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THE SENSITIVITY FUNCTIONALS IN THE BOLTS'S PROBLEM FOR MULTIVARIATE DYNAMIC SYSTEMS DESCRIBED BY ORDINARY INTEGRO-DIFFERENTIAL EQUATIONS

The variational method of calculation of sensitivity functional (connecting first variation of quality functional with variations of variable and constant parameters) for multivariate non-linear dynamic systems, described by ordinary Volterra's the second-kind integro-differential equations by use of the generalized quality functional of a dynamic system (the Bolts problem) is developed.
Key words: variational method; sensitivity functional; integro-differential equation; conjugate equation.

The problem of calculation of sensitivity functional (SF) (connecting first variation of quality functional with variations of variable and constant parameters) and sensitivity coefficients (SC) (components of vector gradient from quality functional according to constant parameters) of dynamic systems are main at the analysis and syntheses of control laws, identification, optimization [1–7]. The first-order sensitivity indexes are most used. Later on we shall examine only SF and SC of the first-order.

Consider a vector output $x(t)$ of dynamic object model under continuous time $t \in [t_0, t^1]$, implicitly depending on vectors parameters $\tilde{\alpha}(t), \overline{\alpha}$, and functional I constructed on the basis of $x(t)$ under $t \in [t_0, t^1]$. The first variation δI and variations $\delta\tilde{\alpha}(t)$ are connected with each other with the help of a single-line functional – SF with respect to variable parameters $\tilde{\alpha}(t)$: $\delta_{\tilde{\alpha}(t)} I = \int_{t_0}^{t^1} V(t)\delta\tilde{\alpha}(t)dt$. SC with respect to constant parameters $\overline{\alpha}$ are called a gradient from I on $\overline{\alpha}$: $(dI / d\overline{\alpha})^T \equiv \nabla_{\overline{\alpha}} I$. SC are a coefficients of single-line relationship between the first variation of functional δI and the variations of constant parameters $\overline{\alpha}$: $\delta_{\overline{\alpha}} I = (dI / d\overline{\alpha})\delta\overline{\alpha} \equiv \sum_{j=1}^{m} (\partial I / \partial\overline{\alpha}_j)\delta\overline{\alpha}_j$.

For simplest classes of dynamic systems it is shown, that at the SC calculation it is possible to pass from a solution of the bulky sensitivity equations to a solution of the conjugate equations – conjugated with respect to object dynamic equations. Method of receipt of conjugate equations (it was offered in 1962) is cumbersome, because it is based on the analysis of sensitivity equations, and it does not get its developments.

Variational method [4], ascending to Lagrange's, Hamilton's, Euler's memoirs, makes possible to simplify the process of determination of conjugate equations and formulas of account of SC and SF. On the basis of this method it is an extension of quality functional by means of including into it object dynamic equations by means of Lagrange's multipliers and obtaining the first variation of extended functional in phase coordinates of object and on interesting parameters. The dynamic equations for Lagrange's multipliers are obtained from a condition of equality to zero of appropriate components (relative to phase coordinates) of the first variation of an extended functional. Given simplification of first variation of extended functional brings at presence in the right part only parameter variations, i.e. it is produced the SF. If all parameters are constant then the parameters variations are carried out from corresponding integrals and at the final result in obtained functional variation the coefficients before parameters variations are the required SC. Given method was used in [8] for dynamic systems described by ordinary continuous Volterra's of the second-kind integral and integro-differential equations.

In this paper the variational method of account of SF develops to more general continuous many-dimensional non-linear dynamic systems circumscribed by the vectorial non-linear continuous Volterra's of the second-kind integro-differential equations with variable $\tilde{\alpha}(t)$ and constant $\overline{\alpha}$ parameters and with reviewing of generalised quality functional (the Bolts problem) and registration of dependencies: 1) disturbing actions of a object model from initial instant; 2) of initial t_0 and final t^1 instants from constant parameters $\overline{\alpha}$.

1. Statement of the problem

Consider that the dynamic object is described by system of non-linear continuous integro-differential equations (I-DE) with integral components of Volterra's type of the second genus and with variable $\tilde{\alpha}(t)$ and constant $\overline{\alpha}$ parameters

$$\dot{x}(t) = f(x(t), y(t), \tilde{\alpha}(t), \overline{\alpha}, t), \ \ t_0 < t \le t^1, \ x(t_0) = x_0(\overline{\alpha}, t_0), \tag{1}$$

$$y(t) = r(\tilde{\alpha}(t), \overline{\alpha}, t_0, t) + \int_{t_0}^{t} K(t, x(s), y(s), \tilde{\alpha}(s), \overline{\alpha}, s)\, ds, \ \ t_0 \le t \le t^1, \ t_0 = t_0(\overline{\alpha}), \ t^1 = t^1(\overline{\alpha}).$$

Here: the magnitudes of initial t_0 and final t^1 and initial values $x(t_0)$ are known functions from constant parameters $\overline{\alpha}$: $t_0 = t_0(\overline{\alpha}), t^1 = t^1(\overline{\alpha})$, $x(t_0) = x_0(\overline{\alpha}, t_0)$; x, y – a vector-columns of phase coordinates; $\tilde{\alpha}(t), \overline{\alpha}$ – vector-columns of interesting variable and constant parameters; $f(\cdot)$, $r(\cdot)$, $K(\cdot)$, $t_0(\overline{\alpha})$, $t^1(\overline{\alpha})$, $x_0(\cdot)$ – known continuously differentiated limited vector-functions.

