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On generalized orthogonal partial metric spaces: α, β-admissible mappings and fixed point results

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Abstract. In this paper, we introduce the notion of α, β-admissible mappings as an extension of the so-called α-admissible mappings. After that, we propose for this class of mappings a new fixed point result in the setting of generalized orthogonal partial metric spaces. At the end of the results, to illustrate wide usability of our findings, we establish the existence and the uniqueness of solutions for a class of functional equations arising in dynamic programming.
Keywords: fixed point, α, β-admissible mapping, generalized orthogonal set, partial metric

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Научная статья

Об обобщенных ортогональных частично метрических пространствах: α, β-допустимые отображения и результаты о неподвижных точках

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Аннотация. Вводится понятие α, β-допустимых отображений как расширение так называемых α-допустимых отображений. Предложен для этого класса отображений новый результат о неподвижной точке в задании обобщенных ортогональных частичных метрических пространств. Чтобы проиллюстрировать широкую применимость полученных выводов, устанавливаются существование и единственность решений для класса функциональных уравнений, возникающих в динамическом программировании.

Ключевые слова: неподвижная точка, α, β-допустимое отображение, обобщенное ортогональное множество, частичная метрика

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1. Introduction

The Banach contraction principle is a fundamental result in the fixed point theory, which has been extended in many directions. Not only contraction mappings but the concept of metric space is also extended in many ways in the literature. The notion of partial metric spaces was initiated in 1994 by Matthews [1] in connection with logic programming semantics. In specific applications of logic programming, it is required to have nonzero selfdistances. Moreover, via this kind of spaces, many papers have been appeared [2, 3].

Very recently, in 2020, the authors in [4] defined the notion of generalized orthogonal sets by extending orthogonal sets introduced in 2017 by Gordji et al. [5]; the reader can see also [6]. In addition, they proved in [4] some fixed point theorems for $\perp_{\psi F}$ -contractions on generalized orthogonal metric spaces which generalize both *F*-contractions and $\perp_{\psi F}$ -contractions on metric spaces defined respectively in [7, 8]. For additional details, please refer to [9–11].

One of the various extensions of the celebrated Banach contraction is the α-admissible which is introduced in 2012 by Samet et al. [12]. In this direction, the authors in [12] established some fixed point theorems for such class of mappings in the setting of complete metric spaces. In other words, they proved some fixed point results in a complete metric space X for the class of mappings T satisfying

$$\alpha(x,y)\geq 1\Rightarrow \alpha(Tx,Ty)\geq 1, \qquad (1.1)$$

where $\alpha: X\times X\to R^+$ is a function. As one can see in (1.1), the authors compared the function α with the constant 1. A very natural idea is to compare α with another function β. In this new proposed extension, (1.1) becomes

$$\alpha(x,y)\geq\beta(x,y)\Rightarrow \alpha(Tx,Ty)\geq\beta(Tx,Ty), \qquad (1.2)$$

where $\alpha,\beta: X\times X\to R^+$ are two functions. Motivated by this fact (i.e., (1.2)) and due to the importance of α-admissible justified by the number of the papers has been published in this direction (see for instance [13–15] and references cited therein), in this paper we introduce the notion of α,β-admissible mappings. Then, a large class of mappings satisfying the fixed point property is added to the literature. It is worth mentioning that the case of partial metric spaces can be regarded as a special type of generalized partial metric spaces, so our results are valid in metric spaces and partial metric spaces as well.

Finally, using α,β-admissible mappings, the proven fixed point theorem are applied to investigate the existence and uniqueness of solutions for a class of functional equations that arise in the context of dynamic programming. Compared with other results, we point out that the study of the above integral equation is done under new weak conditions.

2. Preliminary

In this section, we recite some basic notions and results needed in the rest of the paper. In 1994, Matthews [1] introduced the notion of partial metric spaces as an extension of metric spaces as follows:

Definition 2.1. ([1]) A partial metric on a nonempty set X is a function $p : X \times X \to R^{+}$ such that for all $x, y, z \in X$:

(1) $p(x,x) = p(x,y) = p(y,y)$ if and only if $x = y$;

(2) $p(x,x) \leq p(x,y)$;

(3) $p(x,y) = p(y,x)$;

(4) $p(x,z) + p(y,y) \leq p(x,y) + p(y,z)$.

