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Virtual braids and cluster algebras

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Abstract. In 2015, Hikami and Inoue constructed a representation of the braid group B_n in terms of cluster algebra associated with the decomposition of the complement of the corresponding knot into ideal hyperbolic tetrahedra. This representation leads to the calculation of the hyperbolic volume of the complement of the knot that is the closure of the corresponding braid. In this paper, based on the Hikami–Inoue representation discussed above, we construct a representation for the virtual braid group VB_n. We show that the so-called "forbidden relations" do not hold in the image of the resulting representation. In addition, based on the developed method, we construct representations for the flat braid group FB_n and the flat virtual braid group FVB_n.

Keywords: braid group, virtual braid group, cluster algebra

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Научная статья

Виртуальные косы и кластерные алгебры

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Аннотация. В 2015 г. Хиками и Иноуе построили представление группы кос B_n в терминах кластерной алгебры, связанной с разбиением дополнения соответствующего узла на идеальные гиперболические тетраэдры. Это представление приводит к вычислению гиперболического объема дополнения к узлу, являющемуся замыканием соответствующей косы. В данной работе, основываясь на обсуждаемом выше представлении Хиками–Иноуе, мы строим представление для группы виртуальных кос VB_n. Мы показываем, что в образе полученного представления не будут выполняться так называемые «запрещенные соотношения», которые, как известно, в группе VB_n не выполняются. Кроме того, на основе разработанного метода мы строим представления для группы плоских кос FB_n и группы плоских виртуальных кос FVB_n.

Ключевые слова: группа кос, группа виртуальных кос, кластерные алгебры

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1. Introduction

Let us start with recalling braid groups and related groups. For $n \geq 2$, the braid group B_n is defined as a group with generators $\sigma_1, \ldots, \sigma_{n-1}$ and the following defining relations [1]:

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \qquad i = 1, 2, \ldots, n-2, \tag{1}$$

$$\sigma_i \sigma_j = \sigma_j \sigma_i, \qquad |i - j| \geq 2. \tag{2}$$

A geometric interpretation of B_n is well known, it is isomorphic to a group of geometric braids on n strings, and a mapping class group of an n-punctured disc [2]. By adding the relations

$$\sigma_i^2 = 1, \qquad i = 1, 2, \ldots, n-1. \tag{3}$$

we get the flat braid group FB_n on n strings.

The virtual braid group VB_n on n strings is the group with two families of generators, classical and virtual, denoted by $\sigma_1, \ldots, \sigma_{n-1}$ and $\rho_1, \ldots, \rho_{n-1}$, with the following defining relations: (1) and (2) for classical generators; (4), (5) and (6) for virtual generators,

$$\rho_i \rho_{i+1} \rho_i = \rho_{i+1} \rho_i \rho_{i+1}, \qquad i = 1, 2, \ldots, n-2, \tag{4}$$

$$\rho_i \rho_j = \rho_j \rho_i, \qquad |i - j| \geq 2, \tag{5}$$

$$\rho_i^2 = 1, \qquad i = 1, 2, \ldots, n-1, \tag{6}$$

and mixed relations (7) and (8) for classical and virtual generators both.

$$\sigma_i \rho_j = \rho_j \sigma_i, \qquad |i - j| \geq 2, \tag{7}$$

$$\rho_i \rho_{i+1} \sigma_i = \sigma_{i+1} \rho_i \rho_{i+1}, \qquad i = 1, 2, \ldots, n-2. \tag{8}$$

It was observed in [3] that relations (9) и (10)

$$\rho_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \rho_{i+1}, \tag{9}$$

$$\rho_{i+1} \sigma_i \sigma_{i+1} = \sigma_i \sigma_{i+1} \rho_i \tag{10}$$

do not hold in VB_n, so these relations are called forbidden relations. By adding relation (3) to VB_n, we obtain the flat virtual braid group FVB_n on n strings.

The relation described above between braid groups and virtual braid groups admits to construct representations of VB_n by extending known representations of B_n by corresponding to ρ_i suitable involutions. In particular, Bardakov, Vesnin, and Wiest [4] constructed a representation of VB_n by extending Dynnikov representation [5], and demonstrated that the representation from [4] is faithful for $n = 2$ and distinguish virtual braids on three strings good enough. Gotin [6] constructed a representation of VB_n by extending a representation of B_n through rook algebras given by Bigelow, Ramos, and Yi [7].

In the present note we construct a representation of VB_n by extending a representation of B_n given by Hikami and Inoue in [8] in terms of a cluster algebra (Theorem 3.1.). It was demonstrated in [9] that the representation from [8] allows to compute the volume of a hyperbolic knot which is the closer of a braid. Further, we also construct representations for a flat braid group and virtual flat braid groups (Theorems 5.1. and 6.1.).

2. Cluster mutations

Let V be a complex vector space. An automorphism R of the tensor product $V \otimes V$ is said to be an R-operator if it satisfies the following Yang–Baxter equation

$$\left(R\otimes \mathrm{Id}\right)\left(\mathrm{Id}\otimes R\right)\left(R\otimes \mathrm{Id}\right)=\left(\mathrm{Id}\otimes R\right)\left(R\otimes \mathrm{Id}\right)\left(\mathrm{Id}\otimes R\right),$$

where Id is the identity operator Id: $V \to V$.