The model of a measuring device is given as

$$\eta(t) = \eta(x(t), y(t), \tilde{\alpha}(t), \overline{\alpha}, t), \ t \in [t_0, t^1], \tag{2}$$

where $\eta(\cdot)$ is also continuous, differentiated, limited (together with the first derivatives) vector-function. The required parameters $\tilde{\alpha}(t), \overline{\alpha}$ are inserted also in (2). Dimensionality of vectors x, y and η in general event can be different.

On the basis of output coordinates of a measuring device η the quality functional of a dynamic system is constructed:

$$I(\alpha) = \int_{t_0}^{t^1} f_0(\eta(t), \tilde{\alpha}(t), \overline{\alpha}, t)\, dt + I_1(\eta(t^1), \overline{\alpha}, t^1). \tag{3}$$

Conditions for functions $f_0(\cdot)$, $I_1(\cdot)$ are the same as for $f(\cdot)$, $r(\cdot)$, $K(\cdot)$, $t_0(\cdot)$, $t^1(\cdot)$, $x_0(\cdot)$, etc.

With use of a functional (3) in the optimization problem (in the theory of optimal control) known as the Bolts's problem. From it as the individual variants follows: Lagrange's problem (when there is only integrated component) and Mayer's problem (when there is only second component – function from phase coordinates at a finishing point).

With the purpose of simplification of appropriate deductions with preservation of a generality in all transformations (1)–(3) there are a two vectors of parameters $\tilde{\alpha}(t), \overline{\alpha}$. If in the equations (1)–(3) the parameters are different then it is possible formally to unite them in one vector $\tilde{\alpha}(t)$ or else $\overline{\alpha}$, to use obtained outcomes and then, taking into account a structure of a vectors $\tilde{\alpha}(t), \overline{\alpha}$, to make appropriate simplifications.

The scheme of interaction between variables of object model, measuring device and quality functional is shown in fig. 1.

By obtaining of results the obvious designations:

$$f(t) \equiv f(x(t), y(t), \tilde{\alpha}(t), \overline{\alpha}, t), \ \ r(t) \equiv r(\tilde{\alpha}(t), \overline{\alpha}, t_0, t), K(t,s) \equiv K(t, x(s), y(s), \tilde{\alpha}(t), \overline{\alpha}, s),$$

$$\eta(t) \equiv \eta(x(t), y(t), \tilde{\alpha}(t), \overline{\alpha}, t), f_0(t) \equiv f_0(\eta(t), \tilde{\alpha}(t), \overline{\alpha}, t), \ \ I_1(t^1) \equiv I_1(\eta(t^1), \overline{\alpha}, t^1)$$

are used.

Let's pass from the integro-differential equations (I-DE) to integral equations (IE), we shall apply results of paper [9] and then shall return to initial variables.

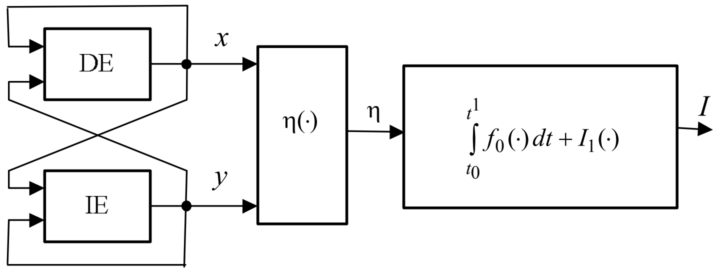

Fig. 1. Scheme of interaction between variables in (1)–(3)

The convenience of a integro-differential model consists in its structural universality. At simplification of a model it is enough in final results to convert in a zero appropriate addends. This reception we shall apply in a final part of this paper.

2. Basic result

In I-DE (1) the we write differential equation in the integral form

$$x(t) = x_0(\overline{\alpha}, t_0) + \int_{t_0}^{t} f(x(s), y(s), \tilde{\alpha}(s), \overline{\alpha}, s)ds, \ \ t_0 \le t \le t^1 . \tag{4}$$

We use notations

$$\tilde{y}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \ \tilde{r}(\tilde{\alpha}(t), \overline{\alpha}, t_0, t) = \begin{pmatrix} x_0(\overline{\alpha}, t_0) \\ r(\tilde{\alpha}(t), \overline{\alpha}, t_0, t) \end{pmatrix} \equiv \begin{pmatrix} x_0(\overline{\alpha}, t_0) \\ r(t) \end{pmatrix} \equiv \tilde{r}(t) ,$$

$$\tilde{K}(t, \tilde{y}(s), \tilde{\alpha}(s), \overline{\alpha}, s) = \begin{pmatrix} f(x(s), y(s), \tilde{\alpha}(s), \overline{\alpha}, s) \\ K(t, x(s), y(s), \tilde{\alpha}(s), \overline{\alpha}, s) \end{pmatrix} \equiv \begin{pmatrix} f(s) \\ K(t,s) \end{pmatrix} \equiv \tilde{K}(t,s) , \tag{5}$$

and obtain IE

$$\tilde{y}(t) = \tilde{r}(\tilde{\alpha}(t), \overline{\alpha}, t_0, t) + \int_{t_0}^{t} \tilde{K}(t, \tilde{y}(s), \tilde{\alpha}(s), \overline{\alpha}, s)ds , \ \ t_0 \le t \le t^1 . \tag{6}$$

In further also a notation

$$\eta(t) \equiv \eta(\tilde{y}(t), \tilde{\alpha}(t), \overline{\alpha}, t) \tag{7}$$

is used for a model of a measuring device.

