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THE NELSON-SIEGEL-SVENSSON YIELDS. PROBABILITY PROPERTIES AND ESTIMATION

Probabilty properties of the yield interest rates that are generated by model of Nelson – Siegel and Nelson – Siegel – Svensson are considered. The paper is directly related to [1]. It is shown that the model of Nelson – Siegel does not differ from the traditional two-factor model of affine yield, the volatility of which does not depend on the market state variables. Accordingly, the model of Nelson – Siegel – Svensson – from a four-factor model. These models generate the interest rates of yield to maturity and the forward yield with a normal distribution, for which the expectations and covariance matrices are found. To estimate the values of the rates of yield to maturity in the current time it is offered a recurrent procedure based on the use of the Kalman filter.
Keywords: term structure; yield curve; affine models; Nelson-Siegel-Svensson yield model; Kalman filter.

The Nelson – Sigel (NS) yield models belong to the affine family of models. The affine models are the factor models where some variables, for example a components of the state vector x, are considered as factors. In affine models of the term structure of interest rates it is assumed that the yield to maturity $y(\tau, x)$ on a zero coupon bond with price $P(\tau, x)$ is defined by relation

$$y(\tau, x) \equiv \frac{-\ln P(\tau,x)}{\tau} = \frac{x^{\mathrm{T}} B(\tau) - A(\tau)}{\tau},$$

where τ – term to maturity and $A(\tau)$, $B(\tau)$ are the functions of term structure. Usually as components of the state vector x some market indexes are taken and then functions of term structure $A(\tau)$ and $B(\tau)$ are determined so to avoid an arbitrage opportunities. In their yield model C. Nelson and A. Sigel [2] do conversely: they set vector $B(\tau)$ and then determine vector x so to fit the yield $y(\tau, x)$ to the market observations. Vector $B(\tau)$ that set by Nelson and Sigel is

$$B(\tau) = \begin{pmatrix} \dfrac{1-e^{-\gamma\tau}}{\gamma} \\ \dfrac{1-e^{-\gamma\tau}}{\gamma} - \tau e^{-\gamma\tau} \end{pmatrix} \quad (1)$$

with parameter $\gamma > 0$. $A(\tau) = -\tau L(\tau)$ where $L(\tau)$ is known as a *level* factor. Vector x has components S_t and C_t: S_t – a *slope* factor and C_t – a *curvature* factor. Factors L, S and C are named too *latent* factors.

Thus in the NS yield models the yield curve $y(\tau) \equiv y(\tau \mid L, S, C)$ and the forward curve $f(\tau) \equiv f(\tau \mid L, S, C)$ are determined by following relations

$$y(\tau) = L(\tau) + S_t \frac{1-e^{-\gamma\tau}}{\gamma\tau} + C_t\left(\frac{1-e^{-\gamma\tau}}{\gamma\tau} - e^{-\gamma\tau}\right), \quad (2)$$

$$f(\tau) = (\tau L(\tau))' + S_t \exp(-\gamma\tau) + C_t \gamma\tau \exp(-\gamma\tau).$$

L. Svensson [3] offered to extend the NS model in order to increase the flexibility and improve the fit models to empirical data. He introduced the additional latent factor but with parameter $\delta \neq \gamma$, $\delta > 0$. Then vectors $B(\tau)$ and x will have four components:

$$B(\tau)=\begin{pmatrix}\dfrac{1-e^{-\gamma\tau}}{\gamma}\\ \dfrac{1-e^{-\gamma\tau}}{\gamma}-\tau e^{-\gamma\tau}\\ \dfrac{1-e^{-\delta\tau}}{\delta}\\ \dfrac{1-e^{-\delta\tau}}{\delta}-\tau e^{-\delta\tau}\end{pmatrix},\quad x=\begin{pmatrix}S_t\\ C_t\\ G_t\\ H_t\end{pmatrix}. \tag{3}$$

Note that in Svensson's proposal $G_t = 0$ (more exactly: G_t is absent) and vectors $B(\tau)$ and x have only three components. However as it is shown [1] in this case the no-arbitrage conditions are not fulfilled. Therefore we will assume that G_t can be not zero.

