УДК 519.7 DOI 10.17223/20710410/68/6

ROLE COLORING OF GRAPHS FROM ROOTED PRODUCTS

M. Komathi, P. Ragukumar

School of Advanced Sciences, Vellore Institute of Technology, Vellore, India

E-mail: komathirk2108@gmail.com, ragukumar2003@gmail.com

A k-role coloring is an assignment of k colors to the vertices of a graph such that if any two vertices receive the same color, then the set of colors assigned to their neighborhood will also be the same. Any graph with n vertices can have n-role coloring. Although it is easy to determine whether a graph with n vertices accepts a 1-role coloring, the challenge of k-role coloring is known to be difficult for $k \geqslant 2$. In fact, k-role coloring is known to be NP-complete for $k \geqslant 2$ on general graphs. In this paper, we determine k-role coloring of the rooted product of various graphs.

Keywords: *role coloring, role graph, rooted product, binary product.*

РОЛЕВАЯ РАСКРАСКА ГРАФОВ ИЗ КОРНЕВЫХ ПРОИЗВЕДЕНИЙ

М. Комати, П. Рагукумар

Школа передовых наук, Технологический институт, г. Веллор, Индия

k-Ролевая раскраска — это назначение k цветов вершинам графа таким образом, что если любые две вершины окрашены в один и тот же цвет, то набор цветов, назначенных их соседям, также будет одинаковым. Любой граф с n вершинами может быть раскрашен n ролями. Легко определить, допускает ли граф с n вершинами 1-ролевую раскраску, но задача k-ролевой раскраски для $k \geqslant 2$ на произвольных графах является NP-полной. В работе описана k-ролевая раскраска корневого произведения различных графов.

Ключевые слова: *ролевая раскраска, ролевой граф, корневое произведение, бинарное произведение.*

1. Introduction

All graphs considered in this paper are simple, finite, and undirected (except the role graph R; it may have loops). The graph $G = (V, E)$ has the vertex set $V(G)$ and the edge set $E(G)$. The (open) neighborhood $N_G(v) = N(v)$ of vertex v in a graph G is the set of all vertices in G that are adjacent to v, $v \in V$. The degree of a vertex v is indicated by $\deg(v)$, and the minimum and maximum degrees of vertices in G are represented by $\delta(G)$ and $\Delta(G)$, respectively. Let $\alpha(v)$ denote the color of the vertex v, and $\alpha(N(v))$ denote the color set of the neighborhood of v. For the standard graph terminology notions, we follow J. A. Bondy and U. S. R. Murty [1].

Social networks are a part of everyone's life these days. A social network is envisioned as a graph where the edges indicate the relationships between the persons and the vertices represent the individuals in order to research their behavior. In 1991, M. G. Everett and S. Borgatti [2] defined role assignment under the term "role coloring" based on graph models for social networks. A k-role coloring for any graph G is the assignment of precisely k colors to its vertices such that if any two vertices get the same color, then the set of colors assigned to their neighborhood is also the same. That is, k-role coloring is a surjective map $\alpha : V(G) \to \{1, \ldots, k\}$ such that, for all $u, v \in V(G)$, if $\alpha(u) = \alpha(v)$, then $\alpha(N(u)) = \alpha(N(v))$ [3]. Figure 1 provides an example of role coloring of a graph G.

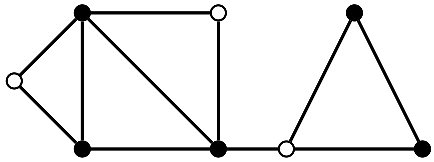

Fig. 1. 2-Role coloring of G

In general, every graph has two trivial role coloring for $k = 1, n$. The color image graph R of a graph G is called a role graph. The role graph R is defined as the graph with $V(R) = \{1, 2, \ldots k\}$ and $E(R) = \{(\alpha(u), \alpha(v)) : (u, v) \in E(G)\}$ and $|V(R)| \leqslant |V(G)|$. Also, for all $v \in V(G)$, $\deg_G(v) \geqslant \deg_R(\alpha(v))$ [3]. Figure 2 displays the possible role graphs for 2-role coloring of connected graphs.