Let us recall the construction of the R-operator from [8]. Denote by $\mathbb{F}_N$ the field of rational functions over $\mathbb{C}$ of N algebraically independent variables $\mathbf{x} = \left(x_1, \ldots, x_N\right)$. A cluster seed is a pair $\left(\mathbf{x}, \mathbf{B}\right)$, where

- $\mathbf{x} = \left(x_1, \ldots, x_N\right)$ is an ordered set of N algebraically independent variables,
- $\mathbf{B} = \left(b_{ij}\right)$ is an antisymmetric $N \times N$-matrix of integers.

For any $k = 1, \ldots, N$ define a mutation μ_k of a seed $\left(\mathbf{x}, \mathbf{B}\right)$ in direction k as follows

$$\mu_k\left(\mathbf{x}, \mathbf{B}\right) = (\tilde{\mathbf{x}}, \tilde{\mathbf{B}})$$

where $\tilde{\mathbf{x}} = (\tilde{x}_1, ..., \tilde{x}_N)$ is defined by the rule

$$\tilde{x}_i = \begin{cases} x_i, & \text{if } i \neq k, \\ \dfrac{1}{x_k}\left(\prod\limits_{j:b_{jk}>0} x_j^{b_{jk}} + \prod\limits_{j:b_{jk}<0} x_j^{-b_{jk}}\right), & \text{if } i = k, \end{cases} \tag{11}$$

and matrix $\tilde{\mathbf{B}} = (\tilde{b}_{ij})$ is calculated by the formula

$$\tilde{b}_{ij} = \begin{cases} -b_{ij}, & \text{if } i = k \text{ or } j = k, \\ b_{ij} + \dfrac{|b_{ik}| b_{kj} + b_{ik} |b_{kj}|}{2}, & \text{otherwise.} \end{cases} \tag{12}$$

A pair $(\tilde{\mathbf{x}}, \tilde{\mathbf{B}})$ is a cluster seed again.

Using cluster variables **x** we define cluster variables $\mathbf{y} = (y_1, \dots, y_N)$ by setting

$$y_j = \prod_{k=1}^{N} x_k^{b_{kj}}. \tag{13}$$

Mutation μ_k induces a mutation of a pair $(\mathbf{y}, \mathbf{B}), \mu_k(\mathbf{y}, \mathbf{B}) = (\tilde{\mathbf{y}}, \tilde{\mathbf{B}})$, where $\tilde{\mathbf{B}}$ is given by formula (12) and $\tilde{\mathbf{y}} = (y_1, ..., y_N)$ is given by the following formulas:

$$\tilde{y}_i = \begin{cases} y_k^{-1}, & \text{if } i = k, \\ y_i\left(1 + y_k^{-1}\right)^{-b_{ki}}, & \text{if } i \neq k \text{ and } b_{ki} \geq 0, \\ y_i\left(1 + y_k\right)^{-b_{ki}}, & \text{if } i \neq k \text{ and } b_{ki} \leq 0. \end{cases} \tag{14}$$

In [8], a matrix B was taken equal to the adjacency matrix of a quiver (oriented graph) Γ presented in figure 1. Graph Γ has $N = 3n + 1$ vertices. Namely B is $(3n+1)\times(3n+1)$ - matrix with enters determined by the quiver Γ:

$$b_{ij} = \begin{cases} 1, & \text{if there is an edge going from vertex } i \text{ to vertex } j, \\ -1, & \text{if there is an edge from vertex } j \text{ to vertex } i, \\ 0, & \text{if vertices } i \text{ and } j \text{ are not adjacent.} \end{cases}$$

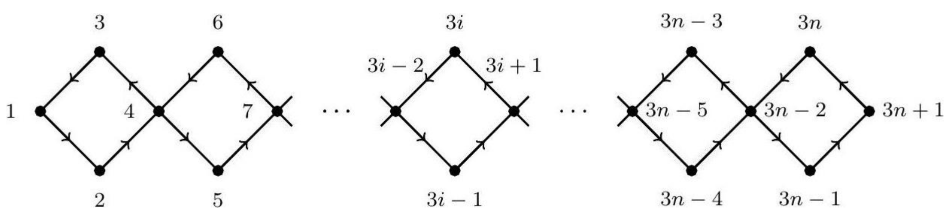

Fig. 1. Quiver Γ with $3n + 1$ vertices

In particular, if $n = 2$, then matrix **B** is of the form

$$B = \begin{pmatrix} 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 \end{pmatrix}. \tag{15}$$