So that in the Nelson – Sigel – Svensson (NSS) model the yield curve $y(\tau) \equiv y(\tau \mid L, S, C, G, H)$ and the forward curve $f(\tau) \equiv f(\tau \mid L, S, C, G, H)$ are determined by following relations

$$y(\tau) = L(\tau) + S_t\frac{1-e^{-\gamma\tau}}{\gamma\tau} + C_t\left(\frac{1-e^{-\gamma\tau}}{\gamma\tau}-e^{-\gamma\tau}\right) + G_t\frac{1-e^{-\delta\tau}}{\delta\tau} + H_t\left(\frac{1-e^{-\delta\tau}}{\delta\tau}-e^{-\delta\tau}\right),$$

$$f(\tau) = (\tau L(\tau))' + S_t \exp(-\gamma\tau) + C_t\,\gamma\tau\exp(-\gamma\tau) + G_t\exp(-\delta\tau) + H_t\,\delta\tau\exp(-\delta\tau).$$

1. Probability properties

It was shown [1] that the yield model is the no-arbitrage Nelson – Sigel (NS) model or the no-arbitrage Nelson – Sigel – Svensson (NSS) model only when the vector of a state variables of financial market $X(t) = (X_1, X_2, \ldots, X_n)^{\mathrm{T}}$ follows to homogeneous for time Markov process generated by the stochastic differential equation with linear function of drift $\mu(x)$ over x, and the volatility matrix $\sigma(x)$ not depended on x.

Under these conditions this equation can be written down in a form

$$dX(t) = [-KX(t) + K\theta]dt + \sigma W(t),\quad X(t_0) = X_0, \tag{4}$$

where K – $(n\times n)$-matrix, θ – n-vector, σ – $(n\times m)$-matrix and $W(t) = \{W(t);\ t \ge t_0\}$ – m-dimensional Wiener process with components, which are independent scalar Wiener processes. As well as any differential equation (10) should be supplied by the entry condition fixing a vector $X(t)$ at some initial time t_0. It is possible to show [4] that the equation (4) can be solved analytically in the following form

$$X(t) = \Theta(t_0,t)\left[X_0 + \int_{t_0}^{t}\Theta^{-1}(t_0,s)K\theta ds + \int_{t_0}^{t}\Theta^{-1}(t_0,s)\sigma dW(s)\right], \tag{5}$$

where $\Theta(t_0,t)$ – $(n\times n)$ fundamental matrix of solutions, satisfying to the initial condition $\Theta(t_0,t_0) = I$ and to the homogeneous matrix differential equation

$$d\,\Theta(t_0,t) = -K\Theta(t_0,t)dt, \tag{6}$$

which can be considered as n the vector differential equations. Note that to solve this equation with respect $\Theta(t_0,t)$ in an explicit form also it is possible. As the equation (4) is the equation with constant coefficients the fundamental matrix of solutions $\Theta(t_0,t)$ will depend not on two arguments, but only from one: $\Theta(t_0,t) = \Theta(t-t_0)$. To stop receiving cumbersome expressions, we present the solution of equations (4)–(6) for NS model, when $n = m = 2$. In this case matrix K is determined for NS model by the formula

$$K = \begin{pmatrix}\gamma & -\gamma\\ 0 & \gamma\end{pmatrix} \tag{7}$$

and the solution of the equation (6) has a form:

$$\Theta(t-t_0) = \begin{pmatrix}e^{-\gamma(t-t_0)} & \gamma(t-t_0)e^{-\gamma(t-t_0)}\\ 0 & e^{-\gamma(t-t_0)}\end{pmatrix},\quad \Theta^{-1}(t-t_0) = \begin{pmatrix}e^{\gamma(t-t_0)} & -\gamma(t-t_0)e^{\gamma(t-t_0)}\\ 0 & e^{\gamma(t-t_0)}\end{pmatrix}.$$

Therefore expression (5) takes the form

$$X(t)=\begin{pmatrix}X_1(t)\\X_2(t)\end{pmatrix}=\begin{pmatrix}e^{-\gamma(t-t_0)}[X_1(t_0)+\gamma(t-t_0)X_2(t_0)]\\e^{-\gamma(t-t_0)}X_2(t_0)\end{pmatrix}+$$

$$+\begin{pmatrix}(1-e^{-(t-t_0)\gamma})\theta_1-\gamma(t-t_0)e^{-(t-t_0)\gamma}\theta_2\\(1-e^{-(t-t_0)\gamma})\theta_2\end{pmatrix}+\xi(t). \qquad (8)$$

