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Bishop's formula for a matrix polyhedron with a non-piecewise smooth boundary

Erkin M. Makhkamov[1], Jurabek T. Bozorov[2]

[1] Non-governmental Organization of Higher Education Perfect University, Tashkent
[1] National University of Uzbekistan, Uzbekistan, Tashkent
[2] Termez State University, Uzbekistan, Termez
[1] erkin_mahkamov83@mail.ru
[2] jorabek.bozorov.89@mail.ru

Abstract. In this work, a matrix polyhedral domain is defined using a matrix ball. In this matrix polyhedral domain, an analogue of Bishop's formula for meromorphic functions of a special form is obtained.
Keywords: holomorphic functions and mappings, matrix polyhedral set, matrix polyhedral domain, generalized matrix ball, meromorphic function, Bishop's integral formula

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Научная статья

Формула Бишопа для матричного полиэдра с не кусочно гладкой границей

Эркин Мусурманович Махкамов[1],
Жўрабек Тоғаймуротович Бозоров[2]

[1] Негосударственная организация высшего образования Perfect University, Узбекистан, Ташкент
[1] Национальный университет Узбекистана, Узбекистан, Ташкент
[2] Термезский государственный университет, Узбекистан, Термез
[1] erkin_mahkamov83@mail.ru
[2] jorabek.bozorov.89@mail.ru

Аннотация. В теории функций многих комплексных переменных интегральные формулы занимают важное место в теории голоморфных и мероморфных функций специального вида. При этом задачи получения новых интегральных формул с помощью локальных вычетов, разложения в ряды голоморфных и мероморфных функций специального типа с помощью интегральных формул считаются целевыми научными исследованиями. В данной работе определена матричная полиэдрическая область с помощью матричного шара. В этой матричной полиэдрической области получен аналог формулы Бишопа для мероморфных функции специального вида.

Ключевые слова: голоморфные функции и отражения, матричное полиэдричиское множество, матричная полиэдрическая область, обобщенный матричный шар, мероморфная функция, интегральная формула Бишопа

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Introduction. Formulation of the problem

In complex analysis of multivariables, integral formulas have been studied by many authors. These results are presented in monographs [1–3] by A.K. Tsikh. A.K. Tsikh proved the Weyl and Bishop integral formulas in a special analytical polyhedron using local residues of multivariables [4]. In the polyhedral domain, A. Weil [5] studied integral formulas with a holomorphic kernel. In the [6–11], a matrix analogue of the integral formula of Cauchy–Weil, Bishop and the Carleman formula was studied.

Recall that an analytic polyhedron is defined by a family of functions $f_1, f_2, ..., f_m \in \mathcal{O}(G)$, $G \subset \mathbb{C}^n$ (or by a mapping $f = (f_1, f_2, ..., f_n): G \to \mathbb{C}^m$), as

$$\Pi_r = \left\{ z \in G : \left| f_1(z) \right| < r_1, \left| f_2(z) \right| < r_2, \ldots, \left| f_m(z) \right| < r_m \right\},$$

if it is relatively compact in G (i.e. $\overline{\Pi}_r \subset G$). If m is equal to n, which is the dimension of space $\mathbb{C}^n$, then the analytic polyhedron Π_r is called special.

Let $f = (f_1, f_2, ..., f_n): D \to G$ – holomorphic mapping of domains $D \subset \mathbb{C}^n_z, G \subset \mathbb{C}^n_w$. Consider meromorphic functions of the form $\dfrac{h(Z)}{J_f(Z)}$, where $h(z)$ is a holomorphic function, and $J_f(z)$ is the Jacobian of the mapping $f: D \to G$ which is of finite type. In [4. P. 43], A.K. Tsikh obtained Bishop's integral formula for a special analytic polyhedron an analogue of which we obtain for a generalized matrix ball.

Theorem 1 [4]. *At every point $z \in \Pi_r$ in which the Jacobian J_f of the mapping f is nonzero, the following integral formula for the meromorphic function $\dfrac{h}{J}$, $h \in \mathcal{O}(\overline{\Pi}_r)$ holds,*

$$\frac{h}{J}\Omega(z,z) = (2\pi i)^{-n} \int_{\Gamma} \frac{h(\xi)\Omega(z,\xi)d\xi}{f(\xi) - f(z)},$$

where Γ is the skeleton of the polyhedron Π_r, weight function $\Omega(z,\xi) \not\equiv 0$, and holomorphic in the neighborhood $(\Pi_r \times \Pi_r)$.