The pair (X, p) is called a partial metric space.

Definition 2.2. ([1]) Let (X, p) be a partial metric space. Then

(i) A sequence $\{x_n\}$ in X converges to a point $x \in X$ if and only if $\lim_{n\to\infty} p(x, x_n) = p(x,x)$.

(ii) A sequence $\{x_n\} \subset X$ is Cauchy if $\lim_{m,n\to\infty} p(x_m, x_n)$ exists and is finite.

(iii) X is complete if every Cauchy sequence $\{x_n\} \subset X$ converges to a point $x \in X$, that is, $\lim_{m,n\to\infty} p(x_m, x_n) = p(x,x) = \lim_{n\to\infty} p(x, x_n)$.

In 2009, Romaguera [16] introduced the following notions as generalization of the above concepts.

Definition 2.3. ([16]) Let (X, p) be a partial metric space.

(i) A sequence $\{x_n\}$ in (X, p) is called 0-Cauchy if $\lim_{m,n\to\infty} p(x_m, x_n) = 0$.

(ii) The space (X, p) is called 0-complete if every 0-Cauchy sequence in X converges to a point $x \in X$ such that $p(x,x) = 0$.

Note that if (X, p) is complete, then it is 0-complete. The author in [13] has given an example which proves that the converse assertion does not hold.

On the other hand, in 2012, Samet et al. [12] introduced the notion of α–ψ-contractive type mappings and established some fixed point theorems for these mappings in complete metric spaces.

Definition 2.4. (see [12]). Let (X, d) be a metric space, $T : X \to X$ and $\alpha : X \times X \to R^{+}$ be two given mappings. Then, T is called an α-admissible mapping if $\alpha(x,y) \geq 1 \Rightarrow \alpha(Tx,Ty) \geq 1$ for all $x, y \in X$.

Theorem 2.5. (Theorem 3.2 [12]). Let be a complete metric space and $T : X \to X$ be an α–ψ-contractive mapping, that is,

$$\alpha(x,y)\, d(Tx,Ty) \leq \psi(d(x,y)), \forall x, y \in X$$

where $\psi \in \Psi$. Assume that

(i) T is α-admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;

(iii) T is continuous.

Then, T has a fixed point.

Here, Ψ is the family of nondecreasing functions $\psi : [0,\infty) \to [0,\infty)$ such that $\sum_{1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$, with ψ^n is the nth iterate of ψ (see [15]).

Lemma 2.6. ([12]) For every function $\psi : [0,\infty) \to [0,\infty)$ the following holds:

if ψ is nondecreasing, then for each $t > 0$, $\lim_{\infty} \psi^n(t) = 0$ implies $\psi(t) < t$.

Very recently, the authors in [4] introduced a new class of sets which generalizes the notion of orthogonal sets defined by Gordji et al. [5].

Definition 2.7. ([4]) Let $X \neq \emptyset$ and let $\perp_g \subset X \times X$ be a binary relation such that $\perp_g$ satisfies the following condition.

$$\exists x_0, \forall y \in X \setminus \{x_0\}, y \perp_g x_0 \ \ or \ \ x_0 \perp_g y, \tag{2.1}$$

then it is called a generalized orthogonal set. We denote it by $(X, \perp_g)$.

The element x_0 is said to be a generalized orthogonal element.

Definition 2.8. ([4]) Let $(X, \perp_g)$ be a generalized orthogonal set. Then, a sequence $\{x_n\} \subset X$ is called a generalized orthogonal sequence if for all $n \in N$,

$$x_n \neq x_{n+1} \Rightarrow x_n \perp_g x_{n+1} \text{ or } x_{n+1} \perp_g x_n.$$

Definition 2.9. ([4]) Let $(X, \perp_g)$ be a generalized orthogonal space and $T : X \to X$ be a selfmapping. T is said to be generalized $\perp_g$ preserving if for all $x, y \in X$,

$$x \perp_g y \Rightarrow Tx \perp_g Ty.$$

Now, inspired by [13] and [4], we introduce the following definitions.