We apply results of paper [9] for models (4), (6) and functional (3) with variables and constant parameters $\tilde{\alpha}(t), \overline{\alpha}$. The conjugate equations for Lagrange's multipliers $\gamma(t)$ have a form:

$$\gamma^T(t) = \Phi(t^1) \frac{\partial \tilde{K}(t^1, t)}{\partial \tilde{y}(t)} + \frac{\partial f_0(t)}{\partial \eta(t)} \frac{\partial \eta(t)}{\partial \tilde{y}(t)} + \int_{t}^{t^1} \gamma^T(s) \frac{\partial \tilde{K}(s,t)}{\partial \tilde{y}(t)} ds , \ \ t_0 \le t \le t^1 . \tag{8}$$

Here: $\gamma(t)$ is column vector; $\dfrac{\partial I_1(t^1)}{\partial \eta(t^1)} \dfrac{\partial \eta(t^1)}{\partial \tilde{y}(t^1)} \equiv \Phi(t^1)$. These conjugate equations are decided in the opposite direction of time (from t^1).

The form of the conjugate equations doesn't change, but they contain variables and constant parameters $\tilde{\alpha}(t), \overline{\alpha}$.

SF for an integral model (6) has the form [9]:

$$\delta I = \delta_{\tilde{\alpha}(t)} I + \delta_{\tilde{\alpha}(t^1)} I + \delta_{\overline{\alpha}} I ,$$

$$\delta_{\tilde{\alpha}(t)} I=\int_{t_0}^{t^1}\Big[\frac{\partial f_0(t)}{\partial\eta(t)}\frac{\partial\eta(t)}{\partial\tilde{\alpha}(t)}+\frac{\partial f_0(t)}{\partial\tilde{\alpha}(t)}+\gamma^T(t)\frac{\partial\tilde{r}(t)}{\partial\tilde{\alpha}(t)}+\Phi(t^1)\frac{\partial\tilde{K}(t^1,t)}{\partial\tilde{\alpha}(t)}+\int_t^{t^1}\gamma^T(s)\frac{\partial\tilde{K}(s,t)}{\partial\tilde{\alpha}(t)}ds\Big]\delta\tilde{\alpha}(t)\,dt\,;$$

$$\delta_{\tilde{\alpha}(t^1)} I=\Big[\frac{\partial I_1(t^1)}{\partial\eta(t^1)}\frac{\partial\eta(t^1)}{\partial\tilde{\alpha}(t^1)}+\Phi(t^1)\frac{\partial\tilde{r}(t^1)}{\partial\tilde{\alpha}(t^1)}\Big]\delta\tilde{\alpha}(t^1)\,;$$

$$\delta_{\overline{\alpha}} I=\left\{\frac{\partial I_1(t^1)}{\partial\eta(t^1)}\frac{\partial\eta(t^1)}{\partial\overline{\alpha}}+\frac{\partial I_1(t^1)}{\partial\overline{\alpha}}+\Phi(t^1)\left[\frac{\partial\tilde{r}(t^1)}{\partial\overline{\alpha}}+\int_{t_0}^{t^1}\frac{\partial\tilde{K}(t^1,s)}{\partial\overline{\alpha}}ds\right]+\right.$$

$$+\int_{t_0}^{t^1}\Big[\frac{\partial f_0(t)}{\partial\eta(t)}\frac{\partial\eta(t)}{\partial\overline{\alpha}}+\frac{\partial f_0(t)}{\partial\overline{\alpha}}+\gamma^T(t)\frac{\partial\tilde{r}(t)}{\partial\overline{\alpha}}+\int_t^{t^1}\gamma^T(s)\frac{\partial\tilde{K}(s,t)}{\partial\overline{\alpha}}ds\Big]dt+$$

$$+\left[\Phi(t^1)\Big[\frac{\partial\tilde{r}(t^1)}{\partial t_0}-\tilde{K}(t^1,t_0)\Big]-f_0(t_0)+\int_{t_0}^{t^1}\gamma^T(t)\Big(\frac{\partial\tilde{r}(t)}{\partial t_0}-\tilde{K}(t,t_0)\Big)dt\right]\frac{dt_0}{d\overline{\alpha}}+$$

$$\left.+\left[\Phi(t^1)\Big[\frac{\partial\tilde{r}(t^1)}{\partial t^1}+\tilde{K}(t^1,t^1)+\int_{t_0}^{t^1}\frac{\partial\tilde{K}(t^1,s)}{\partial t^1}ds\Big]+\frac{\partial I_1(t^1)}{\partial\eta(t^1)}\frac{\partial\eta(t^1)}{\partial t^1}+\frac{\partial I_1(t^1)}{\partial t^1}+f_0(t^1)\right]\frac{dt^1}{d\overline{\alpha}}\right\}d\overline{\alpha}\,. \qquad (9)$$