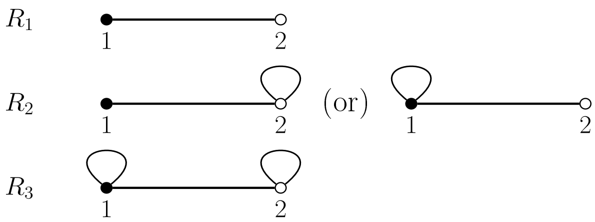

Fig. 2. Role graph

Since each color is assigned to some vertex of G, it is easy to see that if G is connected, the role graph R is also connected. This problem is equivalent to deciding if there exists a locally surjective homomorphism between the graphs G and R [4]. Finding out whether a graph G has a 2-role coloring is NP-complete, as demonstrated by F. S. Roberts and L. Sheng [5]. If the graph is chordal, then k-role assignment can be solved in linear time for $k = 2$ and NP-complete for $k \geqslant 3$ [6]. Role assignments can be computed in polynomial time for proper interval graphs [7]. C. Purcell and P. Rombach [8] proved that k-role coloring is NP-hard for planar graphs, while for trees and cographs it can be solved in polynomial time. They also examined the role coloring for hereditary classes of graphs [9]. Characterization has been done to acquire 3-role coloring in split graphs; it is one of the fascinating graph classes where 2-role coloring is always achievable [10]. S. Pandey and V. Sahlot [3] demonstrated that k-role coloring is NP-complete for bipartite graphs when $k \geqslant 3$. D. Castonguay et al. [11] demonstrated that role assignments restricted to Cartesian products are invariably 2-role colorable.

Based on the work [3], the complexity of 2-role coloring of non-bipartite graphs is evident. So, we are intended to characterize graphs that are 2-role colorable from the rooted product of G and H. Also, we restrict G and H by considering at least one of the graph as non-bipartite.

The rooted product of graphs is one of the well-known binary operations. It was introduced by C. D. Godsil and B. D. McKay [12] in 1978.

Definition 1. The rooted product of two graphs G and H is defined as the graph obtained from G and H by taking one copy of G and $|V(G)|$ copies of H and identifying the i-th vertex of G with the root vertex v in the i-th copy of H for every $i = 1, \dots, |V(G)|$. It is denoted by $G \circ_v H$.

The paper is organised as follows. The results of role coloring the rooted product of cycles with cycles are presented in Section 2. In Section 3, we determine the role coloring of the rooted product of graphs generated by considering at least one graph from G and H as non-bipartite. The conclusion is given in Section 4.

2. Rooted product of C_m and C_n

Theorem 1. Let $G \cong C_m$ and $H \cong C_n$, where $m = 2k$, $k \geqslant 2$ and $n = 2t$, $t \geqslant 2$. Then $G \circ_v H$ is 2-role colorable with role graph R_1.

Proof. Let $\{u_1, \dots, u_m\} = V(C_m)$ and $\{v_1, \dots, v_n\} = V(C_n)$. Let v_r be any arbitrary vertex in C_n. Now we obtain $C_m \circ_v C_n$ by identifying each $u_i \in V(C_m)$ with v_r, this produces m copies of C_n with vertices $\{v_{1,1}, v_{1,2}, \dots, v_{1,n}, v_{2,1}, v_{2,2}, \dots, v_{2,n}, \ \dots, v_{m,1}, v_{m,2}, \dots, v_{m,n}\}$. Let us assume $v_r = v_1$. Now define $\alpha : V(C_m \circ_v C_n) \to \{1, 2\}$ as follows:

$$\alpha(v_{i,1}) = \begin{cases} 1, & \text{if } i \text{ is odd,} \\ 2, & \text{if } i \text{ is even,} \end{cases} \qquad 1 \leqslant i \leqslant m.$$

Now, for all $v_{1,j} \in V(C_n^{(1)})$ we have:

$$\alpha(v_{1,j}) = \begin{cases} 1, & \text{if } j \text{ is odd,} \\ 2, & \text{if } j \text{ is even,} \end{cases} \qquad 1 \leqslant j \leqslant n.$$

In general, for all $v_{i,j} \in V(C_m \circ_v C_n)$ we have:

$$\alpha(v_{i,j}) = \begin{cases} 1, & \text{if } i, j \text{ have the same parity,} \\ 2, & \text{otherwise.} \end{cases}$$

This gives a 2-role coloring of $C_m \circ_v C_n$ with role graph R_1 since every vertex assigned color 1 has color 2 in its neighborhood and every vertex assigned color 2 has color 1 in its neighborhood. ■

Theorem 2. Let $G \cong C_m$ and $H \cong C_n$, where $m \geqslant 3$ and $n = 2t + 1$, $t \geqslant 1$. Then $G \circ_v H$ is 2-role colorable with role graph R_3.