Definition 2.10. The triple $(X, p, \perp_g)$ is called generalized orthogonal partial metric space if $(X, \perp_g)$ is a generalized orthogonal set together with (X, p) is a partial metric space.

Definition 2.11. Let $(X, p, \perp_g)$ be a generalized orthogonal partial metric space. A sequence $\{x_n\} \subset X$ is said to be Cauchy generalized orthogonal sequence if it satisfies both:

(i) $\{x_n\}$ is an O-Cauchy sequence in (X, p),

(ii) $\{x_n\}$ is a generalized orthogonal sequence in $(X, \perp_g)$.

Definition 2.12. Let $(X, p, \perp_g)$ be a generalized orthogonal partial metric space. X is said to be generalized orthogonal complete space if every Cauchy generalized orthogonal sequence $\{x_n\} \subset X$ is convergent.

Definition 2.13. Let $(X, p, \perp_g)$ be a generalized orthogonal partial metric space. A mapping $T : X \to X$ is said to be generalized orthogonal continuous mapping, if for every generalized orthogonal sequence $\{x_n\} \subset X$ satisfying $\lim_n p(x_n,u)=0$ for some $u\in X$ we have $\lim_n p(Tx_n,Tu)=0.$

3. Main results

In this section, we start with the following definitions.

Definition 3.1. Let $(X,p,\perp_g)$ be a generalized orthogonal partial metric space and $T:X\to X$ be a mapping. T is said to be generalized orthogonal α,β–ψ contractive if there exit $\alpha,\beta : X^2 \to (0,\infty)$ such that for all $x,y\in X$

$$x\perp_g y \Rightarrow \alpha(x,y)\,p(Tx,Ty)\le\beta(x,y)\,\psi(p(x,y)),$$

where $\psi\in\Psi$ and α, β are introduced in the next definition.

Definition 3.2. Let $T:X\to X$ be a mapping, T is said to be α,β-admissible. If for all $x,y\in X$ we have

$$\alpha(x,y)\ge\beta(x,y)\Rightarrow\alpha(Tx,Ty)\ge\beta(Tx,Ty). \tag{3.1}$$

Remark 3.3. In the above Definition, if we take $\beta(x,y)=1$, the concept α,β-admissible becomes α-admissible.

Example. Let $X=(0,\infty)$ and $T:X\to X$ be a mapping such that $Tx=\sqrt{x}$ for all $x\in X$. Define $\alpha,\beta : X^2\to R^+$ by

$$\alpha(x,y)=\begin{cases} e^{x-y} & \text{if } x\ge y \\ y-x & \text{else}\end{cases}$$

and

$$\beta(x,y)=\begin{cases} x-y+1 & \text{if } x\ge y \\ \ln(y-x+1) & \text{else}\end{cases}$$

It is easy to see that T has a fixed point, but is not an α-admissible mapping: $\alpha(1,2)\ge1$, however $\alpha(T1,T2)=\sqrt{2}-1<1$. On the other hand, T is an α,β-admissible mapping:

Let $x,y\in X$, so we have:

Case 1: If $x\ge y$, then $\alpha(Tx,Ty)=e^{\sqrt{x}-\sqrt{y}}\ge\sqrt{x}-\sqrt{y}+1=\beta(Tx,Ty)$.

Case 2: If $x<y$, then $\alpha(Tx,Ty)=y-x\ge\ln(y-x+1)=\beta(Tx,Ty)$.

Remark 3.4. The proposed class of α,β-admissible mappings contains a large class of mappings compared with the class of α-admissible mappings.

Now, we are able to discuss the main result.

Theorem 3.5. Let $(X,p,\perp_g)$ be a complete generalized orthogonal partial metric space and $T:X\to X$ be a mapping satisfying

(i) T is a generalized orthogonal α,β–ψ-contractive;

(ii) T is α,β-admissible;

(iii) for a generalized orthogonal element $x_0\in X$ we have $\alpha(x_0,Tx_0)\ge\beta(x_0,Tx_0)$;

(iv) T is a generalized $\perp_g$ -preserving;

(v) T is a generalized orthogonal continuous mapping.