It is necessary in (8), (9) to fulfil matrix transformations with the registration earlier entered notations (6), and also

$$\gamma(t)=\begin{pmatrix}\gamma_x(t)\\ \gamma_y(t)\end{pmatrix},\ \gamma^T(t)=\big(\gamma_x^T(t);\quad \gamma_y^T(t)\big),$$

i.e. $\Phi(t^1)\equiv\frac{\partial I_1(t^1)}{\partial\eta(t^1)}\frac{\partial\eta(t^1)}{\partial\tilde{y}(t^1)}=\left(\frac{\partial I_1(t^1)}{\partial\eta(t^1)}\frac{\partial\eta(t^1)}{\partial x(t^1)};\quad \frac{\partial I_1(t^1)}{\partial\eta(t^1)}\frac{\partial\eta(t^1)}{\partial y(t^1)}\right)\equiv\big(\Phi_x(t^1);\quad \Phi_y(t^1)\big)$,

$$\frac{\partial\tilde{K}(t^1,t)}{\partial\tilde{y}(t)}=\begin{pmatrix}\frac{\partial f(t)}{\partial x(t)}; & \frac{\partial f(t)}{\partial y(t)}\\ \frac{\partial K(t^1,t)}{\partial x(t)}; & \frac{\partial K(t^1,t)}{\partial y(t)}\end{pmatrix},\quad \frac{\partial r(t^1)}{\partial\overline{\alpha}}=\begin{pmatrix}\frac{\partial x_0(\overline{\alpha},t_0)}{\partial\overline{\alpha}}\\ \frac{\partial r(t^1)}{\partial\overline{\alpha}}\end{pmatrix},$$

$$\Phi(t^1)\frac{\partial\tilde{K}(t^1,t)}{\partial\tilde{y}(t)}=\left(\Phi_x(t^1)\frac{\partial f(t)}{\partial x(t)}+\Phi_y(t^1)\frac{\partial K(t^1,t)}{\partial x(t)};\quad \Phi_x(t^1)\frac{\partial f(t)}{\partial y(t)}+\Phi_y(t^1)\frac{\partial K(t^1,t)}{\partial y(t)}\right),$$

$$\frac{\partial f_0(t)}{\partial\eta(t)}\frac{\partial\eta(t)}{\partial\tilde{y}(t)}=\left(\frac{\partial f_0(t)}{\partial\eta(t)}\frac{\partial\eta(t)}{\partial x(t)};\quad \frac{\partial f_0(t)}{\partial\eta(t)}\frac{\partial\eta(t)}{\partial y(t)}\right),$$

$$\gamma^T(s)\frac{\partial\tilde{K}(s,t)}{\partial\tilde{y}(t)}=\left(\gamma_x^T(s)\frac{\partial f(t)}{\partial x(t)}+\gamma_y^T(s)\frac{\partial K(s,t)}{\partial x(t)};\ \gamma_x^T(s)\frac{\partial f(t)}{\partial y(t)}+\gamma_y^T(s)\frac{\partial K(s,t)}{\partial y(t)}\right),$$

$$\Phi(t^1)\frac{\partial\tilde{r}(t^1)}{\partial\overline{\alpha}}=\Phi_x(t^1)\frac{\partial x_0(\overline{\alpha},t_0)}{\partial\overline{\alpha}}+\Phi_y(t^1)\frac{\partial r(t^1)}{\partial\overline{\alpha}},\ \gamma^T(t)\frac{\partial\tilde{r}(t)}{\partial\overline{\alpha}}=\gamma_x^T(t)\frac{\partial x_0(\overline{\alpha},t_0)}{\partial\overline{\alpha}}+\gamma_y^T(t)\frac{\partial r(t)}{\partial\overline{\alpha}}, \qquad (10)$$

$$\Phi(t^1)\frac{\partial\tilde{K}(t^1,t)}{\partial\overline{\alpha}}=\Phi_x(t^1)\frac{\partial f(t)}{\partial\overline{\alpha}}+\Phi_y(t^1)\frac{\partial K(t^1,t)}{\partial\overline{\alpha}},\ \int_t^{t^1}\gamma^T(s)\frac{\partial\tilde{K}(s,t)}{\partial\overline{\alpha}}ds=\int_t^{t^1}\Big[\gamma_x^T(s)\frac{\partial f(t)}{\partial\overline{\alpha}}+\gamma_y^T(s)\frac{\partial K(s,t)}{\partial\overline{\alpha}}\Big]ds,\ \text{etc.}$$

In the total we obtain the conjugate equations for Lagrange's multipliers

$$\gamma_x^T(t)=\Phi_x(t^1)\frac{\partial f(t)}{\partial x(t)}+\Phi_y(t^1)\frac{\partial K(t^1,t)}{\partial x(t)}+\frac{\partial f_0(t)}{\partial\eta(t)}\frac{\partial\eta(t)}{\partial x(t)}+\int_t^{t^1}\Big[\gamma_x^T(s)\frac{\partial f(t)}{\partial x(t)}+\gamma_y^T(s)\frac{\partial K(s,t)}{\partial x(t)}\Big]ds\,, \qquad (11)$$

$$\gamma_y^T(t)=\Phi_x(t^1)\frac{\partial f(t)}{\partial y(t)}+\Phi_y(t^1)\frac{\partial K(t^1,t)}{\partial y(t)}+\frac{\partial f_0(t)}{\partial\eta(t)}\frac{\partial\eta(t)}{\partial y(t)}+\int_t^{t^1}\Big[\gamma_x^T(s)\frac{\partial f(t)}{\partial y(t)}+\gamma_y^T(s)\frac{\partial K(s,t)}{\partial y(t)}\Big]ds\,,\ t_0\le t\le t^1\,.$$