Proof. Let $\{v_{1,1}, v_{1,2}, \dots, v_{1,n}, v_{2,1}, v_{2,2}, \dots, v_{2,n}, \ \dots, v_{m,1}, v_{m,2}, \dots, v_{m,n}\}$ be the vertices of $C_m \circ_v C_n$. Let v_r be any arbitrary vertex in C_n. Let $v_r = v_1$ and $v_{i,1}$ be the root vertices identified with the vertices of C_m. Since C_n is odd and non bipartite, assigning colors with role graph R_1 is not possible. Let us define $\alpha : V(C_m \circ_v C_n) \to V(R_3)$.

C a s e (i). Let $H \cong C_{2t+1}$, where t is an odd positive integer. Let us consider $\alpha(v_{i,1}) = 1$ for all $v_{i,1} \in V(C_m \circ_v C_n)$. Here $2 \notin \alpha(N(v_{i,1}))$, thus we have $\alpha(v_{i,2}) = \alpha(v_{i,3}) = 2$. Again $1 \notin \alpha(N(v_{i,3})$, thus $\alpha(v_{i,4}) = \alpha(v_{i,5}) = 1$. Proceeding in this way we get

$$\alpha(v_{i,j}) = \begin{cases} 1, & \text{if } j \equiv 0 \text{ or } 1 \pmod{4}, \\ 2, & \text{if } j \equiv 2 \text{ or } 3 \pmod{4}. \end{cases}$$

C a s e (ii). Let $H \cong C_{2t+1}$, where t is an even positive integer. If suppose $\alpha(v_{i,1}) = 1$, then there exist two vertices $v_{i,g}, v_{i,h} \in V(C_m \circ_v C_{2t+1})$, where $\alpha(v_{i,g}) = \alpha(v_{i,h})$ but $\alpha(N(v_{i,g})) \neq \neq \alpha(N(v_{i,h}))$. Thus, we have

$$\alpha(v_{i,1}) = \begin{cases} 1, & \text{if } i \text{ is odd}, \\ 2, & \text{if } i \text{ is even}. \end{cases}$$

In general, for all $v_{i,j} \in V(C_m \circ_v C_{2t+1})$, $j > 1$, we have

$$\alpha(v_{i,j}) = \begin{cases} 1, & \text{if } \big(i \text{ is odd},\ j \equiv 1 \text{ or } 2 \pmod 4\big) \text{ or } \big(i \text{ is even},\ j \equiv 0 \text{ or } 3 \pmod 4\big), \\ 2, & \text{if } \big(i \text{ is odd},\ j \equiv 0 \text{ or } 3 \pmod 4\big) \text{ or } \big(i \text{ is even},\ j \equiv 1 \text{ or } 2 \pmod 4\big). \end{cases}$$

Here, each vertex assigned color 1 has both the colors 1 and 2 in its neighborhood; similarly, every vertex assigned color 2 has both the colors 1 and 2 in its neighborhood. This gives a 2-role coloring of $C_m \circ_v C_n$ with role graph R_3. ■

An example illustrating Theorem 2 is shown in Fig. 3.

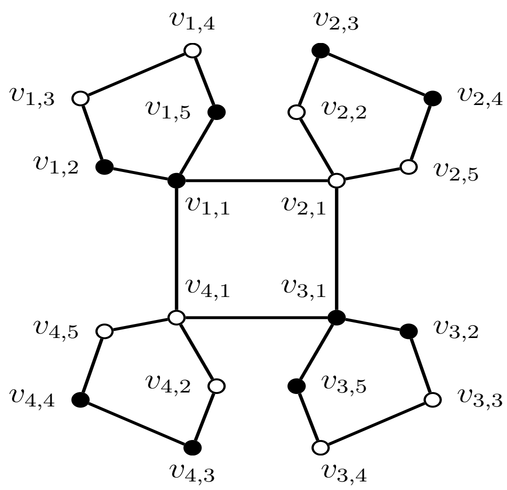

Fig. 3. 2-Role coloring of $C_4 \circ_v C_5$

Theorem 3. Let $G \cong C_m$ and $H \cong C_n$, where $m = 2k+1$, $k \geqslant 1$ and $n = 2t$, $t \geqslant 2$. If n satisfies any of the following conditions:

(i) $n \equiv 0 \text{ or } 6 \pmod{12}$,
(ii) $n \equiv 2 \text{ or } 8 \pmod{12}$,
(iii) $n \equiv 4 \pmod{12}$,

then $G \circ_v H$ is 2-role colorable.

