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ON p^nBext PROJECTIVE ABELIAN p-GROUPS[1]

We introduce the concept of p^nBext projective abelian p-groups and show that they form a class which properly contains the class of all n-balanced projective p-groups. This somewhat enlarges a result due to Keef-Danchev in Houston J. Math. (2012).

Keywords: *balanced projectives*, *n-balanced projectives*, *p^nBext projectives.*

1. Introduction and Background

Everywhere in the text of this brief paper our groups are p-primary abelian, where p is a fixed prime for the duration of the article. The undefined explicitly below notions and notations are in agreement with [5]. For instance, a group G is called *balanced projective* if the equality $\mathrm{Bext}(G,X) = \{0\}$ holds for all groups X. In order to generalize this, imitating [3], for any integer $n \geq 0$, we say that the short exact sequence $0 \to X \to Y \to G \to 0$ is *n-balanced exact* if it represents an element of $p^n\mathrm{Bext}(G,X)$. Thus we will say that a group G is *n-balanced projective* provided every such n-balanced exact sequence splits. Evidently, these two notions coincide when $n = 0$.

It is worthwhile noticing that certain non-trivial properties of these groups are given in [3] (see also [4]). These ideas lead us to the next new concept:

Definition 1.1. Let $n \geq 0$. A group G is said to be *p^nBext-projective* if

$$(\forall X), p^n - \mathrm{Bext}(G,X) = \{0\}.$$

The aim of this note is to prove that each n-balanced projective group is p^n yields Bext-projective but the converse fails. We close the work with a specific question arisen from unexpected difficulties in the proof of the central statement.

2. Main Result and Problem

Theorem 2.1. *Suppose that G is a group and $n < \omega$ is a natural. If G is n-balanced projective, then it is p^nBext-projective.*

Proof. Letting the short exact sequence E defined by

$$0 \to X \to B \overset{f}{\to} G \to 0$$

is in $p^n\mathrm{Bext}(G,X)$, then there is another element E' of $\mathrm{Bext}(G,X)$ given by

$$0 \to X \to B' \overset{f'}{\to} G \to 0$$

such that the following pull-back diagram can be completed:

[1] 2010 Mathematics Subject Classication. 20 K10.

$$\begin{array}{ccccccccc} 0 & \to & X & \to & B & \xrightarrow{f} & G & \to & 0 \\ & & \| & & \downarrow & & \downarrow p^n & & \\ 0 & \to & X & \to & B' & \xrightarrow{f'} & G & \to & 0. \end{array}$$

Let $H(G)$ be the standard functorial group, depending on G, as defined in [7] (cf. [3] too). If $\pi_G : H(G) \to G$ is our usual homomorphism used in defining E_G, then $p^n(\pi_G(G[p^n])) = \{0\}$, so $p^n\pi_G$ induces a homomorphism $H(G)/G[p^n] \to G$. Since E' is balanced exact and $H(G)/G[p^n]$ is totally projective, there is a homomorphism $g' : H(G) \to B'$ such that $f' \circ g' = p^n\pi_G$. Now the well-known universal properties of pull-back diagrams yield that there exists a homomorphism $g : H(G) \to B$ B such that $f \circ g = \pi_G$. However, this means that E is n-balanced exact. Since we are assuming G is n-balanced projective, it follows now that E splits, as wanted. □

Example 2.2. There is a pBext-projective group which is not 1-balanced projective.

Proof. Referring to [6] there exists a summable C_{ω_1}-group A which is a proper p^{ω_1+1}-projective group (thus it is manifestly *not* totally projective by virtue of [5]). Moreover, since it is summable, it follows that it is also not 1-balanced projective.

However, on the other side, since A is a C_{ω_1}-group, it follows that $\mathrm{Bext}(A,X) = p^{\omega_1}\mathrm{Ext}(A,X)$ for all groups X (compare with [5]). And finally, because it is a p^{ω_1+1}-projective as well, we can conclude that A has to be pBext-projective, as claimed. □

A reasonable query is whether or not for any $n \geq 2$ does there exist a p^nBext-projective group that is not n-totally projective? Resuming, we have restricted our attention only on $n = 1$, though essentially the same argument works for larger values of n (see cf. [1] and [2] too). In fact, last argument stated above asserts that any element of $p^n\mathrm{Bext}(A,X)$ will be n-balanced exact, so that every group which is projective with respect to the collection of n-balanced exact sequences will also be projective with respect to the functor p^nBext. The second assertion then implies that there are n-balanced exact sequences that are not elements of $p^n\mathrm{Bext}(A,X)$.

Besides, notice that the totally projective (i.e., the balanced projective) groups are exactly p^0Bext-projective groups, and there are an abundance of them. Nevertheless, it is actually *not* at all clear whether there are enough p^nBext-projectives whenever $n > 0$. So, the following homological question is of some interest:

Problem. Is it true that the collection of n-balanced exact sequences form the largest subfunctor of p^nBext which does have this important homological property?

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Вводится понятие p^nBext проективных абелевых *p*-групп и доказывается, что эти группы образуют собственный подкласс в классе всех *n*-сбалансированных проективных *p*-групп. Данное утверждение улучшает соответствующий полученный результат, опубликованный Кифом и Данчевым в журнале Houston J. Math. (2012).

Ключевые слова: сбалансированная проективность, *n*-сбалансированная проективность, p^nBext проективность.