Here $\xi(t)$ – a two-dimensional normally distributed random vector process with the components having zero mean and covariance matrix

$$\Omega(t-t_0)=E\,[(X(t)-E\,[X(t)])\,(X(t)-E\,[X(t)])^{\mathrm{T}}]=E\,[\xi(t)\,\xi(t)^{\mathrm{T}}].$$

Generally the matrix $\Omega(t-t_0)$ is calculated easy, but its elements $[\Omega(t-t_0)]_{jk}$ are quite cumbersome. Therefore we present here only their analytical expressions for the simplified case where the off-diagonal elements of the volatility matrix σ are zero ($\sigma_{12}=\sigma_{21}=0$, σ_{ki} – matrix elements σ).

$$[\Omega(t-t_0)]_{11}=\frac{1-e^{-2\gamma(t-t_0)}}{4\gamma}(2\sigma_{11}^2+\sigma_{22}^2)-\frac{e^{-2\gamma(t-t_0)}}{4\gamma}2\gamma(t-t_0)(1+\gamma(t-t_0))\sigma_{22}^2,$$

$$[\Omega(t-t_0)]_{12}=[\Omega(t-t_0)]_{21}=\frac{1-e^{-2\gamma(t-t_0)}-2\gamma(t-t_0)e^{-2\gamma(t-t_0)}}{4\gamma}\sigma_{22}^2, \qquad (9)$$

$$[\Omega(t-t_0)]_{22}=\frac{1-e^{-2\gamma(t-t_0)}}{2\gamma}\sigma_{22}^2.$$

Expression (8) is composed of three terms: the first defines the time-decreasing dependence on the initial state X_0, the second approaches over time to a steady mean θ, and the third term is random process $\xi(t)$ with zero mean and tending eventually to its limiting form covariance matrix $\Omega(t-t_0)$. Limiting matrix $\Omega(t-t_0)$ when $t-t_0\to\infty$ looks as follows:

$$\Omega(\infty)=\frac{1}{4\gamma}\begin{pmatrix}2\sigma_{11}^2+2\sigma_{12}^2+2\sigma_{11}\sigma_{21}+2\sigma_{12}\sigma_{22}+\sigma_{21}^2+\sigma_{22}^2 & 2\sigma_{11}\sigma_{21}+2\sigma_{12}\sigma_{22}+\sigma_{21}^2+\sigma_{22}^2\\2\sigma_{11}\sigma_{21}+2\sigma_{12}\sigma_{22}+\sigma_{21}^2+\sigma_{22}^2 & 2(\sigma_{21}^2+\sigma_{22}^2)\end{pmatrix}.$$

Note that from properties of matrixes $\Omega(t-t_0)$ follows that components of a vector of state $X(t)$ – the correlated random processes. This correlation between vector components for some fixed time t. However, the serial correlation of the state vectors for different time points t and $t+s$ is important too.

$$\Omega(t,t+s)=E\,[(X(t)-E\,[X(t)])\,(X(t+s)-E\,[X(t+s)])^{\mathrm{T}}]=$$
$$=E\,[\xi(t)\,\xi(t+s)^{\mathrm{T}}]=Cov\,(X(t),X(t+s)).$$

As well as in the previous case, we present this matrix only in the case a diagonal volatility matrix σ. Besides, we will put that $t_0=0$.

$$[\Omega(t,t+s)]_{11}=\frac{e^{-\gamma s}}{4\gamma}((1-e^{-2\gamma t})(2\sigma_{11}^2+(1+\gamma s)\sigma_{22}^2)-2\gamma te^{-2\gamma t}(1+(t+s))\sigma_{22}^2),$$

$$[\Omega(t,t+s)]_{12}=\frac{e^{-\gamma s}}{4\gamma}(1-e^{-2\gamma t}-2\gamma te^{-2\gamma t})\sigma_{22}^2,$$

$$[\Omega(t,t+s)]_{21}=\frac{e^{-\gamma s}}{4\gamma}((1-e^{-2\gamma t})(1+2\gamma s)-2\gamma te^{-2\gamma t})\sigma_{22}^2,$$

$$[\Omega(t,t+s)]_{22}=\frac{e^{-\gamma s}}{2\gamma}(1-e^{-2\gamma t})\sigma_{22}^2.$$