Proof. Let $\{u_1, \ldots, u_m\} = V(C_m)$ and $\{v_1, \ldots, v_n\} = V(C_n)$. Let $\{v_{1,1}, \ldots, v_{1,n}, v_{2,1}, \ldots, v_{2,n}, \ldots, v_{m,1}, \ldots, v_{m,n}\}$ be the vertices of $C_m \circ_v C_n$. Let v_r be any arbitrary vertex in C_n. Let $v_r = v_1$ and $v_{i,1}$ be the root vertices identified with the vertices of C_m. Now we define $\alpha : V(C_m \circ_v C_n) \to \{1, 2\}$ as follows.

C a s e (i). Let $n \equiv 0 \text{ or } 6 \pmod{12}$, then for all $v_{i,1} \in V(C_{2k+1} \circ_v C_{2t})$ we have $\alpha(v_{i,1}) = 1$. In general, for all $v_{i,j} \in V(C_{2k+1} \circ_v C_{2t})$ we have

$$\alpha(v_{i,j}) = \begin{cases} 2, & \text{if } j \equiv 0 \pmod 3, \\ 1, & \text{otherwise}. \end{cases}$$

C a s e (ii). If suppose $n \equiv 2$ or $8 \pmod{12}$, then

$$\alpha(v_{i,j}) = \begin{cases} 1, & \text{if } j \equiv 0 \text{ or } 1 \pmod 3, \\ 2, & \text{if } j \equiv 2 \pmod 3. \end{cases}$$

Here, every vertex assigned color 1 has both the colors 1 and 2 in its neighborhood. Every vertex assigned color 2 has color 1 in its neighborhood. Thus, it is a 2-role coloring with role graph R_2.

C a s e (iii). Let us consider the case $n \equiv 4 \pmod{12}$. Then for all $v_{i,j} \in V(C_{2k+1} \circ_v C_{2t})$ we have

$$\alpha(v_{i,j}) = \begin{cases} 1, & \text{if } j \equiv 0 \text{ or } 1 \pmod 4, \\ 2, & \text{if } j \equiv 2 \text{ or } 3 \pmod 4. \end{cases}$$

Here, every vertex assigned color 1 has both the colors 1 and 2 in its neighborhood; similarly, every vertex assigned color 2 has both the colors 1 and 2 in its neighborhood. This gives a 2-role coloring of $C_m \circ_v C_n$ with role graph R_3. ■

An example illustrating Theorem 3 is shown in Fig. 4.

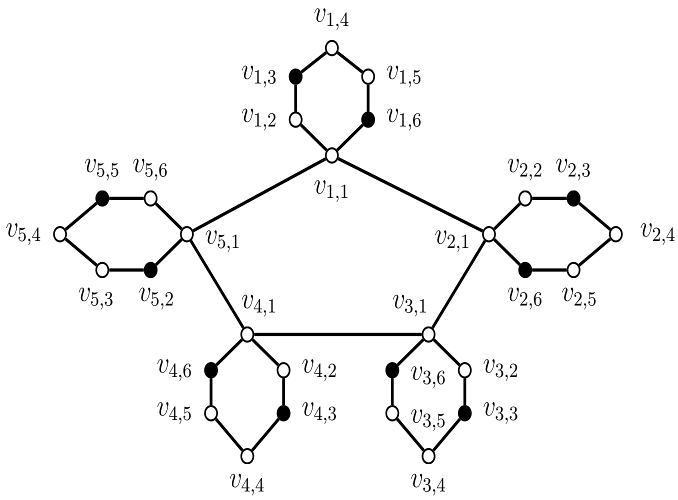

Fig. 4. 2-Role coloring of $C_5 \circ_v C_6$

Theorem 4. Let $G \cong C_m$ and $H \cong C_n$, where $m = 2k+1$, $k \geqslant 1$ and $n = 2t$, $t \geqslant 2$. If $n \equiv 10 \pmod{12}$, then $G \circ_v H$ is not 2-role colorable.