The first variation of a functional I in relation to variable $\tilde{\alpha}(t)$ and constant $\overline{\alpha}$ parameters has three components:

$$\delta I = \delta_{\tilde{\alpha}(t)} I + \delta_{\tilde{\alpha}(t^1)} I + \delta_{\overline{\alpha}} I ; \tag{12}$$

$$\delta_{\tilde{\alpha}(t)} I = \int_{t_0}^{t^1} \Bigg[\frac{\partial f_0(t)}{\partial \eta(t)} \frac{\partial \eta(t)}{\partial \tilde{\alpha}(t)} + \frac{\partial f_0(t)}{\partial \tilde{\alpha}(t)} + \gamma_y^T(t) \frac{\partial r(t)}{\partial \tilde{\alpha}(t)} + \Phi_x(t^1) \frac{\partial f(t)}{\partial \tilde{\alpha}(t)} + \Phi_y(t^1) \frac{\partial K(t^1,t)}{\partial \tilde{\alpha}(t)} +$$

$$+ \int_t^{t^1} [\gamma_x^T(s) \frac{\partial f(t)}{\partial \tilde{\alpha}(t)} + \gamma_y^T(s) \frac{\partial K(s,t)}{\partial \tilde{\alpha}(t)}] ds \Bigg] \delta \tilde{\alpha}(t) \, dt ;$$

$$\delta_{\tilde{\alpha}(t^1)} I = [\frac{\partial I_1(t^1)}{\partial \eta(t^1)} \frac{\partial \eta(t^1)}{\partial \tilde{\alpha}(t^1)} + \Phi_y(t^1) \frac{\partial r(t^1)}{\partial \tilde{\alpha}(t^1)}] \delta \tilde{\alpha}(t^1) ; \; \delta_{\overline{\alpha}} I = \Bigg\{ \frac{\partial I_1(t^1)}{\partial \eta(t^1)} \frac{\partial \eta(t^1)}{\partial \overline{\alpha}} + \frac{\partial I_1(t^1)}{\partial \overline{\alpha}} +$$

$$+ \Phi_x(t^1) \Bigg[\frac{\partial x_0(\overline{\alpha}, t_0)}{\partial \overline{\alpha}} + \int_{t_0}^{t^1} \frac{\partial f(s)}{\partial \overline{\alpha}} ds \Bigg] + \Phi_y(t^1) \Bigg[\frac{\partial r(t^1)}{\partial \overline{\alpha}} + \int_{t_0}^{t^1} \frac{\partial K(t^1,s)}{\partial \overline{\alpha}} ds \Bigg] +$$

$$+ \int_{t_0}^{t^1} \Bigg[\frac{\partial f_0(t)}{\partial \eta(t)} \frac{\partial \eta(t)}{\partial \overline{\alpha}} + \frac{\partial f_0(t)}{\partial \overline{\alpha}} + \gamma_x^T(t) \frac{\partial x_0(\overline{\alpha}, t_0)}{\partial \overline{\alpha}} + \gamma_y^T(t) \frac{\partial r(t)}{\partial \overline{\alpha}} +$$

$$+ \int_t^{t^1} [\gamma_x^T(s) \frac{\partial f(t)}{\partial \overline{\alpha}} + \gamma_y^T(s) \frac{\partial K(s,t)}{\partial \overline{\alpha}}] ds \Bigg] dt + \Bigg[\Phi_x(t^1) [\frac{\partial x_0(\overline{\alpha}, t_0)}{\partial t_0} - f(t_0)] +$$

$$+ \Phi_y(t^1) [\frac{\partial r(t^1)}{\partial t_0} - K(t^1, t_0)] - f_0(t_0) + \int_{t_0}^{t^1} \gamma_x^T(t) dt (\frac{\partial x_0(\overline{\alpha}, t_0)}{\partial t_0} - f(t_0)) + \int_{t_0}^{t^1} \gamma_y^T(t) (\frac{\partial r(t)}{\partial t_0} - K(t, t_0)) dt \Bigg] \frac{dt_0}{d\overline{\alpha}} +$$

$$+ \Bigg[\Phi_x(t^1) f(t^1) + \Phi_y(t^1) [\frac{\partial r(t^1)}{\partial t^1} + K(t^1, t^1) + \int_{t_0}^{t^1} \frac{\partial K(t^1, s)}{\partial t^1} ds] + \frac{\partial I_1(t^1)}{\partial \eta(t^1)} \frac{\partial \eta(t^1)}{\partial t^1} + \frac{\partial I_1(t^1)}{\partial t^1} + f_0(t^1) \Bigg] \frac{dt^1}{d\overline{\alpha}} \Bigg\} d\overline{\alpha} .$$

It is expedient to add the conjugate equations for Lagrange's multipliers (11) too form of the integro-differential equations.

We enter new variable $\Phi_x(t^1) + \int_t^{t^1} \gamma_x^T(s) ds = \lambda_x^T(t)$, either $\gamma_x^T(t) = -\dot{\lambda}_x^T(t), \;\; \lambda_x^T(t^1) = \Phi_x(t^1)$, and change an order of integrating in double integral inside of triangular area [9]: $\int_{t_0}^{t^1} \int_{t_0}^{t} A(t,s) \, ds \, dt = \int_{t_0}^{t^1} \int_{t}^{t^1} A(s,t) \, ds \, dt$. Then conjugate equations (11) are noted as (13) and SF (see (12)) are calculated under the formula (14).