In a stationary case, i.e. at $t_0\to-\infty$, covariance matrix $Cov\,(X(t),X(t+s))$ does not depend from t and looks like

$$Cov(X(t),X(t+s))=\frac{e^{-\gamma s}}{4\gamma}\begin{pmatrix}2\sigma_{11}^2+(1+\gamma s)\sigma_{22}^2 & \sigma_{22}^2\\(1+\gamma s)\sigma_{22}^2 & 2\sigma_{22}^2\end{pmatrix}.$$

Note that

$$Cov\,(X_2(t),X_2(t+s))=\frac{\sigma_{22}^2}{2\gamma}e^{-\gamma s}.$$

Therefore the correlation function of components $X_2(t)$ depends only on one parameter γ, i.e.

$$\frac{Cov[X_2(t), X_2(t+s)]}{Var[X_2(t)]} = e^{-\gamma s}.$$

So if this correlation function can be estimated, then it is possible to estimate and parameter γ.

Because processes $X(t)$ and $\xi(t)$ have normal distribution the knowledge of mathematical expectations and covariance matrixes gives their comprehensive probabilistic description.

For NSS model $n = m = 4$ and in this case vector $B(\tau)$ has the form (3) and matrix K is determined by the formula

$$K = \begin{pmatrix} \gamma & -\gamma & 0 & 0 \\ 0 & \gamma & 0 & 0 \\ 0 & 0 & \delta & -\delta \\ 0 & 0 & 0 & \delta \end{pmatrix}.$$

The matrixes $\Theta(t-t_0)$, $\Omega(t-t_0)$, $\Omega(t, t+s)$ will be four-dimensional too. The four-dimensional process $X(t) = (X_1, X_2, X_3, X_4)^{\mathrm{T}}$ will be normal and will have the similar properties as such process of NS model. However the explicit expressions for this case will be considerable more bulky and here not cited.

It is shown [1] that in no-arbitrage affine models between the state variables of financial market $X(t)$ and latent factors of NS model $\{S_t, C_t\}$ there are an accordance

$$S_t = \phi_1 X_1(t) + \phi_2 X_2(t) = r(t),\ C_t = \phi_1 X_2(t),\ L_t = -\left.\frac{A(\tau)}{\tau}\right|_{\tau=T-t}. \tag{10}$$

So that the factor of level L_t in the explicit form does not depend on current time, and depends only on term to maturity and is not random. A vector $\phi = (\phi_1\ \ \phi_2)^{\mathrm{T}}$ is defined by following general limiting property

$$\lim_{\tau\to 0} y(\tau, r) = \lim_{\tau\to 0} \frac{x^{\mathrm{T}}B(\tau) - A(\tau)}{\tau} = x^{\mathrm{T}}\phi = r,$$

which is fulfiled uniformly over x. Therefore $\lim_{\tau\to 0}\frac{-A(\tau)}{\tau} = L(0) = 0$, and $B'(0) = \phi$. r is the short-term (riskless) interest rate. From this and from the expression (1) it is follows that $\phi = B'(0) = (1\ 0)$, i.e. $\phi_1 = 1$, $\phi_2 = 0$.

The slope and curvature factors $\{S_t, C_t\}$ are accurately the state variables $\{X_1(t), X_2(t)\}$. Hence all probability properties of state variables $\{X_1(t), X_2(t)\}$ are properties of latent factors $\{S_t, C_t\}$ also.