Proof. Let $\{v_{1,1}, \ldots, v_{1,n}, v_{2,1}, \ldots, v_{2,n}, \ldots, v_{m,1}, \ldots, v_{m,n}\}$ be the vertices of $G \circ_v H$. Let v_r be any arbitrary vertex in C_n. Let $v_r = v_1$ and $v_{i,1}$ be the root vertices identified with the vertices of C_m. Let C_n be an even cycle, thus assigning colors with role graph R_1 results in a contradiction, since the graph C_m is not bipartite. Hence, it can be role colored with the role graph R_2 or R_3. By Theorem 3, the only case left is $n \equiv 10 \pmod{12}$. Let us assume $\alpha : V(C_n) \to V(R_2)$ with a loop on 1 such that, given the vertices $v_1, v_2, v_n \in V(H)$, we have $\alpha(v_1) = \alpha(v_2) = \alpha(v_n) = 1$, where $2 \notin \alpha(N(v_1))$. Now consider $v_{1,j} \in V(C_n^{(1)})$ from $V(C_m \circ_v C_n)$, thus we have

$$\alpha(v_{1,j}) = \begin{cases} 2, & \text{if } j \equiv 0 \pmod 3, \\ 1, & \text{otherwise.} \end{cases}$$

Here $\alpha(v_{1,1}) = 1$, where $2 \notin \alpha(N(v_{1,1}))$. Thus, we assign $\alpha(v_{2,1}) = 2$, which satisfies the neighborhood condition. Hence, for all $v_{2,j} \in V(C_m \circ_v C_n)$, $1 \leqslant j \leqslant n$, we have

$$\alpha(v_{2,j}) = \begin{cases} 1, & \text{if } j \equiv 0 \text{ or } 2 \pmod 3, \\ 2, & \text{otherwise.} \end{cases}$$

Here $\alpha(v_{2,1}) = \alpha(v_{2,m}) = 2$ and $v_{2,1}$ is adjacent to $v_{2,m}$, it follows that $\alpha(N(v_{2,1})) = \alpha(N(v_{2,m})) \in \{1, 2\}$ but this is not true for other vertices colored 2. Hence, it is not 2-role colorable with role graph R_2. Now we define $\alpha : V(C_n) \to V(R_3)$. Consider the vertices $v_1, v_2, v_{n-1}, v_n \in V(C_n)$ such that $\alpha(v_1) = \alpha(v_2) = \alpha(v_{n-1}) = \alpha(v_n) = 1$, where $2 \notin \alpha(N(v_1))$ and $2 \notin \alpha(N(v_n))$. Let us consider $v_{1,j} \in V(C_n^{(1)})$. Thus, we have

$$\alpha(v_{1,j}) = \begin{cases} 1, & \text{if } j \equiv 0 \text{ or } 1 \pmod 4, \\ 2, & \text{otherwise.} \end{cases}$$

Here $1 \notin \alpha(N(v_{1,1}))$, therefore $\alpha(v_{2,1}) = 1$. But $\alpha(v_{1,n}) = 2$, where $2 \notin \alpha(N(v_{1,n}))$ since $\alpha(v_{1,1}) = \alpha(v_{1,(n-1)}) = 1$. Hence, $C_m \circ_v C_n$ is not 2-role colorable with role graph R_3 when $n \equiv 10 \pmod{12}$. ■

The following table summarizes the results from Theorem 1–4.

Role coloring of rooted product of cycles with cycles

Cycles (C_m)	Cycles (C_n)	k-Role coloring of cycles C_m and C_n
When m is even	When n is even	$k = 2$
When m is even	When n is odd	$k = 2$
When m is odd	When n is odd	$k = 2$
When m is odd	When n is even	$k = 2$ when $n \not\equiv 10 \pmod{12}$

3. Rooted product on other graph classes

In this section, we find the role coloring of graphs that are obtained from rooted product of other graph classes.

Theorem 5. Let G be any graph and $H \cong K_n$ or W_n. Then $G \circ_v H$ is 2-role colorable.

Proof. Let $\{v_{1,1}, \ldots, v_{1,n}, v_{2,1}, \ldots, v_{2,n}, \ldots, v_{m,1}, \ldots, v_{m,n}\}$ be the vertices of $G \circ_v H$.

C a s e (i). If suppose $H \cong W_n$, then the root can be either a universal vertex or any vertex in a cycle. Let v_r be any arbitrary vertex in W_n or K_n. Let $v_{i,1}$ be the universal vertex in W_n and $v_r = v_k$, then $v_{i,k}$ be the root vertices identified with the vertices of G. Now define $\alpha : V(G \circ_v H) \to \{1, 2\}$ as follows:

$$\alpha(v_{i,j}) = \begin{cases} 1, & \text{if } j = 1, \\ 2, & \text{otherwise.} \end{cases}$$

Let us assume $v_r = v_1$. Then again $\alpha(v_{i,1}) = 1$ and $\alpha(v_{i,j}) = 2$ for $j \neq 1$. If suppose v_r is a universal vertex, then every vertex assigned color 1 has both the colors 1 and 2 in its neighborhood; similarly, every vertex assigned color 2 has both the colors 1 and 2 in its neighborhood. Thus, we obtain a 2-role coloring with role graph R_3. Otherwise, it can have 2-role coloring with role graph R_2.