The main part

Let $Z = (Z_1, Z_2, \ldots, Z_n)$ be a vector, whose entries are quadratic matrices Z_j, $1 \le j \le n$, of order m over the field of complex numbers $\mathbb{C}$. It can be assumed that Z is the element of the space $\mathbb{C}^n[m \times m] \cong \mathbb{C}^{nm^2}$ [3].

Define matrix “scalar” multiplication for $Z,W \in \mathbb{C}^n[\mathrm{m}\times\mathrm{m}]$ as [3]:

$$\langle Z,W\rangle = Z_1W_1^* + \ldots + Z_nW_n^*,$$

where W_j^* is a matrix, which is conjugate and transpose of W.

The domain $B_{m,n} = \{Z \in \mathbb{C}^n[\mathrm{m}\times\mathrm{m}] : I - \langle Z,Z\rangle > 0\}$, is called a *matrix ball*, where I is the identity matrix of order *m*.

The skeleton of this domain is a manifold of the form:

$$X_{m,n} = \{Z : \langle Z,Z\rangle = I\}.$$

Obviously, the dimension of the skeleton is $m^2(2n-1)$.

When $m = n = 1$, $B_{1,1}$ is the identity disc from $\mathbb{C}$, and $X_{1,1}$ is the identity circle.

Let *D* be a bounded complete circular convex domain with Shilov boundary *S*, which is smooth (of class C^1) manifold.

Define the family $H^1(D)$ of all functions *f*, holomorphic in *D*, for which

$$\sup_{0<r<1} \int_S |f(r\zeta)| d\mu < +\infty,$$

where $r\zeta = (r\zeta_1,\ldots,r\zeta_n)$ and $d\mu$ is the normalized Lebesgue measure on a manifold *S*, invariant under rotations.

Theorem 2 [3]. *For any function* $f \in H^1(B_{m,n})$, *the following formula holds:*

$$f(Z) = \int_{X_{m,n}} \frac{f(W)d\sigma(W)}{\det^{mn}\left(I^{(m)} - \langle Z,W\rangle\right)}, \quad Z \in B_{m,n} \tag{1}$$

where $d\sigma(W)$ *is the normalized Lebesgue measure on the skeleton* $X_{m,n}$.

Take a mapping $f = (f_1,\ldots,f_{nm^2}) : G \to \mathbb{C}^{nm^2}$, which is holomorphic in some domain $G \subset \mathbb{C}^{nm^2}$.

In what follows, the mapping $f = (f_1,\ldots,f_{nm^2}) : G \to \mathbb{C}^{nm^2}$ will be considered in the form

$$f(Z) = \left(\begin{pmatrix} f_{11}^1(Z) & \cdots & f_{1m}^1(Z) \\ \vdots & \ddots & \vdots \\ f_{m1}^1(Z) & \cdots & f_{mm}^1(Z) \end{pmatrix}, \ldots, \begin{pmatrix} f_{11}^n(Z) & \cdots & f_{1m}^n(Z) \\ \vdots & \ddots & \vdots \\ f_{m1}^n(Z) & \cdots & f_{mm}^n(Z) \end{pmatrix} \right) : G \to \mathbb{C}^n[m\times m].$$

Definition 1. A matrix polyhedral set defined by a holomorphic mapping $f : G \to \mathbb{C}^n[m\times m]$ is the set

$$f^{-1}(B_{\mathrm{m,n}}) = \{Z \in G : r^2 I - \langle f(Z), f(Z)\rangle > 0, r > 0\},$$

which is relatively compact in *G*, i.e., $f^{-1}(B_{\mathrm{m,n}}) \Subset G$.