Conjugate equations have the form

$$- \dot{\lambda}_x^T(t) = \lambda_x^T(t) \frac{\partial f(t)}{\partial x(t)} + \Phi_y(t^1) \frac{\partial K(t^1,t)}{\partial x(t)} + \frac{\partial f_0(t)}{\partial \eta(t)} \frac{\partial \eta(t)}{\partial x(t)} + \int_t^{t^1} \gamma_y^T(s) \frac{\partial K(s,t)}{\partial x(t)} ds , \; t \in [t_0, t^1) , \; \lambda_x^T(t^1) = \Phi_x(t^1) ,$$

$$\gamma_y^T(t) = \lambda_x^T(t) \frac{\partial f(t)}{\partial y(t)} + \Phi_y(t^1) \frac{\partial K(t^1,t)}{\partial y(t)} + \frac{\partial f_0(t)}{\partial \eta(t)} \frac{\partial \eta(t)}{\partial y(t)} + \int_t^{t^1} \gamma_y^T(s) \frac{\partial K(s,t)}{\partial y(t)} ds , t_0 \le t \le t^1 . \tag{13}$$

Here $\Phi_x(t^1) \equiv \frac{\partial I_1(t^1)}{\partial \eta(t^1)} \frac{\partial \eta(t^1)}{\partial x(t^1)}$, $\Phi_y(t^1) \equiv \frac{\partial I_1(t^1)}{\partial \eta(t^1)} \frac{\partial \eta(t^1)}{\partial y(t^1)}$.

SF are calculated under the formula:

$$\delta I=\delta_{\tilde{\alpha}(t)}I+\delta_{\tilde{\alpha}(t^1)}I+\delta_{\bar{\alpha}}I\,;\;\;\delta_{\tilde{\alpha}(t)}I=\int_{t_0}^{t^1}\left[\frac{\partial f_0(t)}{\partial\eta(t)}\frac{\partial\eta(t)}{\partial\tilde{\alpha}(t)}+\frac{\partial f_0(t)}{\partial\tilde{\alpha}(t)}+\gamma_y^T(t)\frac{\partial r(t)}{\partial\tilde{\alpha}(t)}+\lambda_x^T(t)\frac{\partial f(t)}{\partial\tilde{\alpha}(t)}+\right. \tag{14}$$

$$\left.+\Phi_y(t^1)\frac{\partial K(t^1,t)}{\partial\tilde{\alpha}(t)}+\int_t^{t^1}\gamma_y^T(s)\frac{\partial K(s,t)}{\partial\tilde{\alpha}(t)}ds\right]\delta\tilde{\alpha}(t)\,dt\,;\;\;\delta_{\tilde{\alpha}(t^1)}I=[\frac{\partial I_1(t^1)}{\partial\eta(t^1)}\frac{\partial\eta(t^1)}{\partial\tilde{\alpha}(t^1)}+\Phi_y(t^1)\frac{\partial r(t^1)}{\partial\tilde{\alpha}(t^1)}]\delta\tilde{\alpha}(t^1)\,;$$

$$\delta_{\bar{\alpha}}I=\left\{\frac{\partial I_1(t^1)}{\partial\eta(t^1)}\frac{\partial\eta(t^1)}{\partial\bar{\alpha}}+\frac{\partial I_1(t^1)}{\partial\bar{\alpha}}+\lambda_x^T(t_0)\frac{\partial x_0(\bar{\alpha},t_0)}{\partial\bar{\alpha}}+\int_{t_0}^{t^1}\lambda_x^T(t)\frac{\partial f(t)}{\partial\bar{\alpha}}dt+\Phi_y(t^1)\left[\frac{\partial r(t^1)}{\partial\bar{\alpha}}+\int_{t_0}^{t^1}\frac{\partial K(t^1,s)}{\partial\bar{\alpha}}ds\right]+\right.$$

$$+\int_{t_0}^{t^1}[\frac{\partial f_0(t)}{\partial\eta(t)}\frac{\partial\eta(t)}{\partial\bar{\alpha}}+\frac{\partial f_0(t)}{\partial\bar{\alpha}}+\gamma_y^T(t)\frac{\partial r(t)}{\partial\bar{\alpha}}+\int_t^{t^1}\gamma_y^T(s)\frac{\partial K(s,t)}{\partial\bar{\alpha}}ds]\,dt+\left[\lambda_x^T(t_0)\left(\frac{\partial x_0(\bar{\alpha},t_0)}{\partial t_0}-f(t_0)\right)+\right.$$

$$\left.+\Phi_y(t^1)[\frac{\partial r(t^1)}{\partial t_0}-K(t^1,t_0)]-f_0(t_0)+\int_{t_0}^{t^1}\gamma_y^T(t)(\frac{\partial r(t)}{\partial t_0}-K(t,t_0))dt\right]\frac{dt_0}{d\bar{\alpha}}+$$

$$\left.+\left[\Phi_x(t^1)f(t^1)+\Phi_y(t^1)[\frac{\partial r(t^1)}{\partial t^1}+K(t^1,t^1)+\int_{t_0}^{t^1}\frac{\partial K(t^1,s)}{\partial t^1}ds]+\frac{\partial I_1(t^1)}{\partial\eta(t^1)}\frac{\partial\eta(t^1)}{\partial t^1}+\frac{\partial I_1(t^1)}{\partial t^1}+f_0(t^1)\right]\frac{dt^1}{d\bar{\alpha}}\right\}\delta\bar{\alpha}\,.$$

The obtained form of representation of the conjugate equations and SF allows easily to write out outcomes with reviewing separately of differential and integrated models or their various combinations.

3. SF with use of variants of an integro-differential models

Variant 3.1. In initial statement of the task (1)–(2) the equations of a model of object and measuring device vary:

$$\dot{x}(t)=f(x(t),\tilde{\alpha}(t),\bar{\alpha},t),\;\;t_0<t\le t^1,\;\;x(t_0)=x_0(\bar{\alpha},t_0),\;\;t_0=t_0(\bar{\alpha}),\;\;t^1=t^1(\bar{\alpha}),$$

$$y(t)=r(\tilde{\alpha}(t),\bar{\alpha},t_0,t)+\int_{t_0}^{t}K(t,x(s),y(s),\tilde{\alpha}(s),\bar{\alpha},s)\,ds,\;\;t_0\le t\le t^1\,; \tag{15}$$

$$\eta(t)=\eta(y(t),\tilde{\alpha}(t),\bar{\alpha},t),\;t\in[t_0,t^1]\,.$$

The scheme of interaction between variables of a object model, measuring device and quality functional is shown in fig. 2.