Thus for NS model the factor $L(\tau)$ is determinate and vector of factors $\{S_t, C_t\}$ are the normally distributed two-dimensional random variable with the vector of mathematical expectation $\theta = (\theta_1\ \ \theta_2)^{\mathrm{T}}$ and the covariance matrix $\Omega(t - t_0)$ with elements that are computed by formulae (9). From this it follows that yield rate $y(\tau)$ and the forward rate $f(\tau)$ have mathematical expectations and variances

$$E[y(t, \tau)] = L(\tau) + \frac{1}{\tau}\theta^{\mathrm{T}}B(\tau),\ E[f(t, \tau)] = \frac{d[\tau L(\tau)]}{d\tau} + \theta^{\mathrm{T}}\frac{d[B(\tau)]}{d\tau}.$$

$$\mathrm{VAR}[Y(T, \tau)] = \frac{1}{\tau^2}B(\tau)^{\mathrm{T}}\Omega(T - T_0)B(\tau),\ \mathrm{VAR}[F(T, \tau)] = \frac{d[B(\tau)]}{d\tau}^{\mathrm{T}}\Omega(T - T_0)\frac{d[B(\tau)]}{d\tau}.$$

These expectations and variances can be computed in explicit form both for NS model and NSS model. Only for NS model the vectors and the matrixes are two-dimensional and for NSS model these vectors and matrixes are four-dimensional. Because of bulkiness these expressions here not cited.

2. Estimation

If to enter designations $\tilde{S}_t = S_t - \theta_1$, $\tilde{C}_t = C_t - \theta_2$ then representation (8) can be written down as well in more compact kind:

$$\begin{pmatrix} \tilde{S}_t \\ \tilde{C}_t \end{pmatrix} = \begin{pmatrix} \tilde{S}_s + \gamma(t-s)\tilde{C}_s \\ \tilde{C}_s \end{pmatrix} e^{-\gamma(t-s)} + \zeta(t, s) = e^{-\gamma(t-s)}\begin{pmatrix} 1 & \gamma(t-s) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \tilde{S}_s \\ \tilde{C}_s \end{pmatrix} + \zeta(t, s),$$

From representations (8) and (9) follows that if the pair $\{S_s, C_s\}$ is known for $s < t$ then the least squares estimator of pair $\{S_t, C_t\}$ is expression

$$\begin{pmatrix} \hat{S}_t \\ \hat{C}_t \end{pmatrix} = e^{-\gamma(t-s)} \begin{pmatrix} 1 & \gamma(t-s) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} S_s \\ C_s \end{pmatrix} + \begin{pmatrix} 1-e^{-(t-s)\gamma} & -\gamma(t-s)e^{-(t-s)\gamma} \\ 0 & 1-e^{-(t-s)\gamma} \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}. \tag{11}$$

This estimate is unbiased, and the error covariance matrix is $\Sigma(t-s) = E[\zeta(t, s)\, \zeta(t, s)^{\mathrm{T}}]$.

According to equalities (2) the yield $y_{\mathrm{NS}}(\tau, r)$ at the point time t is determined by expression

$$y_t(\tau) = L_t(\tau) + S_t \frac{1-e^{-\gamma\tau}}{\gamma\tau} + C_t \left(\frac{1-e^{-\gamma\tau}}{\gamma\tau} - e^{-\gamma\tau} \right) =$$

$$= L(\tau) + (b_1(\tau)\ \ b_2(\tau)) \times \begin{pmatrix} S_t \\ C_t \end{pmatrix} = L(\tau) + B(\tau)\, g(t). \tag{12}$$

Here it is designated $b_1(\tau) = \frac{B_1(\tau)}{\tau} = \frac{1-e^{-\gamma\tau}}{\gamma\tau}$, $b_2(\tau) = \frac{B_2(\tau)}{\tau} = \frac{1-e^{-\gamma\tau}}{\gamma\tau} - e^{-\gamma\tau}$, and also considered that the factor $L_t(\tau)$, as mentioned previously, is independent on current time t. Functions $b_1(\tau)$, $b_2(\tau)$ form a row vector $b(\tau) = (b_2(\tau), b_3(\tau))$, depending only on the τ and does not depend on t. Other vector, a vector-column of factors $g(t) = (S_t, C_t)^{\mathrm{T}}$, depends only from t and does not depend from τ.