Then T has a fixed point.

Proof. Let $x_0 \in X$ be the generalized orthogonal element satisfying (iii), so for all $x_0 \neq y \in X$

$$x_0 \perp_g y \text{ or } y \perp_g x_0. \tag{3.2}$$

We can construct a sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for all $n \in N \bigcup \{0\}$.

If there exists $n_0 \in N \bigcup \{0\}$ such that $x_{n_0} = x_{n_0+1}$, so x_{n_0} is the fixed point for T. Otherwise, we suppose that $x_n \neq x_{n+1}$ for all $n \in N \bigcup \{0\}$, from (3.2) we obtain that $x_0 \perp_g Tx_0$ or $Tx_0 \perp_g x_0$ and by the fact that T is a generalized $\perp_g$ -preserving mapping we can conclude that $\{x_n\}$ is a generalized orthogonal sequence.

Now, since T is α,β-admissible, by (iii) we have

$$\alpha(x_n, x_{n+1}) \geq \beta(x_n, x_{n+1}), \tag{3.3}$$

for all $n \in N \bigcup \{0\}$.

Applying (i), we obtain

$$\alpha(x_n, x_{n+1})\, p(x_{n+1}, x_{n+2}) \leq \beta(x_n, x_{n+1})\, \psi\left(p(x_n, x_{n+1})\right) \tag{3.4}$$

for all $n \in N \bigcup \{0\}$.

We have $p(x_n, x_{n+1}) > 0$ for all $n \in N \bigcup \{0\}$, this fact with the monotony of ψ imply that

$$p(x_{n+1}, x_{n+2}) \leq \psi^{n+1}\left(p(x_0, x_1)\right) \tag{3.5}$$

for all $n \in N \bigcup \{0\}$.

In this step, let $n, m \in N$ be such that $m \geq 2$, so we have

$$p(x_n, x_{n+m}) \leq \sum_{n}^{n+m-1} p(x_k, x_{k+1}) - \sum_{n+1}^{n+m-1} p(x_k, x_k), \tag{3.6}$$

which implies by (3.5) and the definition of the function ψ that $\{x_n\}$ is a Cauchy generalized orthogonal sequence. Since X is a complete generalized orthogonal partial space, there exists $u \in X$ such that $\lim_{n\to\infty} p(u, x_n) = p(u,u) = \lim_{n\to\infty} p(x_m, x_n) = 0$. Now by (v), we have $\lim_{n\to\infty} p(Tu, Tx_n) = 0$, and hence

$$\begin{aligned} p(u, Tu) &\leq p(u, x_n) + p(x_n, x_{n+1}) + p(x_{n+1}, Tu) \\ &\leq p(u, x_n) + p(x_n, x_{n+1}) + p(Tx_n, Tu), \end{aligned} \tag{3.7}$$

for all $n \in N$. Letting $n \to \infty$, we get $p(u, Tu) = 0$ which implies that $u = Tu$. □

In the above Theorem, if we take $p(x, x) = 0$ for all $x \in X$ (in other words, since every metric space is a partial metric space), we obtain the following result

Corollary 3.6. Let $(X, d, \perp_g)$ be a complete generalized orthogonal metric space and $T: X \to X$ be a mapping satisfying

(i) T is a generalized orthogonal α,β–ψ-contractive mapping;
(ii) T is α,β-admissible;
(iii) for a generalized orthogonal element $x_0 \in X$ we have

$$\alpha(x_0, Tx_0) \geq \beta(x_0, Tx_0);$$

(iv) T is a generalized $\perp_g$ -preserving;
(v) T is a generalized orthogonal continuous mapping.
Then T has a fixed point.

If we replace the condition "generalized orthogonal α,β–ψ-contractive" by "α,β–ψ-contractive," we obtain the following result.