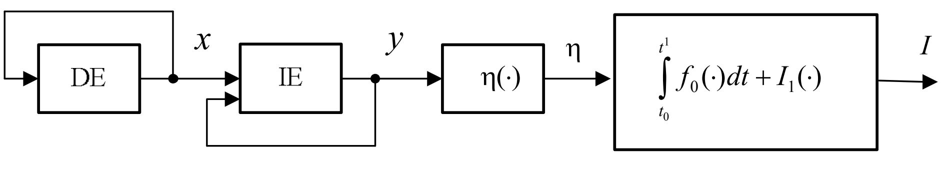

Fig. 2. The scheme of interaction between variables in (15), (3)

In total results (13), (14) it is necessary to take into account that $\partial\eta/\partial x=0$, $\Phi_x(t^1)=0$, $\partial f(t)/\partial y(t)=0$.

Variant 3.2. The equations of a object model and measuring device look like:

$$\dot{x}(t)=f(y(t),\tilde{\alpha}(t),\bar{\alpha},t),\;\;t_0<t\le t^1,\;\;x(t_0)=x_0(\bar{\alpha},t_0),$$

$$y(t)=r(\tilde{\alpha}(t),\bar{\alpha},t_0,t)+\int_{t_0}^{t}K(t,x(s),y(s),\tilde{\alpha}(s),\bar{\alpha},s)\,ds,\;\;t_0\le t\le t^1,t_0=t_0(\bar{\alpha}),\,t^1=t^1(\bar{\alpha}), \tag{16}$$

$$\eta(t)=\eta(y(t),\tilde{\alpha}(t),\bar{\alpha},t),\;t\in[t_0,t^1]\,.$$

The scheme of interaction between variables of a object model, measuring device and quality functional is represented in fig. 3.

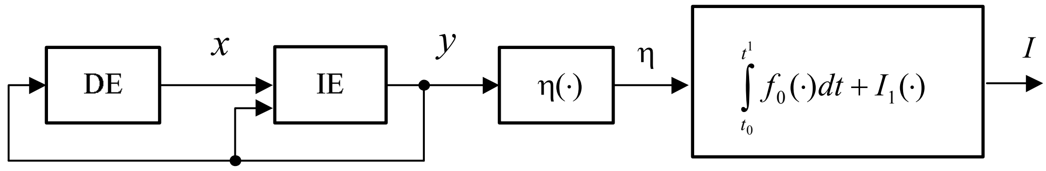

Fig. 3. The scheme of interaction between variables in (16), (3)

In total results (13), (14) it is necessary to take into account that $\partial\eta/\partial x = 0$, $\Phi_x(t^1) = 0$, $\partial f(t)/\partial x(t) = 0$.

Variant 3.3. The equations of a object model and measuring device look like:

$$\dot{x}(t) = f(x(t), y(t), \tilde{\alpha}(t), \overline{\alpha}, t),\ t_0 < t \le t^1,\ x(t_0) = x_0(\overline{\alpha}, t_0),$$

$$y(t) = r(\tilde{\alpha}(t), \overline{\alpha}, t_0, t) + \int_{t_0}^{t} K(t, s, y(s), \tilde{\alpha}(s), \overline{\alpha})\, ds,\ t_0 \le t \le t^1,\ t_0 = t_0(\overline{\alpha}),\ t^1 = t^1(\overline{\alpha}), \qquad (17)$$

$$\eta(t) = \eta(x(t), \tilde{\alpha}(t), \overline{\alpha}, t),\ t \in [t_0, t^1].$$

x is a basic variable. It satisfies to the differential equation. The variable y represents itself as input for a basic differential model and y satisfies to an independent integral equation. The scheme of interaction between variables is shown in fig. 4.

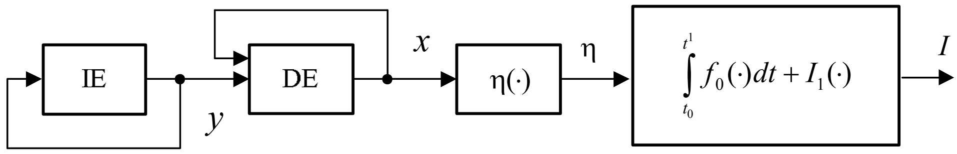

Fig. 4. The scheme of interaction between variables in (17), (3)

In total results (13), (14) it is necessary to take into account that $\partial\eta/\partial y = 0$, $\Phi_y(t^1) = 0$, $\partial K(s,t)/\partial x(t) = 0$.

Variant 3.4. The equations of a object model have the form:

$$\dot{x}(t) = f(x(t), y(t), \tilde{\alpha}(t), \overline{\alpha}, t),\ t_0 < t \le t^1,\ x(t_0) = x_0(\overline{\alpha}, t_0),$$

$$y(t) = r(\tilde{\alpha}(t), \overline{\alpha}, t_0, t) + \int_{t_0}^{t} K(t, x(s), \tilde{\alpha}(s), \overline{\alpha}, s)\, ds,\ t_0 \le t \le t^1,\ t_0 = t_0(\overline{\alpha}),\ t^1 = t^1(\overline{\alpha}), \quad (18)$$

$$\eta(t) = \eta(x(t), \tilde{\alpha}(t), \overline{\alpha}, t),\ t \in [t_0, t^1].$$

The variable y is auxiliary. It reflects rather simple integrated connection from a basic variable x. The exit of a measuring device also depends only on a basic variable.