Let quotes zero-coupon bonds are produced for n maturity terms τ_i, $i = 1, 2..., n$, for business dates t, $t = 1, 2..., N$. Assume also that the quotes $y_t(\tau_i)$ may contain some random errors ε_{it}. Then the structure of the sample can be described by the following relations

$$y_t(\tau_i) = L(\tau_i) + b(\tau_i)\, g(t) + \varepsilon_{it},\ i = 1, 2..., n,\ t = 1, 2..., N. \tag{13}$$

Let's form by means these equalities a n-vector Y_t from the quotations received in date t. Its structure will be represented by means of n-vector $\mathcal{L}$, $(n\times2)$-matrix $\mathcal{B}$ composed from vectors-row $b(\tau_i)$ of functions of term structure and a n-vector of errors ε_t.

$$\mathcal{B} \equiv \begin{pmatrix} \frac{1-e^{-\gamma\tau_1}}{\gamma\tau_1} & \frac{1-e^{-\gamma\tau_1}}{\gamma\tau_1} - e^{-\gamma\tau_1} \\ \dots & \dots \\ \frac{1-e^{-\gamma\tau_n}}{\gamma\tau_n} & \frac{1-e^{-\gamma\tau_n}}{\gamma\tau_n} - e^{-\gamma\tau_n} \end{pmatrix},\ \mathcal{L} \equiv \begin{pmatrix} L(\tau_1) \\ L(\tau_2) \\ \dots \\ L(\tau_n) \end{pmatrix},\ V \equiv E\,[\varepsilon_t \varepsilon_t^{\mathrm{T}}].$$

So that

$$Y_t = \mathcal{L} + \mathcal{B} g(t) + \varepsilon_t,\ t = 1, 2..., N. \tag{14}$$

From the resulting equation it is possible to estimate a vector $g(t)$ of factors as follows. We multiply equation (14) by the matrix $\mathcal{B}^{\mathrm{T}}$. In this case a multiplier at $g(t)$ becomes square (2×2)-matrix $\mathcal{B}^{\mathrm{T}}\mathcal{B}$, non-singular at $n > 3$. Then we find an inverse matrix $(\mathcal{B}^{\mathrm{T}}\mathcal{B})^{-1}$ and multiply on it at the left the modified equality (14) then the structure of a vector of factors $g(t)$ is easily found in form

$$g(t) = (\mathcal{B}^{\mathrm{T}}\mathcal{B})^{-1}\mathcal{B}^{\mathrm{T}}(Y_t - \mathcal{L}) - (\mathcal{B}^{\mathrm{T}}\mathcal{B})^{-1}\mathcal{B}^{\mathrm{T}}\varepsilon_t.$$

If random errors ε_{it} are independent for various indexes i and t, the vector of errors ε_t has a diagonal covariance matrix $\mathrm{Var}\,[\varepsilon_t] = \mathrm{Diag}\,[\sigma^2_{\varepsilon i} \mid i = 1, 2..., n]$ and a zero mathematical expectation. Therefore the least squares (LS) estimate of a vector of factors

$$\hat{g}(t) = (\mathcal{B}^{\mathrm{T}}\mathcal{B})^{-1}\mathcal{B}^{\mathrm{T}}(Y_t - \mathcal{L}) \tag{15}$$

is unbiased and has a covariance matrix

$$(\mathcal{B}^{\mathrm{T}}\mathcal{B})^{-1}\mathcal{B}^{\mathrm{T}} Y_t\, \mathrm{Var}\,[\varepsilon_t]\, Y_t^{\mathrm{T}} \mathcal{B}\, (\mathcal{B}^{\mathrm{T}}\mathcal{B})^{-1}.$$

If ε_{it} – normally distributed vector then this estimate will be also maximum likelihood (ML) estimate. Note that matrix $\mathcal{B}$ is known, as its rows are $b(\tau)$ with elements $b_1(\tau) = \frac{1-e^{-\gamma\tau}}{\gamma\tau}$, $b_2(\tau) = \frac{1-e^{-\gamma\tau}}{\gamma\tau} - e^{-\gamma\tau}$. As follows from (10), for determination $L(\tau)$ need the only function $A(\tau)$ with the opposite sign. Equality for function $A(\tau)$ actually is not the equation [1] when functions $B_1(\tau)$ and $B_2(\tau)$ are known and determined by expressions (1). If to take the elementary case when a volatility matrix σ in the equation (9) is constant and diagonal, matrix K is set in the form of (7), and risk market prices λ – constants then the derivative of function A (τ) will be such [1]

$$-\frac{dA(\tau)}{d\tau} = \frac{d[\tau L(\tau)]}{d\tau} = (\gamma\theta_1 - \gamma\theta_2 - \sigma_1\lambda_1)\frac{1-e^{-\gamma\tau}}{\gamma} + $$
$$+ (\gamma\theta_2 - \sigma_2\lambda_2)\left(\frac{1-e^{-\gamma\tau}}{\gamma} - \tau e^{-\gamma\tau}\right) - $$
$$- \frac{\sigma_1^2}{2}\left(\frac{1-e^{-\gamma\tau}}{\gamma}\right)^2 - \frac{\sigma_2^2}{2}\left(\frac{1-e^{-\gamma\tau}}{\gamma} - \tau e^{-\gamma\tau}\right)^2. \tag{16}$$

