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ON SOLVABILITY OF REGULAR EQUATIONS IN THE VARIETY OF METABELIAN GROUPS[1]

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We study the solvability of equations over groups within a given variety or another class of groups. The classes of nilpotent and solvable groups were considered as main classes to investigate from such point of view. The natural analogues of the famous Kervaire — Laudenbach and Levin conjectures were raised to the challenge. It was also noted that the "solvable" version of the known theorem by Brodskiǐ is not true. In this paper, for each $n \in \mathbb{N}, n \geqslant 2$, we prove that every regular equation over the free metabelian group M_n is solvable in the class $\mathcal{M}$ of all metabelian groups. Moreover, there is a metabelian group $\tilde{M}_n$ that contains a solution of every unimodular equation over M_n. These results are extended to the class of rigid metabelian groups. Also, we give an example showing that there exists an equation over a locally indicable torsion-free metabelian group G that has no solution in any solvable overgroup of G. It follows that solvable versions of the Levin conjecture are not true. Another example presents an unimodular equation over a locally indicable torsion-free metabelian group G that has no solution in any metabelian overgroup of G. Hence, the Kervaire — Laudenbach conjecture is not valid for the variety of all metabelian groups. We prove that there is an unimodular equation over a finite metabelian group G that has no solutions in any finite metabelian overgroup of G. This means that analog of the famous theorem by Gerstenhaber and Rothaus (about solvability of each unimodular equation over a finite group G in some finite overgroup of G) is not valid for the class of finite metabelian groups.

Keywords: *Kervaire — Laudenbach conjecture, Levin conjecture, solvable group, metabelian group, rigid group, nilpotent group, locally indicable group, regular equation, solvability over group.*

Introduction

The purpose of this note is to study solvability of equations over groups in varieties or other classes of groups, emphasizing the variety of metabelian groups.

The process of solving equations is central to much of mathematics. In general setting, there are two questions to answer when presented with an equation. The first question is about existing of a solution, the second one is about finding a solution if it exists. Equations in groups are the old and well-established area of group theory. For instance, see the surveys [1–4].

An equation in k variables over a group G is an expression of the form

$$u(x_1, \ldots, x_k) = 1, \tag{1}$$

where

$$u(x_1, \ldots, x_k) = g_0 x_{i_1}^{\varepsilon_1} g_1 \ldots g_{n-1} x_{in}^{\varepsilon_n} \in F(X) * G.$$

Here, each g_j is a group element, each exponent ε_j is 1 or -1, each x_{i_j} is taken from an alphabet of variables $X = \{x_1, \ldots, x_k, \ldots\}$, and $F(X)$ denotes the free group with basis X.

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For brevity, we often use the word "equation" to refer a word $u(x_1,\dots,x_k)$, i. e., the left-hand side of an equation of the form (1). Also, an equation can be written in the form $u(x_1,\dots,u_k) = v(x_1,\dots,x_k)$, where $u(x_1,\dots,x_k), v(x_1,\dots,x_k) \in F(X) * G$. This equation is equivalent to $u(x_1,\dots,x_k)v(x_1,\dots,x_k)^{-1} = 1$.

By the preceding convention, the free product $G_X = F(X) * G$ is the space of all equations with variables in X and coefficients in G. We assume that $n \geqslant 1$ and if $i_j = i_{j+1}$ and $\varepsilon_{i_j} + \varepsilon_{i_{j+1}} = 0$, then $g_j \neq 1$. Also, we assume that if $g_0 = 1$ and $i_1 = i_n$, then $\varepsilon_1 + \varepsilon_n \neq 0$. If not, we can conjugate the both sides of (1) by $x_{i_1}^{-\varepsilon_1}$ to get an equivalent equation with a shorter length. We exclude the case that u is conjugate to an element of G. The g_j's are called the *coefficients* or *constants* of the equation and n is called the *length* of the equation. For every positive integer k, let $X_k = \{x_1,\dots,x_k\} \subseteq X$. The free product $G_{X_k} = F(X_k) * G$ is the space of all equations over G in k variables.