Definition 2. The connected component of a matrix polyhedral set $f^{-1}(B_{\mathrm{m,n}})$ is called *a matrix polyhedron (generalized matrix ball)* which is denoted as $\Theta_{f,r}$. The skeleton of the domain $\Theta_{f,r}$ is defined as

$$\Gamma_{f,r} = \{Z \in G : \langle f(Z), f(Z)\rangle = r^2 I, r > 0\}.$$

Let $f(Z): D \to G$ be a holomorphic mapping of domains $D \subset \mathbb{C}_Z^n[m \times m]$, $G \subset \mathbb{C}_W^n[m \times m]$ and $H(Z) = \frac{\varphi(Z)}{\psi(Z)}$ meromorphic function in *D*.

Definition 3 [4]. A mapping $f(Z)$ *has a finite type*, if for any $W \in G$ the equation $f(Z) = W$ has the same finite number of roots (*taking into account multiplicities*) in domain *D*.

Definition 4 [4]. *The trace of a function* $H(\mathrm{Z})$ *related to a mapping* $f(\mathrm{Z})$ *is the function*

$$[\mathrm{Tr}\, H](\mathrm{W}) = \sum_{\nu} H(\mathrm{Z}^{(\nu)}(\mathrm{W})),\ W \in G \setminus f(\psi = 0),$$

where the summation is carried out over all roots (taking into account multiplicities) of the equation $f(\mathrm{Z}) = W$.

In this work, using formula (1), we obtain the Bishop integral formula in the domain $\Theta_{f,r}$ for a special function of the form $\frac{h(Z)}{J_f(Z)}$, where $h(Z) \in H^1(\Theta_{f,r})$, $J_f(Z)$ is the Jacobian of the mapping $f(\mathrm{Z})$, which has a finite type.

Let $f: D \to G$ be a holomorphic mapping of a finite type of the domain $D \subset \mathbb{C}_Z^n[m \times m]$ into $G \subset \mathbb{C}_W^n[m \times m]$ and $W^0 \in G$ be an arbitrary point. Consider a domain $B_{m,n,r}(W^0)$ in *G* with the center at the point W^0:

$$B_{m,n,r}(W^0) = \left\{ W : r^2 I - \left\langle W - W^0, W - W^0 \right\rangle > 0 \right\} \Subset G .$$

Theorem 3. *Let* $H(Z) \in H^1(D)$. *Then for the trace* $[\mathrm{Tr}\, H](W)$ *in the domain* $B_{m,n,r}(W^0)$ *the following integral formula holds:*

$$[Tr\, H](W) = \int_{\Gamma_{f,r}} \frac{H d\sigma(f(Z))}{\det^{mn}\left(I^{(m)} - \left\langle f(Z), W \right\rangle\right)}, \qquad (2)$$

where $\Gamma_{f,r} = \left\{ Z \in D : \left\langle f(Z), f(Z) \right\rangle = r^2 I^{(m)} \right\}$.

Proof. For simplicity, we prove the theorem when $W^0 = 0$.

According to the proposition in [4. P. 26], for almost all $W \in B_{m,n,r}(W^0)$ the roots of the system of equations $f(Z) - W = 0$ are simple; denote them as $Z^{(1)}(W), \dots, \mathrm{Z}^{(\mu)}(W)$. Let, $U_\nu \subset D$ be the family of disjoint neighborhoods of points $Z^{(\nu)}(W)$ and

$$\Gamma_\nu = \Gamma_{f(Z)-W,\delta} = \left\{ Z \in D : \left\langle f(Z) - W, f(Z) - W \right\rangle = \delta^2 I^{(m)} \right\}$$

is a cycle in U_ν.

Then, by the definition of the trace and Cauchy–Szegő formula (1), we have

$$[Tr\, H](W) = \sum_{\nu=1}^{\mu} \int_{\Gamma_{Z^{(\nu)}(W),\delta}} \frac{H\, d\sigma(f(Z))}{\det^{mn}\left(I^{(m)} - \left\langle f(Z), W \right\rangle\right)}.$$

Moreover, the sum $\sum_{\nu=1}^{\mu}\Gamma_\nu$ is homologic to the cycle $\Gamma_{f,r}$ in the domain of regularity of the subintegral form of (2). Hence, applying the Stokes formula, we obtain

$$\sum_{\nu=1}^{\mu}\int_{\Gamma_{Z^{(\nu)}(W),\delta}}\frac{H\,d\sigma\left(f\left(Z\right)\right)}{\det^{mn}\left(I^{(m)}-\left\langle f\left(Z\right),W\right\rangle\right)}=\int_{\Gamma_{f,r}}\frac{H\,d\sigma\left(f\left(Z\right)\right)}{\det^{mn}\left(I^{(m)}-\left\langle f\left(Z\right),W\right\rangle\right)}.$$

The proof is complete.