Let us denote by $\Phi : \mathbb{F}_{3n+1} \to \mathbb{F}_{3n+1}$, $n \geq 2$, the operator defined in [8, Formula 2-13] as a composition of mutations. If $n = 2$ then we get $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5, x_6, x_7)$ and Φ is of the form

$$\Phi(\mathbf{x})=\begin{pmatrix}\Phi_1(\mathbf{x})\\ \Phi_2(\mathbf{x})\\ \Phi_3(\mathbf{x})\\ \Phi_4(\mathbf{x})\\ \Phi_5(\mathbf{x})\\ \Phi_6(\mathbf{x})\\ \Phi_7(\mathbf{x})\end{pmatrix}^T=\begin{pmatrix} x_1\\ x_5\\ \dfrac{x_1x_3x_5+x_3x_4x_5+x_1x_2x_6}{x_2x_4}\\ \dfrac{x_1x_3x_4x_5+x_3x_4^2x_5+x_1x_3x_5x_7+x_3x_4x_5x_7+x_1x_2x_6x_7}{x_2x_4x_6}\\ \dfrac{x_1x_3x_5+x_3x_4x_5+x_1x_2x_6}{x_4x_6}\\ x_3\\ x_7\end{pmatrix}^T.$$

We denote by $\Psi:\mathbb{F}_{3n+1}\to\mathbb{F}_{3n+1}, n\geq 2$, the operator inverse to Φ. If $n = 2$ then

$$\Psi(\mathbf{x})=\begin{pmatrix}\Psi_1(\mathbf{x})\\ \Psi_2(\mathbf{x})\\ \Psi_3(\mathbf{x})\\ \Psi_4(\mathbf{x})\\ \Psi_5(\mathbf{x})\\ \Psi_6(\mathbf{x})\\ \Psi_7(\mathbf{x})\end{pmatrix}^T=\begin{pmatrix} x_1\\ \dfrac{x_1x_3x_5+x_1x_2x_6+x_2x_4x_6}{x_3x_4}\\ x_6\\ \dfrac{x_1x_2x_4x_6+x_2x_4^2x_6+x_1x_3x_5x_7+x_1x_2x_6x_7+x_2x_4x_6x_7}{x_3x_4x_5}\\ x_2\\ \dfrac{x_2x_4x_6+x_3x_5x_7+x_2x_6x_7}{x_4x_5}\\ x_7\end{pmatrix}^T.$$

Following [8, Formula 2-13] we go from **x**-variables to **y**-variables. If $n = 2$, then $\mathbf{y}=(y_1,y_2,y_3,y_4,y_5,y_6,y_7)$ and *R*-operator Φ will take a form φ, where

$$\varphi(\mathbf{y})=\begin{pmatrix}\varphi_1(\mathbf{y})\\ \varphi_2(\mathbf{y})\\ \varphi_3(\mathbf{y})\\ \varphi_4(\mathbf{y})\\ \varphi_5(\mathbf{y})\\ \varphi_6(\mathbf{y})\\ \varphi_7(\mathbf{y})\end{pmatrix}^T=\begin{pmatrix} y_1(1+y_2+y_2y_4)\\ \dfrac{y_2y_4y_5y_6}{1+y_2+y_6+y_2y_6+y_2y_4y_6}\\ \dfrac{1+y_2+y_4+y_2y_6+y_2y_4y_6}{y_2y_4}\\ \dfrac{y_4}{(1+y_2+y_2y_4)(1+y_6+y_4y_6)}\\ \dfrac{1+y_2+y_6+y_2y_6+y_2y_4y_6}{y_4y_6}\\ \dfrac{y_2y_3y_4y_6}{1+y_2+y_6+y_2y_6+y_2y_4y_6}\\ (1+y_6+y_4y_6)y_7\end{pmatrix}^T, \tag{16}$$

as well as Ψ will takes a form ψ, where

$$\psi(\mathbf{y}) = \begin{pmatrix} \psi_1(\mathbf{y}) \\ \psi_2(\mathbf{y}) \\ \psi_3(\mathbf{y}) \\ \psi_4(\mathbf{y}) \\ \psi_5(\mathbf{y}) \\ \psi_6(\mathbf{y}) \\ \psi_7(\mathbf{y}) \end{pmatrix}^T = \begin{pmatrix} \dfrac{y_1 y_3 y_4}{1+y_4+y_3y_4} \\ \dfrac{y_5}{1+y_4+y_3y_4+y_4y_5+y_3y_4y_5} \\ (1+y_4+y_3y_4+y_4y_5+y_3y_4y_5)y_6 \\ \dfrac{(1+y_4+y_3y_4)(1+y_4+y_4y_5)}{y_3y_4y_5} \\ y_2(1+y_4+y_3y_4+y_4y_5+y_3y_4y_5) \\ \dfrac{y_3}{1+y_4+y_3y_4+y_4y_5+y_3y_4y_5} \\ \dfrac{y_4y_5y_7}{1+y_4+y_4y_5}. \end{pmatrix}^T . \tag{17}$$

The following property easily follows from the above formulae.