Corollary 3.7. Let (X, p) be a complete partial metric space and $T: X \to X$ be a mapping satisfying
(i) T is α,β–ψ-contractive;
(ii) T is α,β-admissible;
(iii) for some $x_0 \in X$ we have

$$\alpha(x_0, Tx_0) \geq \beta(x_0, Tx_0);$$

(iv) T is a continuous mapping.
Then T has a fixed point.

If we take $\beta = 1,$ we obtain the famous result

Corollary 3.8. (Theorem 2.2 [12]) Let (X, d) be a complete metric space and $T: X \to X$ be an α–ψ-contractive mapping satisfying the following conditions:
(i) T is α-admissible;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;
(iii) T is continuous.
Then, T has a fixed point.

Example. Let $X = [0, \infty),$ we endow X with the partial metric $p(x, y) = \max\{x, y\}$.
Define a mapping $T: X \to X$ by

$$T(x) = \begin{cases} 2x & \text{if } x \in [0,1] \cup (2, \infty) \\ x & \text{if } x \in (1, 2] \end{cases}$$

and two functions

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x, y \in [0,1] \\ 0 & \text{if } x, y \in (1, 2] \\ \frac{2}{3} & \text{else,} \end{cases}$$

$$\beta(x, y) = \begin{cases} 3 & \text{if } x, y \in [0,1] \\ 0 & \text{if } x, y \in (1, 2] \\ \max\{2x, 2y\} & \text{else.} \end{cases}$$

Since for all $x, y \in X$ such that $\alpha(x, y) \geq \beta(x, y)$ implies

$$\alpha(Tx, Ty) = \alpha(x, y) = 0 \geq 0 = \beta(x, y) = \beta(Tx, Ty).$$

So, T is an α,β-admissible mapping.
Now, we define a binary relation on $X\times X$ by

$$x\perp_g y \multimap \Leftrightarrow \multimap\multimap x<y$$

and hence $(X,p,\perp_g)$ is a complete generalized orthogonal partial metric space and $\frac{3}{2}$ is a generalized orthogonal element satisfying

$$\alpha\left(\frac{3}{2},T\frac{3}{2}\right)=\alpha\left(\frac{3}{2},\frac{3}{2}\right)=0\geq 0=\beta\left(\frac{3}{2},\frac{3}{2}\right)=\beta\left(\frac{3}{2},T\frac{3}{2}\right).$$

Clearly, T is a generalized $\perp_g$ -preserving and a generalized orthogonal continuous mapping.

Finally, if we take $\psi(t)=\frac{2}{3}t$ for all $t\in R^{+}$, then T is a generalized orthogonal α,β–ψ-contractive. Indeed: let $x,y\in X$ such that $x\perp_g y$, and hence $x<y$, so we have the following cases:

Case 1: if $x,\ y\in[0,1]$, then

$$\begin{aligned}\alpha(x,y)\,p(Tx,Ty)&=\max\{Tx,Ty\}\\&=\max\{2x,2y\}\\&=2y\\&\leq 3\times\frac{2}{3}y\\&=\beta(x,y)\times\frac{2}{3}\max\{x,y\}\\&=\beta(x,y)\psi(p(x,y))\end{aligned}$$

Case 2: if $x\in[0,1]$ and $y>2$, so

$$\begin{aligned}\alpha(x,y)\,p(Tx,Ty)&=\frac{2}{3}\max\{Tx,Ty\}\\&=\frac{2}{3}\max\{2x,2y\}\\&=\frac{4}{3}y\\&\leq\frac{4}{3}y^{2}\\&=\beta(x,y)\times\frac{2}{3}\max\{x,y\}\\&=\beta(x,y)\psi(p(x,y)).\end{aligned}$$

Case 3: if $x,y>2$, then

$$\alpha(x,y)\,p(Tx,Ty)=\frac{2}{3}\max\{Tx,Ty\}$$

$$=\frac{2}{3}\max\{2x,2y\}$$
$$=\frac{4}{3}y$$
$$\le\frac{4}{3}y^2$$
$$\le 2y\frac{2}{3}y$$
$$=\beta(x,y)\times\frac{2}{3}\max\{x,y\}$$
$$=\beta(x,y)\psi(p(x,y)).$$