Now, we present an integral representation for the trace of a meromorphic function of a special form.

Corollary 1. *Let* $h(Z)\in \mathrm{H}^1\left(D\right)$, J_f be the *Jacobian of the mapping* $f:D\to G$, *which has finite type. Then for the trace of the meromorphic function* $H=h/J_f$ *in* $B_{n,m,r}\left(W^0\right)$ *the following formula holds:*

$$[\mathrm{Tr}\,h/J_f]\left(W\right)=\int_{\Gamma_{f,r}}\frac{h\left(Z\right)d\sigma\left(Z\right)}{\det^{mn}\left(I^{(m)}-\left\langle f\left(Z\right),W\right\rangle\right)}. \tag{3}$$

Proof. $d\sigma$ is a normalized Lebesgue measure on $\Gamma_{f,r}$; therefore [6. P. 153],

$$d\sigma\left(f\left(Z\right)\right)=J_f\,d\sigma\left(Z\right).$$

By formula (2) for $W\in B_{m,n,r}\left(W^0\right)$ we have

$$\left[\mathrm{Tr}\,h/J_f\right]\left(W\right)=\sum_{\nu}h/J_f\left(Z^{(\nu)}\left(W\right)\right)=$$

$$=\sum_{\nu}\int_{\Gamma_{Z^{(\nu)}(W),\delta}}\frac{h/J_f\,d\sigma\left(f\left(Z\right)\right)}{\det^{mn}\left(I^{(m)}-\left\langle f\left(Z\right),W\right\rangle\right)}=\int_{\Gamma_{f,r}}\frac{h\,d\sigma\left(Z\right)}{\det^{mn}\left(I^{(m)}-\left\langle f\left(Z\right),W\right\rangle\right)}.$$

The proof is complete.

If the equation $f\left(Z\right)-W=0$ has only one root with multiplicity of 1, we can rewrite the formula (4) as

$$\frac{h\left(W\right)}{J_f\left(W\right)}=\int_{\Gamma_{f,r}}\frac{h\left(Z\right)d\,\sigma\left(Z\right)}{\det^{mn}\left(I^{(m)}-\left\langle f\left(Z\right),W\right\rangle\right)}. \tag{4}$$

Corollary 1 allows us to obtain an analogue of Bishop's formula in the matrix polyhedron $\Theta_{f,r}=\left\{Z\in D:r^2I^{(m)}-\left\langle f\left(Z\right),f\left(Z\right)\right\rangle>0,r>0\right\}$ for the meromorphic function h/J_f.

Теорема 4. *If* $h\left(Z\right)\in H^1\left(\Theta_{f,r}\right)$, $Z\in\Theta_{f,r}$ *and* $J_f\left(Z\right)\neq 0$ *at this point, then the following integral representation holds for the holomorphic function* $\frac{h}{J_f}$:

$$\frac{h\left(Z\right)}{J_f\left(Z\right)}=\int_{\Gamma_{f,r}}\frac{h\left(X\right)\Omega\left(Z,X\right)d\sigma\left(X\right)}{\Omega\left(Z,Z\right)\det^{mn}\left(I-\left\langle f\left(Z\right),f\left(X\right)\right\rangle\right)}. \tag{5}$$

Proof. From the definition of trace it follows that, when $W=f(X)$, the integral in formula (3) is equal to the sum of the values of the function $\frac{h}{J_f}$ at points $Z=X$, and of the values of this function in $X^{\nu}(f(X)),\ \nu=2,...,\mu$ at points $W=f(X)$. Consider a weight function $\Omega(Z,X)\not\equiv 0$ having the following properties: for any fixed *Z*, form

$$\Theta_{f,r}=\left\{Z\in D: r^2I^{(m)}-\langle f(Z),f(Z)\rangle>0, r>0\right\}\Subset D,$$