Lemma 2.1. By setting $y_1 = y_4 = y_7 = -1$ in formulae (16) and (17), we get

$$\varphi_1 = \varphi_4 = \varphi_7 = -1 \text{ and } \psi_1 = \psi_4 = \psi_7 = -1$$

3. Virtual braid groups

For a vector $\mathbf{z} = (z_1, z_2, z_3, z_4) = (y_2, y_3, y_5, y_6)$ of length four, we define two operators

$$S\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix}^T = \begin{pmatrix} -\dfrac{z_1z_3z_4}{1+z_1+z_4} \\ -\dfrac{1+z_1+z_4}{z_1} \\ -\dfrac{1+z_1+z_4}{z_4} \\ -\dfrac{z_1z_2z_4}{1+z_1+z_4} \end{pmatrix}^T, \quad S^{-1}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix}^T = \begin{pmatrix} -\dfrac{z_3}{z_2+z_3+z_2z_3} \\ -(z_2+z_3+z_2z_3)z_4 \\ -z_1(z_2+z_3+z_2z_3) \\ -\dfrac{z_2}{z_2+z_3+z_2z_3} \end{pmatrix}^T \tag{18}$$

and an involution

$$T(z_1, z_2, z_3, z_4) = (z_3, z_4, z_1, z_2). \tag{19}$$

Now for $n \geq 2$ we define operators $S_i^{\pm 1}$ and $T_i, i = 1, \ldots, n-1$, which act on vector $\mathbf{z} = (z_1, z_2, \ldots, z_{2n})$ of length $2n$ by the following rule. Operators $S_i^{\pm 1}$ and T_i act on 4-tuple $(z_{2i-1}, z_{2i}, z_{2i+1}, z_{2i+2})$ in the same way as operators $S^{\pm 1}$ and T act on 4-tuple (z_1, z_2, z_3, z_4), and do not change other components of $\mathbf{z}$:

$$S_i^{\pm 1} = I^{2i-2} \otimes S^{\pm 1} \otimes I^{2n-2i-2}, \; T_i = I^{2i-2} \otimes T \otimes I^{2n-2i-2} .$$

For $n \geq 2$, we denote by Θ_n the group generated by $S_i, T_i, i = 1, \ldots, n-1$, with composition as a group operation. Define a map $F : VB_n \to \Theta_n$ by setting

$$F(\sigma_i) = S_i, \; F(\rho_i) = T_i, \; i = 1, \ldots, n-1. \tag{20}$$

Lemma 3.1. Let w be a word in VB_n. Then for a vector of algebraically independent variables $\mathbf{z}=(z_1,z_2,\ldots,z_{2n})$ in the image of $F(w)(\mathbf{z})$ no coordinate turns into zero or infinity.

Proof. Consider $2n$-tuple $\mathbf{z}'=(-1,-1,\ldots,-1)$. It is easy to see from (18) and (19) that $S_i^{\pm 1}(\mathbf{z}')=\mathbf{z}'$ and $T_i(\mathbf{z}')=\mathbf{z}'$ for each i. Hence $F(w)(\mathbf{z}')=\mathbf{z}'=(-1,-1,\ldots,-1)$. Therefore, in the image of $F(w)(\mathbf{z})$ no coordinate can turn into zero or infinity because for $z_i=-1, i=1,\ldots,2n$, all coordinates of the image will be equal to -1.

Theorem 3.1. Map $F: VB_n \to \Theta_n$, $n\geq 2$, defined by (20) is a homomorphism.

Proof. Let us check that the operators S_i and $T_i, i=1,\ldots,n-1$, act on $\mathbf{z}$ in such a way that the following identities hold:

(1) $S_iS_{i+1}S_i = S_{i+1}S_iS_{i+1}$, where $i=1,2,\ldots,n-2$.

(2) $S_iS_j = S_jS_i$, where $|i-j|\geq 2$.

(3) $T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$, where $i=1,2,\ldots,n-2$.

(4) $T_iT_j = T_jT_i$, where $|i-j|\geq 2$.

(5) $T_i^2=1$, where $i=1,2,\ldots,n-1$.

(6) $T_iT_{i+1}S_i = S_{i+1}T_iT_{i+1}$, where $i=1,2,\ldots,n-2$.

Obviously, it is enough to consider the case $i=1$. Identities (1) and (2) are particular cases of [8, Theorem 2.3]. Nevertheless, we present a straightforward proof of (1) for the reader's convenience. Let $\mathbf{z}=(z_1,z_2,z_3,z_4,z_5,z_6)$. Consider the left-side part of (1)

$$S_1S_2S_1(\mathbf{z}) = S_1S_2S_1(z_1,z_2,z_3,z_4,z_5,z_6) =$$

$$=\left(\frac{z_1z_3z_5z_6}{1-z_1z_3+z_6},\frac{1-z_1z_3+z_6}{z_1z_3},\frac{z_4(1-z_1z_3+z_6)}{1+z_1-z_4z_6},\frac{z_3(1+z_1-z_4z_6)}{1-z_1z_3+z_6},\frac{1+z_1-z_4z_6}{z_4z_6},\frac{z_1z_2z_4z_6}{1+z_1-z_4z_6}\right).$$

The right-side part of (1) is equal

$$S_2S_1S_2(\mathbf{z}) = S_2S_1S_2(z_1,z_2,z_3,z_4,z_5,z_6) =$$

$$=\left(\frac{z_1z_3z_5z_6}{1-z_1z_3+z_6},\frac{1-z_1z_3+z_6}{z_1z_3},\frac{z_4(1-z_1z_3+z_6)}{1+z_1-z_4z_6},\frac{z_3(1+z_1-z_4z_6)}{1-z_1z_3+z_6},\frac{1+z_1-z_4z_6}{z_4z_6},\frac{z_1z_2z_4z_6}{1+z_1-z_4z_6}\right).$$

Thus, the identity (1) holds.