Thus, matrix $\mathcal{B}$ in the ratio (15) for an estimation of a vector of factors $\hat{g}(t)$ can be specified analytically, however integration of expression (16) leads to bulky expression.

Note that in representation (12) $L_t(\tau) \equiv L(\tau)$ on the one hand does not depend on t, and on other – is determined analytically. Therefore this representation can be rewritten as

$$y_t(\tau) - L(\tau) = S_t\frac{1-e^{-\gamma\tau}}{\gamma\tau} + C_t\left(\frac{1-e^{-\gamma\tau}}{\gamma\tau} - e^{-\gamma\tau}\right) = (b_2(\tau), b_3(\tau)) \times \begin{pmatrix} S_t \\ C_t \end{pmatrix}.$$

Let $\tilde{y}_t(\tau) \equiv y_t(\tau) - L(\tau)$. Then the vector equation of observations (14) will be transformed to a form

$$\tilde{Y}_t = \mathcal{B}\, g(t) + \varepsilon_t, \quad t = 1, 2..., N. \tag{17}$$

The relations obtained allow us to construct the optimal mean-squares procedure of estimation the variables $\{S_t, C_t\}$, i.e. sequentially over time to estimate the yield curves in the form (12). This can be done by using a Kalman filter. Suppose that the yield to maturity (13) is listed over regular intervals Δ, so that at each time point $t = k\Delta$, $k = 0, 1, 2...$, is declared n returns $\{y_t(\tau_i), i = 1, 2..., n\}$.

Without stopping on the details of the output of the Kalman filter [5], we present the structure that adapted to our problem.

Let

$$F = e^{-\gamma\Delta}\begin{pmatrix} 1 & \gamma\Delta \\ 0 & 1 \end{pmatrix}.$$

Then the estimate (11) can be written down in a form

$$\begin{pmatrix} \hat{S}_t \\ \hat{C}_t \end{pmatrix}^{-} = F\begin{pmatrix} \hat{S}_{t-\Delta} \\ \hat{C}_{t-\Delta} \end{pmatrix} + (I - F)\begin{pmatrix} \overline{S} \\ \overline{C} \end{pmatrix},$$

where I – the identity matrix, and a minus in the top index means that the estimation is determined under the information which are available till the previous moment $t - \Delta$ inclusive. A covariance matrix P_t^- of this estimation satisfies to a relation

$$P_t^- = F P_{t-\Delta} F^{\mathrm{T}} + \Sigma.$$

Here availability of a minus in the top index P_t^- means the same, as in previous, and absence of such index in $P_{t-\Delta}$ means that the covariance matrix was determined under the information which are available till the moment, specified in the bottom index inclusive. Σ – a covariance matrix $E\,[\zeta(t, t-\Delta)\,\zeta(t, t-\Delta)^{\mathrm{T}}]$.

Suppose further that the vector equation of observations has the form (17)

Let's enter into matrix consideration

$$\mathcal{K}_t = P_t^- \mathcal{B}^{\mathrm{T}} (\mathcal{B} P_t^- \mathcal{B}^{\mathrm{T}} + V)^{-1}, \; P_t = (I - \mathcal{K}_t \mathcal{B}) P_t^-.$$

