$, $i = 1, 2, ..., n-1$, for the examples given are presented in the table. The data in the 2nd and 3rd columns refer to the first example; the data in the 4th and 5th columns, to the second example; and data in the 6th and 7th columns, to the third example.

Numerical results on the determination of the function $v(x)$

x_i	$v(x) = 3 + 2\cos 2\pi x$		$v(x) = 20x - 20x^2$		$v(x) = 15e^{-0.2x}$	
	Exact	Calculated	Exact	Calculated	Exact	Calculated
0.05	4.902	4.899	0.950	0.952	14.851	14.844
0.10	4.618	4.616	1.800	1.799	14.703	14.699
0.15	4.176	4.174	2.550	2.548	14.557	14.552
0.20	3.618	3.617	3.200	3.198	14.412	14.407
0.25	3.000	3.000	3.750	3.748	14.268	14.264
0.30	2.382	2.383	4.200	4.198	14.126	14.122
0.35	1.824	1.826	4.550	4.548	13.986	13.982
0.40	1.382	1.384	4.800	4.798	13.847	13.842
0.45	1.098	1.100	4.950	4.948	13.709	13.705
0.50	1.000	1.003	5.000	4.998	13.573	13.568
0.55	1.098	1.100	4.950	4.948	13.438	13.433
0.60	1.382	1.384	4.800	4.798	13.304	13.299
0.65	1.824	1.826	4.550	4.548	13.171	13.167
0.70	2.382	2.383	4.200	4.198	13.040	13.036
0.75	3.000	3.000	3.750	3.748	12.911	12.906
0.80	3.618	3.617	3.200	3.198	12.782	12.778
0.85	4.176	4.174	2.550	2.548	12.655	12.655
0.90	4.618	4.616	1.800	1.799	12.529	12.535
0.95	4.902	4.899	0.950	0.952	12.404	12.411

The results of numerical experiments indicate that the values of the desired functions $u(x,t)$ and $v(x)$ are determined with a sufficiently high accuracy. At the same time, the maximum relative error in determining the desired function $v(x)$ in the first example does not exceed 0.08%; in the second example, 0.3%; and in the third example, 0.06%. Analysis of results of the numerical experiments shows that to increase the accuracy of solutions, it is sufficient to use small steps of the difference grid.

Conclusion

The problem of determining the initial condition for the time derivative for a one-dimensional wave equation, according to an additionally specified condition at a finite time, is considered. The proposed computational algorithm, based on the discretization of the problem by a spatial variable and the use of a special representation to solve the resulting differential-difference problem, allows us to find by an explicit formula the approximate value of the desired function for each discrete value of the spatial variable. The proposed computational algorithm can also be used to restore the initial condition in time for the one-dimensional wave equation.

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