A *solution* of (1) in G is an assignment $x_j \mapsto h_j \in G$ such that

$$g_0 h_{i_1}^{\varepsilon_1} g_1 \dots g_{n-1} h_{i_n}^{\varepsilon_n} = 1.$$

Let $u = u(x_1,\dots,x_k) \in G_X$ and let U denote the normal closure of u in the group G_X. We say that (the equation) $u = 1$ has a *solution* (equivalently, is *solvable*) over the group G if there is an overgroup H of G such that $u = 1$ has a solution in H. In this case, we say that $u = 1$ has a solution or $u = 1$ is *solvable*. Clearly, $u = 1$ has a solution if and only if the canonical map from G to the group $H = G_X/U$ is an embedding.

An one-variable equation of the form $g_0 x^{\varepsilon_1} g_1 \dots g_{n-1} x^{\varepsilon_n} = 1$ of length n with the exponent sum (*exponent*) $\varepsilon = \sum_{i=1}^{n} \varepsilon_i$ is called *regular* or *non-singular* if $\varepsilon \neq 0$, *unimodular* if $\varepsilon = 1$, and *singular* if $\varepsilon = 0$. Similarly, we call the underlined word $u = g_0 x^{\varepsilon_1} g_1 \dots g_{n-1} x^{\varepsilon_n}$ *regular, unimodular* or *singular*, respectively.

Not every singular equation is solvable. If $g, f \in G$ and $|g| \neq |f|$ (by $|v|$ we denote the order of element v), then the equation $gxfx^{-1} = 1$ is obviously non-solvable.

For any word $w = w(x_1,\dots,x_k) \in G_X$ and for each $j = 1,\dots,k$, we denote the exponent sum of x_j in w by $\exp_j(w)$. Let

$$\{u_1 = 1, \dots, u_m = 1\} \tag{2}$$

be a system of equations in variables $x_1,\dots,x_k$. Let $\sigma_{ij} = \exp_j(u_i)$ for $i = 1,\dots,m$; $j = 1,\dots,k$. The system (2) is called *independent* if the matrix (σ_{ij}) has rank m.

There are the following three major results on solvability of equations in groups.

Theorem 1 (M. Gerstenhaber, O. S. Rothaus, 1962 [5]). Every independent system of equations over a compact connected Lie group has a solution.

It follows that every independent system of equations over a finite group or, more generally, over a locally residually finite group [6] has a solution. In particular, every regular equation over a finite or, more generally, locally residually finite group has a solution.

Theorem 2 (A. A. Klyachko, 1993 [7]). If G is a torsion-free group, then every unimodular equation in one variable over G is solvable.

Theorem 3. (S. D. Brodskiĭ, 1980 [8], 1984 [9]; J. Howie, 1982 [10]; S. M. Gersten, 1985 [11]; S. Krstić, 1985 [12]). If G is locally indicable, then every independent system of equations over G is solvable. Moreover, if G is locally p-indicable for some prime number p, then every p-independent system of equations over G is solvable.

Recall that G is *locally indicable* if every finitely generated subgroup of G maps onto the infinite cyclic group C, and *locally p-indicable* (p is a prime) if every finitely generated subgroup of G maps onto the cyclic group C_p of order p. A system of m equations is called *p-independent* if the corresponding matrix (σ_{ij}) has rank m mod p.

S. D. Brodskiĭ proved the locally indicable statement in the case of a single equation. In fact, he proved that quotient $G{*}H/\mathrm{ncl}(w)$ of a free product $G{*}H$ of two non-trivial locally indicable groups by a normal closure $\mathrm{ncl}(w)$ of element $w \in G{*}H$, which is not conjugate to element of $G \cup H$, contains the natural images of the factors. Independently, J. Howie proved the locally indicable statement for arbitrary independent system. S. M. Gersten established the locally p-indicable version, and S. Krstić gave an alternative proof of the locally p-indicable statement.

Not every singular equation is solvable. If $g, f \in G$ and $|g| \neq |f|$, then the equation $gxfx^{-1} = 1$ is obviously non-solvable.

Since not every equation is solvable over an arbitrary group, some restrictions on group or equation are required. The best known restrictions are the Kervaire — Laudenbach conjecture and the Levin conjecture.