C a s e (ii). Let us consider $H \cong K_n$. Let $v_r = v_k$ be any arbitrary vertex and $v_{i,k}$ be the root vertices identified with the vertices of G. Then we have

$$\alpha(v_{i,j}) = \begin{cases} 1, & \text{if } j = k, \\ 2, & \text{otherwise.} \end{cases}$$

Hence, $G \circ_v K_n$ is 2-role colorable with role graph R_3 since every vertex colored 1 has both the colors 1 and 2 in its neighborhood; similarly, every vertex colored 2 has both the colors 1 and 2 in its neighborhood. ■

An example illustrating Theorem 5 is shown in Fig. 5.

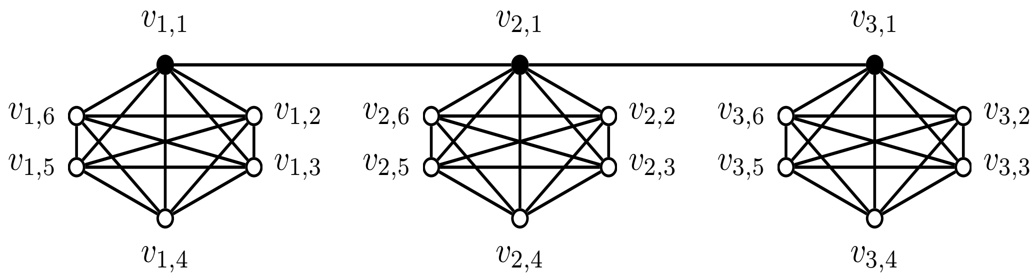

Fig. 5. 2-Role coloring of $P_3 \circ_v K_6$

Theorem 6. Let $G \cong W_m$ or K_m and $H \cong C_n$, where $n = 2t + 1$, $t \geqslant 1$. Then $G \circ_v H$ is 2-role colorable with role graph R_3.

Proof. Let $\{u_1, \ldots, u_m\} = V(G)$ and $\{v_1, \ldots, v_n\} = V(H)$. Let $\{v_{1,1}, \ldots, v_{1,n}, v_{2,1}, \ldots, v_{2,n}, \ldots, v_{m,1}, \ldots, v_{m,n}\}$ be the vertices of $G \circ_v H$. Let v_r be any arbitrary vertex of C_n. Let $v_r = v_1$ be the root. Let $\{v_{1,1}, \ldots, v_{m,1}\} \in V(G)$ in the graph $G \circ_v H$. Here we have two cases based on t.

C a s e (i). Let us consider the case where t is an odd positive integer. Now we define $\alpha : V(G \circ_v H) \to \{1, 2\}$ as follows:

$$\alpha(v_{i,j}) = \begin{cases} 1, & \text{if } j \equiv 0 \text{ or } 1 \pmod 4, \\ 2, & \text{if } j \equiv 2 \text{ or } 3 \pmod 4. \end{cases}$$

C a s e (ii). Let us consider the case where t is an even positive integer. Let $v_{i,1}$ be a universal vertex in W_m and any arbitrary vertex in K_m. Then we have

$$\alpha(v_{1,j}) = \begin{cases} 1, & \text{if } j \equiv 1 \text{ or } 2 \pmod 4, \\ 2, & \text{if } j \equiv 0 \text{ or } 3 \pmod 4, \end{cases}$$

$$\alpha(v_{i,j}) = \begin{cases} 1, & \text{if } j \equiv 0 \text{ or } 3 \pmod 4, \\ 2, & \text{if } j \equiv 1 \text{ or } 2 \pmod 4, \end{cases} \quad i > 1.$$

Here, every vertex assigned color 1 has both the colors 1 and 2 in its neighborhood; similarly, every vertex assigned color 2 has both the colors 1 and 2 in its neighborhood. Hence, this is a 2-role coloring of $G \circ_v H$ with role graph R_3. ■

Lemma 1. Let P_n be a path, where $n \geqslant 2$. If P_n is 2-role colorable with role graph R_2, then $|E(P_n)| = 3k$, where k is a positive integer.