The scheme of interaction between variables is represented in fig. 5.

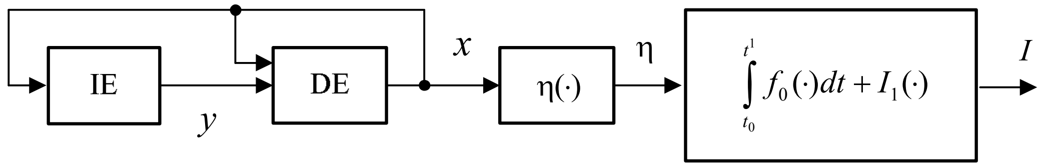

Fig. 5. The scheme of interaction between variables in (18), (3)

In total results for conjugate equations (13) and SC (14) it is necessary to take into account that $\partial\eta/\partial y = 0$, $\Phi_y(t^1) = 0$, $\partial K(s,t)/\partial y(t) = 0$.

This article continues research in [7–9].

Conclusion

In this paper the variational method of calculation SF and SC for the multivariate nonlinear dynamic systems described by general continuous vectorial Volterra's of the second-kind integro-differential equations is developed.

Novelty and generality of results consists in a generality of dynamic object model , of the measuring device model and of quality functional (in the Bolts problem). In models both variables and constant parameters are present. In a basis of calculation of sensitivity indexes the decision of the integro-differential equations of object model in a forward direction of time and obtained integro-differential equations for Lagrange's multipliers in the opposite direction of time lays.

Received in paper SF are more general in comparison with known in the scientific literature.

It is shown, that by consideration of more simple dynamic systems it is enough to put in the received results to zero corresponding additives – look section 3.

Examples of reception from final result sensitivity functionals for the objects described by ordinary integral equations, by ordinary differential equations and by 4-th variants of an integro-differential models are resulted.

Variational method of calculation of SC and SF allows to do a generalization on more complex dynamic system classes, described by: integral and integro-differential equations with additional some time lag and different classes of discontinuous dynamic equations.

Results are applicable at the decision of problems of identification, adaptive optimal control and optimization of dynamic systems, i.e. they allow to create precision systems and devices.

REFERENCES

1. Ostrovskiy, G.M. & Volin, Yu.M. (1967) *Metody optimizatsii khimicheskikh reaktorov* [Methods of optimization of chemical reactors]. Moscow: Khimiya.
2. Speedy, C.B., Brown, R.F. & Goodwin, G.C. (1973) *Teoriya upravleniya. Identifikatsiya i optimal'noe upravlenie* [Control theory: identification and optimal control]. Tranlsated from English by Yu.F. Kichatov. Moscow: Mir. (Edinburg, Oliverand Boyd, 1970).
3. Rosenvasser, E.N. & Yusupov, R.M. (eds) (1971) *Metody teorii chuvstvitel'nosti v avtomaticheskom upravlenii* [Methods of sensitivity theory in automatic control]. Leningrad: Energiya.
4. Bryson, A.E. & Ho, Ju-Chi. (1972) *Prikladnaya teoriya optimal'nogo upravleniya* [Applied optimal control]. Moscow: Mir.
5. Ruban, A.I. (1975) *Identifikatsiya nelineynykh dinamicheskikh ob"ektov na osnove algoritma chuvstvitel'nosti* [Nonlinear dynamic object identification on the base of sensitivity algorithm]. Tomsk: Tomsk State University.
6. Rosenvasser, E.N. & Yusupov, R.M. (1981) *Chuvstvitel'nost' sistem upravleniya* [Sensitivity of control systems]. Moscow: Nauka.
7. Ruban, A.I. (1982) *Identifikatsiya i chuvstvitel'nost' slozhnykh sistem upravleniya* [Identification and sensitivity of complex systems]. Tomsk: Tomsk State University.
8. Rouban, A.I. (1999) Coefficients and functionals of sensitivity for multivariate systems described by integral and integro-differetial equations. *AMSE Journals, Series Advances A.* 35(1). pp. 25–34.
9. Rouban, A.I. (2017) The Sensitivity Functionals in the Bolts's Problem for Multivariate Dynamic Systems described by Ordinary Integral Equations. *Vestnik Tomskogo gosudarstvennogo universiteta. Upravlenie, vychislitel'naya tekhnika i informatika – Tomsk State University Journal of Control and Computer Science.* 38. pp. 30–36. (In Russian). DOI: 10.17223/19988605/38/5

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Рубан Анатолий Иванович (Сибирский Федеральный университет, г. Красноярск, Российская Федерация).

Функционалы чувствительности в задаче Больца для многомерных динамических систем, описываемых обыкновенными интегро-дифференциальными уравнениями.

Ключевые слова: вариационный метод; функционал чувствительности; обыкновенное интегро-дифференциальное уравнение; функционал качества работы системы; задача Больца; сопряженное уравнение.

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Вариационный метод применен для расчета функционалов чувствительности, которые связывают первую вариацию функционалов качества работы систем с вариациями переменных и постоянных параметров для многомерных нелинейных динамических систем, описываемых обобщенными обыкновенными интегро-дифференциальными уравнениями Вольтерра второго рода.