Case 4: if $x,y\in(1,2]$, then

$$\alpha(x,y)\,p(Tx,Ty)=0$$
$$\le\beta(x,y)\psi(p(x,y)).$$

Remark 3.9. It is clear to see that Corollary 3.8 (Theorem 2.2 in [12]) does not ensure the existence of a fixed point since T is not an α-admissible mapping. Indeed:

$$\alpha\left(\frac{3}{4},\frac{7}{8}\right)\ge 1,\ \text{however}\ \alpha\left(T\frac{3}{4},T\frac{7}{8}\right)=\alpha\left(\frac{3}{2},\frac{7}{4}\right)=0<1. \quad (3.8)$$

4. Application in Dynamic Programming

The theory of dynamic programming traces its origins to the study of multistage decision processes. Inspired by [17, 18], we study the existence and uniqueness of a solution of a class of functional equations arising in dynamic programming. For this purpose, suppose that X and Y are Banach spaces, $S\subset X$ is the state space, and $D\subset Y$ is the decision space. Let $\rho: S\times D\to S$, $g: S\times D\to\mathbb{R}^+$ and $G: S\times D\times\mathbb{R}\to\mathbb{R}^+$, where $\mathbb{R}$ is the field of real numbers and $\mathbb{R}^+$ is the set of non-negative real numbers. $B(S)^+$ denotes the set of all continuous bounded and non-negative real-valued functions on S. For $h,k\in B(S)^+$, let

$$p(h,k)=\sup_{x\in S}\{|h(x)-k(x)|\}+\max\{\sup_{x\in S}|h(x)|,\sup_{x\in S}|k(x)|\}.$$

It is easy to see that $(B(S)^+,p,\perp_g:=\le)$ is a complete generalized orthogonal partial metric space with $x_0\equiv 0$.

Now, consider the following functional equation

$$f(x)=\sup\left\{g(x,y)+G\left(x,y,f(\rho(x,y))\right)\right\}, \quad (4.1)$$

where g and G are bounded. We define $T: B(S)^+\to B(S)^+$ by

$$Tf(x)=\sup\left\{g(x,y)+G\left(x,y,f(\rho(x,y))\right)\right\}. \quad (4.2)$$

In the following, we prove the existence and uniqueness of the solution for functional (4.1). This is equivalent to proving the existence and uniqueness of the mapping T defined in (4.2).

Theorem 4.2. Suppose that there exist $\theta: B(S)^+\times B(S)^+\to\mathbb{R}$ and $\psi\in\Psi$ such that for all $h,k\in B(S)^+$ we have the following:

(i) $\theta(x_0, Tx_0) \leq 0$,

(ii) $\theta(h,k) \leq 0 \Rightarrow \theta(Th, Tk) \leq 0$,

(iii) $h \perp_g k \Rightarrow$

$$\begin{cases} |G(x,y,h(\rho(x,y))) - G(x,y,k(\rho(x,y)))| + \max\{\sup_{x\in S}|Th(x)|, \sup_{x\in S}|Tk(x)|\} \leq \psi(p(h,k)), \\ G(x,y,h(\rho(x,y))) \leq G(x,y,k(\rho(x,y))), \end{cases}$$

for all $(x,y) \in S \times D$.

Then, T has a unique fixed point.

Proof. We will proceed by adhering to the following steps:

- Let ε be an arbitrary positive number, let $x \in S$ and $h,k \in B(S)^+$ with $h \perp_g k$; therefore there exist $y, z \in D$ such that

$$Th(x) < g(x,y) + \mathrm{G}\left(x,y,h\big(\rho(x,y)\big)\right) + \varepsilon, \tag{4.3}$$

$$Tk(x) < g(x,z) + \mathrm{G}\left(x,z,k\big(\rho(x,z)\big)\right) + \varepsilon. \tag{4.4}$$