Proof. Let $\{v_1, \ldots, v_n\} = V(P_n)$. Let us assume that $|E(P_n)| \neq 3k$, $k = 1, 2, \ldots$ Let us define $\alpha : V(P_n) \to V(R_2)$. Thus, $|V(P_n)| \geqslant 4$, and hence the length of P_n should be at least 3. Now assume $|E(P_n)| > 3k$ and $|V(P_n)| > 3k + 1$. Let us define $\alpha : V(P_n) \to \{1, 2\}$ as follows:

$$\alpha(v_i) = \begin{cases} 2, & \text{if } i \equiv 0 \text{ or } 2 \pmod 3, \\ 1, & \text{otherwise.} \end{cases}$$

Thus, each vertex assigned color 1 must have color 2 in its neighborhood, and each vertex assigned color 2 must have both the colors 1 and 2 in its neighborhood. Here, $\alpha(v_n) = 2$ and either $2 \notin \alpha(N(v_n))$ or $1 \notin \alpha(N(v_n))$, since $n \equiv 0$ or $2 \pmod 3$. This gives a contradiction that P_n is 2-role colorable. Hence, $|E(P_n)| = 3k$. ■

Theorem 7. Let G be a non-bipartite graph and $H \cong P_n$ be a path where $n \geqslant 2$. Let v_r be a root vertex in P_n. If P_n satisfies any of the following conditions:

(i) $|V(P_n)| = 3k + 1$, where k is a positive integer and $v_r = v_s$, $s \equiv 0$ or $2 \pmod 3$;

(ii) $|V(P_n)| = 3k + 2$ and $v_r = v_n$,

then $G \circ_v H$ is 2-role colorable with role graph R_2.

Proof. Let $\{u_1, \ldots, u_m\} = V(G)$ and $\{v_1, \ldots, v_n\} = V(P_n)$. Let $\{v_{1,1}, \ldots, v_{1,n}, v_{2,1}, \ldots, v_{2,n}, \ldots, v_{m,1}, \ldots, v_{m,n}\}$ be the vertices of $G \circ_v P_n$. Let us define a mapping $\alpha : V(G \circ_v P_n) \to \{1, 2\}$. Here, assigning colors to P_n with role graph R_1 is not possible because G is non-bipartite. Hence, we consider the role graph R_2.

Case (i). Let $|V(P_n)| = 3k + 1$ and $v_r = v_s$, $s \equiv 0$ or $2 \pmod 3$. By Lemma 1, it is obvious that P_n of length $3k$ is 2-role colorable by role graph R_2. Thus, for all $v_i \in V(P_n)$, if $\alpha(v_i) = 1$, then $\alpha(N(v_i)) = 2$, and if $\alpha(v_i) = 2$, then $\alpha(N(v_i)) \in \{1, 2\}$. If suppose $v_r = v_1$, then $\alpha(v_1) = 1$. Now for all $v_{i,1} \in V(G \circ_v P_n)$ we have $\alpha(v_{i,1}) = 1$, where $\alpha(N(v_{i,1})) \in \{1, 2\}$, but for all $v_{i,j} \in V(G \circ_v P_n)$, $j \neq 1, s$, we have $\alpha(v_{i,j}) = 1$, where $1 \notin \alpha(N(v_{i,j}))$. Thus, we consider $v_r = v_s$, here $\alpha(v_s) = 2$ for all $s \equiv 0$ or $2 \pmod 3$ such that for all $v_{i,s} \in V(G \circ_v P_n)$ we have $\alpha(v_{i,s}) = 2$ and $\alpha(N(v_{i,s})) \in \{1, 2\}$. This gives a 2-role coloring of $G \circ_v P_n$ with role graph R_2.

Case (ii). Let $|V(P_n)| = 3k + 2$ and $v_r = v_n$. Here $2 \notin \alpha(N(v_n))$, but $v_{i,n} \in V(G \circ_v P_n)$ be the root vertices identified with vertices of G such that $\alpha(v_{i,n}) = 2$ and $\alpha(N(v_{i,n})) \in \{1, 2\}$, which satisfies the adjacency condition with role graph R_2. ■

Theorem 8. Let G be a non-bipartite graph and $H \cong S_n$ be a star where $n \geqslant 2$. Let v_r be a root vertex in S_n. Then $G \circ_v H$ is 2-role colorable if and only if v_r is the central vertex.