Let us demonstrate that identity (6) holds. Indeed, on the one hand,

$$T_1T_2S_1(\mathbf{z}) = T_1T_2S_1(z_1,z_2,z_3,z_4,z_5,z_6) = T_1T_2(S(z_1),S(z_2),S(z_3),S(z_4),z_5,z_6) =$$

$$=T_1(S(z_1),S(z_2),z_5,z_7,S(z_3),S(z_4)) = (z_5,z_6,S(z_1),S(z_2),S(z_3),S(z_4)).$$

and, on the other hand,

$$S_2T_1T_2(\mathbf{z}) = S_2T_1T_2(z_1,z_2,z_3,z_4,z_5,z_6) = S_2T_1(z_1,z_2,z_5,z_6,z_3,z_4) =$$

$$=S_2(z_5,z_6,z_1,z_2,z_3,z_4) = (z_5,z_6,S(z_1),S(z_2),S(z_3),S(z_4)).$$

Remaining identities (2), (3), (4), and (5) hold obviously.

Theorem 3.1. allows to distinguish elements of the virtual braid group *VBn* by computing their images which are vectors of lengths $2n$.

Example 3.1. It is known [10] that a generalized Burau representation does not distinguish a braid $w_2 = \left(\sigma_1^2\rho_1\sigma_1^{-1}\rho_1\sigma_1^{-1}\rho_1\right)^2 \in VB_2$ from a trivial braid. By acting $F\left(w_2\right)$ on the vector $\left(1,2,2,1\right)$ we get

$$F\left(w_2\right)\left(1,2,2,1\right) = \left(-\frac{44}{19}, -\frac{19}{22}, -\frac{19}{22}, -\frac{44}{19}\right) \neq \left(1,2,2,1\right).$$

Therefore, the homomorphism F distinguishes w_2 from a trivial braid.

Example 3.2. Consider

$$w_3 = \sigma_1\rho_2\sigma_1\sigma_2^{-1}\sigma_1\sigma_2\sigma_1^{-1}\rho_1\sigma_2\rho_1\sigma_1\rho_2\sigma_1^{-1}\rho_2\sigma_2^{-1}\sigma_1^{-1}\sigma_2\sigma_1^{-1}\rho_2\sigma_1^{-1} \in VB_3.$$

It is known that a representation from [4] does not distinguish w_3 from a trivial braid. By acting $F\left(w_3\right)$ on $\left(1,2,2,1,1,2\right)$ we get

$$F\left(w_3\right)\left(1,2,2,1,1,2\right) = \left(\frac{2488285076682521504}{1290542656863845663}, \frac{1290542656863845663}{1244142538341260752},\right.$$

$$\left.\frac{1290542656863845663}{563568067426145589}, \frac{1127136134852291178}{1290542656863845663}, \frac{574648281}{1268603408}, \frac{2537206816}{574648281}\right) \neq$$

$$\neq \left(1,2,2,1,1,2\right).$$

Therefore, the homomorphism F distinguishes w_3 from a trivial braid.

4. Forbidden relations

In this section we demonstrate that the forbidden relations do not hold in the group Θ_n.

Lemma 4.1. Let $\mathbf{z} = \left(z_1, z_2, \ldots, z_{2n-1}, z_{2n}\right)$ and $S_i, S_{i+1}, T_i, T_{i+1} \in \Theta_n$.

(1) The forbidden relation

$$T_i S_{i+1} S_i(\mathbf{z}) = S_{i+1} S_i T_{i+1}(\mathbf{z}) \tag{21}$$

does not hold if and only if the vector **z** is such that $z_j \neq -1$ for $j = 2i-1, 2i+2, 2i+4$.

(2) The forbidden relation

$$T_{i+1} S_i S_{i+1}(\mathbf{z}) = S_i S_{i+1} T_{i+1}(\mathbf{z}) \tag{22}$$

does not hold if and only if the vector **z** is such that $z_j \neq -1$ for $j = 2i-1, 2i+1, 2i+4$.