the function $\Omega(Z,X)$ is equal to zero at all points $Z=X^{(\nu)}$, except for $Z=X$. This function indeed exists. Assume W^0 – non-critical value of the mapping *f*, and $g(Z)$ – linear function i.e., $g\left(X^{\nu}\left(W^0\right)\right)$ – are various. Then we can take the function

$$\Omega(Z,X)=\prod_{\nu=2}^{\mu}\left[g(Z)-g\left(X^{(\nu)}\right)\right]=\left(g(Z)-g\left(X^{(2)}\right)\right)\cdot\ldots\cdot\left(g(Z)-g\left(X^{(\mu)}\right)\right) \quad (6)$$

where the numbering is taken as $X^{(\nu)}=X^{(\nu)}(Z)$, with the convention that $X^{(1)}(Z)=Z$. Thus, the product in (6) is a polynomial of $g(Z)$, with coefficients holomorphically dependent on *X*. As a result, we have

$$\Omega(Z,X)=\sum_{k=1}^{\mu-1}c_k(X)g^k(Z),$$

where $c_k(X)$ are holomorphic functions in $\overline{\Theta}_{f,r}$. By construction, we have $\Omega\left(Z,X^{(\nu)}(Z)\right)=0$, for points $X^{(\nu)}(Z)\neq Z$.

By corollary and the constructed weight function $\Omega(Z,X)$, we obtain Bishop's formula in $\Theta_{f,r}$.

Indeed,

$$\sum_{\nu}\frac{h\left(Z^{(\nu)}(X)\right)\Omega\left(Z,Z^{(\nu)}(X)\right)}{J_f\left(Z^{(\nu)}(X)\right)}=$$

$$=\frac{h(Z)\,\Omega(Z,Z)}{J_f(Z)}+\frac{h\left(Z^{(2)}(X)\right)\Omega\left(Z,Z^{(2)}(X)\right)}{J_f\left(Z^{(2)}(X)\right)}+\ldots=$$

$$=\frac{h(Z)\,\Omega(Z,Z)}{J_f(Z)}=\int_{\Gamma_{f,r}}\frac{h(X)\Omega(Z,X)d\sigma(X)}{\det^{mn}\left(I^{(m)}-\langle f(Z),f(X)\rangle\right)}.$$

The proof is complete.

Corollary 2. When $m=n=1$ and $f(z)=z$, the formula (5) yields the Cauchy formula for a circle with a radius *r* in the complex plane $\mathbb{C}$.

Proof. When $m=n=1$ and $f(z)=z$, the Jacobian of the mapping $f(z)$ is $J_f=1$, and in this case by formula (4) the formula (5) yield the Cauchy formula for a circle.

The proof is complete.

When $n = 1$, from the generalized matrix ball $\Theta_{f,r}$ we get the matrix polyhedron in the space $\mathbb{C}[m \times m]$. In this field, B.A. Shaimkulov derived the Bishop integral formula with a different kernel ([3. P. 227]).

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***Information about the authors*:**
Makhkamov Erkin M. (Doctor of Philosophy (PhD) in Physical and Mathematical Sciences, Associate Professor of the Department of Digital Technologies, Non-Governmental Organization of Higher Education Perfect University; Associate Professor of the Department of Mathematical Analysis, National University of Uzbekistan, Uzbekistan, Tashkent). E-mail: erkin_mahkamov83@mail.ru
Bozorov Jurabek T. (Doctor of Philosophy (PhD) in Physical and Mathematical Sciences, Associate Professor of the Department of Mathematical Analysis, Termez State University, Uzbekistan, Termez). E-mail: jorabek.bozorov.89@mail.ru

***Сведения об авторах*:**
Махкамов Эркин Мусурманович – доктор философии (PhD) по физико-математическим наукам, доцент кафедры цифровых технологий Негосударственной организации высшего образования Perfect University; доцент кафедры математического анализа Национального университета Узбекистана им. Мирзо Улугбека (Ташкент, Узбекистан). E-mail: erkin_mahkamov83@mail.ru
Бозоров Жўрабек Тоғаймуротович – доктор философии (PhD) по физико-математическим наукам, заведующий кафедрой математического анализа Термезского государственного университета (Термез, Узбекистан). E-mail: jorabek.bozorov.89@mail.ru

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