Proof.

$\Rightarrow$: Let $G \circ_v S_n$ be 2-role colorable. On the contrary, we assume $v_r = v_1$ as the leaf vertex in S_n. Let us define a mapping $\alpha : V(G \circ_v S_n) \to \{1, 2\}$. Let $v_{i,1}$ be the root vertices identified with the vertices of G and $v_{i,2}$ be the central vertex of $S_n^{(i)}$. Let $(v_{i,j})$, $j \neq 1, 2$, be the leaf vertices of $S_n^{(i)}$. Now, we assume that $\alpha(v_{i,j}) = 1$ for $j \neq 1, 2$ and $\alpha(v_{i,2}) = 2$. Here, $v_{i,1}$ cannot be colored with role graph R_1, since the graph G is non-bipartite. Thus, if we assign color 1 to $v_{i,1}$, then $\alpha(N(v_{i,1})) \in \{1, 2\}$ but this is not true for all $(v_{i,j})$, since $1 \notin \alpha(N(v_{i,j}))$ for $j \neq 1, 2$. And if we assign color 2 to $v_{i,1}$ then $1 \notin \alpha(N(v_{i,1}))$. Thus, the color of $v_{i,1}$ cannot be the same as the color of $(v_{i,j})$ for $j \neq 1$, which contradicts the assumption that $G \circ_v S_n$ is 2-role colorable. Hence, v_r must be the central vertex.

$\Leftarrow$: Let $v_r = v_1$ be the central vertex. Let us define $\alpha : V(G \circ_v S_n) \to \{1, 2\}$ as follows:

$$\alpha(v_{i,j}) = \begin{cases} 1, & j = 1, \\ 2, & \text{otherwise.} \end{cases}$$

Here, every vertex assigned color 1 has both the colors 1 and 2 in its neighborhood. Every vertex assigned color 2 has color 1 in its neighborhood. Hence, it is a 2-role coloring with role graph R_2. ■

4. Conclusion

In this paper, we explored the role coloring of non-bipartite graphs generated by rooted products between various generic graph classes. Since k-role coloring is NP-complete on non-bipartite graphs when $k = 2$, we characterized graphs obtained from rooted product that are 2-role colorable.

Acknowledgement

The first author expresses her gratitude to the Vellore Institute of Technology, Vellore, for providing financial support that enabled the author to carry out the research work.

REFERENCES

1. *Bondy J. A. and Murty U. S. R.* Graph Theory. London, Springer, 2008.
2. *Everett M. G. and Borgatti S.* Role colouring a graph. Math. Social Sci., 1991, vol. 21, no. 2, pp. 183–188.
3. *Pandey S. and Sahlot V.* Role coloring bipartite graphs. Discrete Appl. Math., 2022, vol. 322, pp. 276–285.
4. *Fiala J. and Paulusma D.* A complete complexity classification of the role assignment problem. Theor. Comput. Sci., 2005, vol. 349, no. 1, pp. 67–81.
5. *Roberts F. S. and Sheng L.* How hard is it to determine if a graph has a 2 role assignment? Networks, 2001, vol. 37, no. 2, pp. 67–73.
6. *Van't Hof P., Paulusma D., and van Rooij J. M.* Computing role assignments of chordal graphs. Theor. Comput. Sci., 2010, vol. 411, no. 40–42, pp. 3601–3613.
7. *Heggernes P., van't Hof P., and Paulusma D.* Computing role assignments of proper interval graphs in polynomial time. LNCS, 2011, vol. 6460, pp. 167–180.
8. *Purcell C. and Rombach P.* On the complexity of role colouring planar graphs, trees and cographs. J. Discrete Algorithms, 2015, vol. 35, pp. 1–8.
9. *Purcell C. and Rombach P.* Role colouring graphs in hereditary classes. Theor. Comput. Sci., 2021, vol. 876, pp. 12–24.
10. *Dourado M. C.* Computing role assignments of split graphs. Theor. Comput. Sci., 2016, vol. 635, pp. 74–84.
11. *Castonguay D., Dias E. S., Mesquita F. N., and Nascimento J. R.* Computing role assignments of cartesian product of graphs. RAIRO-Operations Research, 2023, vol. 57, no. 3, pp. 1075–1086.
12. *Godsil C. D. and McKay B. D.* A new graph product and its spectrum. Bull. Australian Math. Society, 1978, vol. 18, no. 1, pp. 21–28.