Proof. (a) Without loss of generality, we can assume i = 1. The left-hand side of (21) is

$$T_1 S_2 S_1(\mathbf{z}) = T_1 S_2 S_1(z_1, z_2, z_3, z_4, z_5, z_6) =$$

$$= \left(-\frac{(1+z_1+z_4)z_5 z_6}{1+z_1-z_4 z_6}, -\frac{1+z_1-z_4 z_6}{1+z_1+z_4}, -\frac{z_1 z_3 z_4}{1+z_1+z_4}, -\frac{1+z_1+z_4}{z_1}, \frac{1+z_1-z_4 z_6}{z_4 z_6}, \frac{z_1 z_2 z_4 z_6}{1+z_1-z_4 z_6}\right)$$

and the right-hand side is equal

$$S_2 S_1 T_2(\mathbf{z}) = S_2 S_1 T_2(z_1, z_2, z_3, z_4, z_5, z_6) =$$

$$= \left(-\frac{z_1 z_5 z_6}{1+z_1+z_6}, -\frac{1+z_1+z_6}{z_1}, -\frac{(1+z_1+z_6)z_3 z_4}{1+z_1-z_4 z_6}, -\frac{1+z_1-z_6 z_4}{1+z_1+z_6}, \frac{1+z_1-z_6 z_4}{z_6 z_4}, \frac{z_1 z_2 z_6 z_4}{1+z_1-z_6 z_4}\right).$$

Here we used formulae for $S_2 S_1\left(\mathbf{z}\right)$ from Theorem 3.1. The fifth and sixth coordinates are equal. Comparison of the third and fourth coordinates leads to the equation

$$z_1\left(1+z_1-z_4z_6\right)=\left(1+z_1+z_4\right)\left(1+z_1+z_6\right), \tag{23}$$

which is equivalent to

$$\left(z_1+1\right)\left(z_4+1\right)\left(z_6+1\right)=0. \tag{24}$$

Therefore, to obtain the relation (a), the necessary condition is that at least one of numbers z_1, z_4, or z_6 is equal to −1. But if at least one of the numbers z_1, z_4, or z_6 is equal to −1, then the left- and right-hand sides of (a) coincide. Indeed, if $z_1=-1$, then

$$T_1S_2S_1\left(-1,z_2,z_3,z_4,z_5.z_6\right)=S_2S_1T_2\left(-1,z_2,z_3,z_4,z_5,z_6\right)=\left(z_5,z_6,z_3,z_4,-1,z_2\right).$$

Similarly, if $z_4=-1$, then

$$T_1S_2S_1(z_1,z_2,z_3,-1,z_5,z_6)=S_2S_1T_2(z_1,z_2,z_3,-1,z_5,z_6)=$$
$$=\left(-\frac{z_1z_5z_6}{1+z_1+z_6},-\frac{1+z_1+z_6}{z_1},z_3,-1,-\frac{1+z_2+z_6}{z_6},-\frac{z_1z_2z_6}{1+z_1+z_6}\right),$$

and if $z_6=-1$, then

$$T_1S_2S_1(z_1,z_2,z_3,z_4,z_5,-1)=S_2S_1T_1(z_1,z_2,z_3,z_4,z_5,-1)=$$
$$=\left(z_5,-1,-\frac{z_1z_3z_4}{1+z_1+z_4},-\frac{1+z_2+z_4}{z_1},-\frac{1+z_1+z_4}{z_4},-\frac{z_1z_2z_4}{1+z_1+z_4}\right).$$

Therefore, the above necessary condition is also sufficient.

(b) The left-hand side of relation (22) is equal to

$$T_2S_1S_2(\mathbf{z})=T_2S_1S_2(z_1,z_2,z_3,z_4,z_5,z_6)=$$
$$=\left(\frac{z_1z_3z_5z_6}{1-z_1z_3+z_6},\frac{1-z_1z_3+z_6}{z_1z_3},-\frac{1+z_3+z_6}{z_6},-\frac{z_3z_4z_6}{1+z_3+z_6},-\frac{1-z_1z_3+z_6}{1+z_3+z_6},-\frac{z_1z_2(1+z_3+z_6)}{1-z_1z_3+z_6}\right)$$

and the right-hand side is equal to

$$S_1S_2T_1(\mathbf{z})=S_1S_2T_1(z_1,z_2,z_3,z_4,z_5,z_6)=$$
$$=\left(\frac{z_3z_1z_5z_6}{1-z_3z_1+z_6},\frac{1-z_3z_1+z_6}{z_3z_1},-\frac{1-z_1z_3+z_6}{1+z_1+z_6},-\frac{z_3z_4(1+z_1+z_6)}{1-z_3z_1+z_6},-\frac{1+z_1+z_6}{z_6},-\frac{z_1z_2z_6}{1+z_1+z_6}\right).$$

Here we used formulae for $S_1S_2\left(\mathbf{z}\right)$ from Theorem 3.1. By comparing the third and fourth coordinates, we get the equation

$$\left(1+z_3+z_6\right)\left(1+z_1+z_6\right)=z_6\left(1-z_1z_3+z_6\right), \tag{25}$$

which is equivalent to

$$\left(z_1+1\right)\left(z_3+1\right)\left(z_6+1\right)=0. \tag{26}$$

Therefore, to obtain (b), the necessary condition is that at least one of z_1, z_3, or z_6 is equal to −1. But if at least one of these numbers is equal to −1, then left- and right-hand sides of (22) coincide. Indeed, if $z_1=-1$, then

$$T_2S_1S_2(-1,z_2,z_3,z_4,z_5,z_6)=S_2S_1T_2(-1,z_2,z_3,z_4,z_5,z_6)=$$
$$=\left(-\frac{z_3z_5z_6}{1+z_3+z_6},-\frac{1+z_3+z_6}{z_3},-\frac{1+z_3+z_6}{z_6},-\frac{z_3z_4z_6}{1+z_3+z_6},-1,z_2\right).$$