At the same time, from the definition of T, we get

$$Th(x) \geq g(x,z) + G\left(x,z,h\big(\rho(x,z)\big)\right), \tag{4.5}$$

$$Tk(x) \geq g(x,y) + G\left(x,y,h\big(\rho(x,y)\big)\right). \tag{4.6}$$

It follows from (4.3) and (4.6) that

$$Th(x) - Tk(x) < \mathrm{G}\left(x,y,h\big(\rho(x,y)\big)\right) - \mathrm{G}\left(x,y,k\big(\rho(x,y)\big)\right) + \varepsilon$$
$$< |\mathrm{G}\left(x,y,h\big(\rho(x,y)\big)\right) - \mathrm{G}\left(x,y,k\big(\rho(x,y)\big)\right)| + \varepsilon.$$

Thus by condition (iii), we obtain

$$Th(x) - Tk(x) < \psi\big(p(h,k)\big) - \max\{\sup_{x\in S}|Th(x)|, \sup_{x\in S}|Tk(x)| + \varepsilon.$$

Similarly, from (4.4) and (4.5), we have

$$Tk(x) - Th(x) < \psi\big(p(h,k)\big) - \max\{\sup_{x\in S}|Th(x)|, \sup_{x\in S}|Tk(x)| + \varepsilon.$$

Then

$$|Th(x) - Tk(x)| < \psi\big(p(h,k)\big) - \max\{\sup_{x\in S}|Th(x)|, \sup_{x\in S}|Tk(x)| + \varepsilon,$$

Equivalently

$$|Th(x) - Tk(x)| + \max\{\sup_{x\in S}|Th(x)|, \sup_{x\in S}|Tk(x)| < \psi\big(p(h,k)\big) + \varepsilon,$$

This implies that

$$p(Th, Tk) < \psi\big(p(h,k)\big) + \varepsilon.$$

Since ε is taken arbitrary, we obtain

$$p(Th, Tk) \leq \psi\big(p(h,k)\big). \tag{4.7}$$

Now, let

$$\alpha(h,k) = \begin{cases} 1 & \text{if } \theta(h,k) \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\beta(h,k) = \begin{cases} 1 & \text{if } \theta(h,k) \leq 0, \\ 2 & \text{otherwise.} \end{cases}$$

Then

$$\alpha(h,k)p(Th,Tk) \leq \beta(h,k)\psi\big(p(h,k)\big).$$

Therefore, T is generalized orthogonal α,β–ψ-contractive.

- If $\alpha(h,k) \geq \beta(h,k)$, then $\theta(h,k) \leq 0$. Then, from condition (ii), we obtain $\theta(Th,Tk) \leq 0$ which implies that $\alpha(Th,Tk) \geq \beta(Th,Tk)$. Hence, T is α,β-admissible.
- If $\theta(x_0, Tx_0) \leq 0$, then $\alpha(x_0, Tx_0) = 1 \geq 1 = \beta(x_0, Tx_0)$. Then condition (iii) from Theorem 3.5 holds.

- From condition (iii), we have $h \perp_g k$ leads to

$$G\Big(x, y, h\big(\rho(x, y)\big)\Big) \leq G\Big(x, y, k\big(\rho(x, y)\big)\Big),$$

and so by the definition of T we get $Th \perp_g Tk$. This means that the mapping T is generalized $\perp_g$ -preserving.

It remains to prove that T is a generalized orthogonal continuous mapping.

- Let $\{h_n\}$ be a generalized orthogonal sequence of functions of X such that $\lim_n p(h_n, u) = 0$ for some $u \in B(S)^+$. Since $\perp_g = \leq$, we have

$$h_n \leq h_{n+1} \text{ or } h_{n+1} \leq h_n, \text{ then } h_n \leq u \text{ or } u \leq h_n.$$

Therefore, according to (4.7), we have

$$\begin{aligned} p(Th_n, Tu) &\leq \psi\big(p(h_n, u)\big) \\ &< p(h_n, u). \end{aligned}$$

Then $\lim_n p(Th_n, Tu) = 0$ and T is a generalized orthogonal continuous mapping.

Finally, all conditions of Theorem 3.5 hold and T has a unique fixed point. □

Remark 4.3. In a future work, we will compare our results with the famous Caristi's fixed point theorem [19]. Moreover, we will study the existence of solutions for certain differential equations via the proven results.

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