Matrix $\mathcal{K}_t$ determines the optimum compromise of contributions in an estimation of variables $\{S_t, C_t\}$: by observation Y_t arrived at the time point t and by estimates of these variables under the information, which are available till the time point $t - \Delta$ inclusive. The covariance matrix P_t is necessary for realization of the following iteration of algorithm at the point of time $t + \Delta$. Note that as matrixes F, Σ, H, V, $\mathcal{L}$ do not change with time and assumed to be known prior to the beginning of procedure of estimation, then sequence of matrixes P_t^-, $\mathcal{K}_t$, P_t can be calculated in advance prior to the beginning of recurrent procedure of estimation. The final estimate of variables $\{S_t, C_t\}$ at time t is calculated by the formula

$$\begin{pmatrix} \hat{S}_t \\ \hat{C}_t \end{pmatrix} = \begin{pmatrix} \hat{S}_t \\ \hat{C}_t \end{pmatrix}^{-} + \mathcal{K}_t \tilde{Y}_t - \mathcal{K}_t \mathcal{B} \begin{pmatrix} \hat{S}_t \\ \hat{C}_t \end{pmatrix}^{-} = \mathcal{K}_t \tilde{Y}_t + (I - \mathcal{K}_t \mathcal{B}) \begin{pmatrix} \hat{S}_t \\ \hat{C}_t \end{pmatrix}^{-} =$$

$$= \mathcal{K}_t (Y_t - \mathcal{L}) + (I - \mathcal{K}_t \mathcal{B}) \left[F \begin{pmatrix} \hat{S}_{t-\Delta} \\ \hat{C}_{t-\Delta} \end{pmatrix} + (I - F) \begin{pmatrix} \overline{S} \\ \overline{C} \end{pmatrix} \right].$$

This formula allows recurrently during the receiving of observations Y_t compute the best mean-square estimate of factors $\{S_t, C_t\}$. Unfortunately, for realization of such procedure it is necessary to know all model parameters, i.e. γ, θ_1, θ_2, λ_1, λ_2, $\{\sigma_{ik}\}$, or to implement procedure of their estimation, which is a separate problem. An initial vector of estimations of factors $\{S_0, C_0\}$ can be obtained using (15) on the initial observation Y_0:

$$\begin{pmatrix} \hat{S}_0 \\ \hat{C}_0 \end{pmatrix} = (\mathcal{B}^{\mathrm{T}} \mathcal{B})^{-1} \mathcal{B}^{\mathrm{T}} (Y_0 - \mathcal{L}).$$

Conclusion

The description of the Nelson – Siegel (NS) yield models and the Nelson – Siegel – Svensson (NSS) yield models as a traditional multi-dimensional affine yield models is presented. NS model it turns out the two-factor model, in the case of NSS model – the four-factor. This description differs from the representations of F. Diebold and G. Rudebusch [6] so that the dimension of the models is reduced by one, which simplifies the calculations. It was found that the Nelson – Siegel latent variables coincide with the state variables of the traditional models. The explicit representation of the probability distribution and the first two moments of these «latent variables» are obtained. It is computed the expectations and covariance matrices of interest rates of the yield to maturity and forward rates. It is formulated the procedure of latent variables recursion estimation based on the Kalman filter.

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Доходности Нельсона – Сигеля – Свенссона. Вероятностные свойства и оценивание.

Ключевые слова: временная структура; кривая доходности; аффинные модели; модель доходности Нельсона – Сигеля – Свенссона; фильтр Калмана.

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Рассмотрены вероятностные свойства процентной ставки доходности, порождаемой моделями Нельсона – Сигеля (NS) и Нельсона – Сигеля – Свенссона (NSS). Описание моделей доходностей Нельсона – Сигеля и Нельсона – Сигеля – Свенссона представлено как изложение традиционных многомерных аффинных моделей доходности. Показано, что модель Нельсона – Сигеля практически не отличается от традиционной двухфакторной модели аффинной доходности с волатильностью, не зависящей от переменных состояния рынка, соответственно, модель Нельсона – Сигеля – Свенссона – от четырехфакторной такой модели. Такое описание отличается от известных представлений Ф. Диболда и Г. Радебуша тем, что размерность представленных в статье моделей уменьшается на единицу, что приводит к упрощению вычислений. Выяснено, что скрытые

переменные Нельсона – Сигеля полностью совпадают с переменными состояния традиционной модели. Эти модели порождают процентные ставки доходности до погашения и форвардной доходности с нормальным распределением, для которого найдены математические ожидания и ковариационные матрицы в явном виде. Для оценивания значений ставок доходности до погашения в текущем времени сформулирована рекуррентная процедура, основанная на применении фильтра Калмана.

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