Similarly, if $z_3=-1$, then

$$T_2S_1S_2(z_1,z_2,-1,z_4,z_5,z_6)=S_2S_1T_2(z_1,z_2,-1,z_4,z_5,z_6)=$$
$$=\left(-\frac{z_1z_5z_6}{1+z_3+z_6},-\frac{1+z_1+z_6}{z_1},-1,z_4,-\frac{1+z_1+z_6}{z_6},-\frac{z_1z_2z_6}{1+z_1+z_6}\right)$$

and if $z_6=-1$, then

$$T_2S_1S_2\left(z_1,z_2,z_3,z_4,z_5,-1\right)=S_2S_1T_2\left(z_1,z_2,z_3,z_4,z_5,-1\right)=\left(z_5,-1,z_3,z_4,z_1,z_2\right).$$

Therefore, the above necessary condition is also sufficient.

The obvious consequence of this lemma is the following theorem, which concludes the section.

Theorem 4.1. Let $S_i,S_{i+1},T_i,T_{i+1}\in\Theta_n$.

(a) Operators $T_iS_{i+1}S_i$ and $S_{i+1}S_iT_{i+1}$ are different.

(b) Operators $T_{i+1}S_iS_{i+1}$ and $S_iS_{i+1}T_{i+1}$ are different.

So, the forbidden relations do not hold in Θ_n.

5. Flat braid groups

Let us consider vector $\mathbf{z}$ of the form $\left(z_1,\frac{1}{z_1},z_3,\frac{1}{z_3}\right)$. Notice that

$$S\left(\mathbf{z}\right)=\left(\zeta_1,\frac{1}{\zeta_1},\zeta_3,\frac{1}{\zeta_3}\right),$$

where

$$\zeta_1=-\frac{z_1z_3}{1+z_3+z_1z_3},\ \zeta_3=-\left(1+z_3+z_1z_3\right).$$

Also notice that $S^2\left(\mathbf{z}\right)=\mathbf{z}$. These observations inspire to obtain the representation for flat braids.

Consider a vector of algebraically independent variables $\mathbf{t}=\left(t_1,t_2,\ldots,t_n\right)$. Let us define the operators $R_i, i=1,\ldots,n-1$, according to the rule

$$R_i:\begin{cases}t_i\to & -\dfrac{t_it_{i+1}}{1+t_{i+1}+t_it_{i+1}},\\ t_{i+1}\to & -\left(1+t_{i+1}+t_it_{i+1}\right).\end{cases}$$

Let F_{FB} be a map that match operators R_i with generators $\sigma_i, i=1,\ldots,n-1$, of the flat braid group T_n:

$$F_{FB}\left(\sigma_i\right)=R_i.$$

For $n\ge 2$, denote by Ω_n the group generated by operators $R_i, i=1,\ldots,n-1$, with composition as a group operation.

Lemma 5.1. Let w be a word in FB_n. Then for a vector of algebraically independent variables $\mathbf{t}=\left(t_1,t_2,\ldots,t_n\right)$ in the image of $F_{FB}\left(w\right)\left(\mathbf{t}\right)$ no coordinate turns into zero or infinity.

Proof. Consider n-tuple $\mathbf{t}' = (-1,-1,\dots,-1)$. It is easy to see that $R_i^{\pm1}(\mathbf{t}') = \mathbf{t}'$ for each i. Hence $F_{FB}(w)(\mathbf{t}') = \mathbf{t}'$. Hence, in the image of $F_{FB}(w)(\mathbf{t})$ no coordinate can turn into zero or infinity because for $t_i = -1, i = 1,\dots,n$, all coordinates of the image will be equal to −1.

Theorem 5.1. Correspondence $F_{FB}: FB_n \to \Omega_n$ is a homomorphism for any $n \ge 2$.

Proof. Let us check that for the operators $R_i, i = 1,\dots,n-1$, act on $\mathbf{t}$ in such a way that the following identities hold.

(1) $R_i^2 = 1$, where $i = 1,2,\dots,n-2$.

(2) $R_i R_{i+1} R_i = R_{i+1} R_i R_{i+1}$, where $i = 1,2,\dots,n-2$.

(3) $R_i R_j = R_j R_i$, where $|i-j| \ge 2$.

We present a proof for the case of $i = 1$, which also works for an arbitrary $i = 1,\dots,n-1$. Consider $\mathbf{t} = (t_1,t_2,t_3)$. Relation (1) is easily verified. Indeed,

$$R_1^2(\mathbf{t}) = R_1^2(t_1,t_2,t_3) = R_1\left(-\frac{t_1t_2}{1+t_1+t_1t_2}, -(1+t_2+t_1t_2), t_3\right) = ((t_1,t_2,t_3)).$$

Let us now prove identity (2). Its left-hand side is

$$R_1R_2R_1(\mathbf{t}) = R_1R_2R_1(t_1,t_2,t_3) = \left(\frac{t_1t_2t_3}{1+t_3-t_1t_2t_3}, \frac{1+t_3-t_1t_2t_3}{-1+t_2t_3+t_1t_2t_3}, -1+t_2t_3+t_1t_2t_3\right).$$

The right-hand side is

$$R_2R_1R_2(\mathbf{t}) = R_2R_1R_2(t_1,t_2,t_3) = \left(\frac{t_1t_2t_3}{1+t_3-t_1t_2t_3}, \frac{1+t_3-t_1t_2t_3}{-1+t_2t_3+t_1t_2t_3}, -1+t_2t_3+t_1t_2t_3\right).$$

Thus, identity (2) holds. The fulfillment of identity (3) is obvious.

6. Flat virtual braid groups

Consider a vector of algebraically independent variables $\mathbf{t} = (t_1,t_2,\dots,t_n)$. In addition to the operators $R_i, i = 1,\dots,n-1$ introduced in the previous section, we define the operators $V_i, i = 1,\dots,n-1$, according to the rule:

$$V_i : \begin{cases} t_i \to & t_{i+1}, \\ t_{i+1} \to & t_i. \end{cases}$$

Let F_{FVB} be a map that match operators R_i and V_i with generators σ_i and $\rho_i, i = 1,\dots,n-1$, of the virtual flat braid group T_n:

$$F_{FVB}(\sigma_i) = R_i,\ F_{FVB}(\rho_i) = V_i.$$

For $n \ge 2$, denote by Δ_n the group generated by operators $R_i, V_i, i = 1,\dots,n-1$, with composition as a group operation.

Lemma 6.1. Let w be a word in FVB_n. Then for a vector of algebraically independent variables $\mathbf{t} = (t_1,t_2,\dots,t_n)$ in the image of $F_{FVB}(w)(\mathbf{t})$ no coordinate turns into zero or infinity.

Proof. Consider n-tuple $\mathbf{t}'=(-1,-1,\dots,-1)$. It is easy to see that $R_i^{\pm1}(\mathbf{t}')=\mathbf{t}'$ and $V_i^{\pm1}(\mathbf{t}')=\mathbf{t}'$ for each i. Hence $F_{FVB}(w)(\mathbf{t}')=\mathbf{t}'$. Hence, in the image of $F_{FVB}(w)(\mathbf{t})$ no coordinate can turn into zero or infinity, because for $t_i=-1, i=1,\dots,n$, all coordinates of the image will be equal to –1.

Theorem 6.1. Correspondence $F_{FVB}:FVB_n\to\Delta_n$ is a homomorphism for any $n\geq 2$.

Proof. Let us check that the operators R_i and $V_i, i=1,\dots,n-1$, act on $\mathbf{t}$ in such a way that the following identities hold.

(1) $R_i^2=1$, where $i=1,2,\dots,n-2$;

(2) $R_iR_{i+1}R_i=R_{i+1}R_iR_{i+1}$, where $i=1,2,\dots,n-2$;

(3) $R_iR_j=R_jR_i$, where $|i-j|\geq 2$;

(4) $V_iV_{i+1}V_i=V_{i+1}V_iV_{i+1}$, where $i=1,2,\dots,n-2$;

(5) $V_iV_j=V_jV_i$, where $|i-j|\geq 2$;

(6) $V_i^2=1$, where $i=1,2,\dots,n-1$;

(7) $V_iV_{i+1}R_i=R_{i+1}V_iV_{i+1}$, where $i=1,2,\dots,n-2$.

Identities (1), (2), and (3) are proved in Theorem 5.1. The fulfillment of identities (5) and (6) is obvious. It remains to prove the relations (4) and (7). We present a proof for the case of $i=1$, which also works for an arbitrary $i=1,\dots,n-1$. Consider $\mathbf{t}=(t_1,t_2,t_3)$. Let us now prove the identity (4). Its left-hand side is

$$V_1V_2V_1(\mathbf{t})=V_1V_2V_1(t_1,t_2,t_3)=V_1V_2(t_2,t_1,t_3)=V_1(t_2,t_3,t_1)=(t_3,t_2,t_1).$$

The right-hand side is

$$V_2V_1V_2(\mathbf{t})=V_2V_1V_2(t_1,t_2,t_3)=V_2V_1(t_1,t_3,t_2)=V_2(t_3,t_1,t_2)=(t_3,t_2,t_1).$$

So, identity (4) holds. Let us now prove identity (7). Its left-hand side is

$$V_1V_2R_1(\mathbf{t})=V_1V_2R_1(t_1,t_2,t_3)=\left(t_3,-\frac{t_1t_2}{1+t_1+t_1t_2},-(1+t_2+t_1t_2)\right).$$

The right-hand side is

$$R_2V_1V_2(\mathbf{t})=R_2V_1V_2(t_1,t_2,t_3)=\left(t_3,-\frac{t_1t_2}{1+t_1+t_1t_2},-(1+t_2+t_1t_2)\right).$$

So, identity (8) holds.

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