Conjecture 1 (**Kervaire — Laudenbach conjecture**). Every regular equation in one variable over an arbitrary group is solvable.

Conjecture 2 (**Levin conjecture**). Every equation over arbitrary torsion-free group is solvable.

In general, these two conjectures are still open.

At this point, we need to introduce the following notation: $[g, f]$ means the commutator $gfg^{-1}f^{-1}$, g^f stands for conjugate gfg^{-1}, G' means the derived subgroup (commutant) of group G, $\mathbb{Z}$ is the ring of integers, $\mathbb{N}$ is the set of positive integers, $\mathbb{Q}$ denotes the field of rationals. By $\mathbb{Z}[G]$, we denote the group ring of group G.

Recall that a group G is said to be *metabelian* if there exists a short exact sequence $A \hookrightarrow G \twoheadrightarrow B$ of groups with abelian A and B. In particular, one can take $A = G'$ and $B = G/G'$. Hence, metabelian groups are precisely the solvable groups of derived length at least two. We think of A (in particular, G') as a $\mathbb{Z}[B]$-module via conjugation; that is, we define $a \circ q = gag^{-1}$, where $g \mapsto q$ under the standard homomorphism $G \twoheadrightarrow B$. Since A is abelian, this is well-defined.

1. Solvability of equations over a group in a given class of groups

In [4], a new direction was initiated to study solvability of equations over groups within a given variety or another class of groups. The classes of nilpotent and solvable groups were considered as main classes to investigate from such point of view. The natural analogues of the Kervaire — Laudenbach and Levin conjectures were raised to the challenge.

When we study the solvability of equations over a group G in a given class $\mathcal{C}$ of groups, there is an important issue — what is the proper space of equations for a given group G? The free product G_X seems suitable if the class $\mathcal{C}$ coincides with the class $\mathcal{G}$ of all groups. On the other hand, if $\mathcal{C}$ is the class $\mathcal{A}$ of abelian groups (and G is an abelian group), then it is natural to consider an equation $g_1x_1^{-1}g_2x_2x_1 = 1$ as essentially the same as $g_1g_2x_2 = 1$. More generally, it is natural to consider two equations as essentially the same if one can be transformed into the other by applying identities of the variety of abelian groups. So the natural space of equations in variables X over an abelian group G, when we restrict ourselves by the class $\mathcal{A}$, is the direct product $G \times A(X)$ of G and a free abelian group $A(X)$ with basis X. Similarly, if G is in a variety $\mathcal{V}$, and we study solvability of equations over G in $\mathcal{V}$, then the space of equations in variables X over G *relative to* $\mathcal{V}$ is the free product in the variety $\mathcal{V}$ of G and a $\mathcal{V}$-free group with basis X. Following this approach to a metabelian group G, we define the space of *metabelian* equations as the set of elements of a free product $G*_{\mathcal{M}}F_{\mathcal{M}}(X)$, where $\mathcal{M}$ is the variety of all metabelian groups, $*_{\mathcal{M}}$ is the free product in $\mathcal{M}$, and $F_{\mathcal{M}}(X)$ is a free metabelian group with basis X. Let M_n denote the free metabelian group of rank n.

2. Solvability of equations over a group in the variety $\mathcal{M}$ of metabelian groups: negative results

We are interesting in solvability of equations over metabelian groups in the class $\mathcal{M}$. We say that the equation $u = 1$ has a *solution* (equivalently, is *solvable*) over the metabelian group G in the class $\mathcal{M}$ if there is an overgroup $H \in \mathcal{M}$ of G such that $u = 1$ has a solution in H.

It has been noted in [4] that the "solvable" version of theorem 3 is not true. Indeed, an equation of the form

$$xgx^{-1} = [g, hgh^{-1}], \tag{3}$$

where g and h are elements of a torsion-free solvable group G, for which $[g, hgh^{-1}] \neq 1$ (such elements exist in every non-abelian solvable group), has no solution in any solvable overgroup of G. It also follows that solvable versions of the Levin conjecture are not true. Hence, these conjectures are not true in the class $\mathcal{M}$.

Indeed, let the equation (3) has a solution in a solvable group H of the derived length r and $H \geqslant G$. Suppose $g \in H^{(k)}$, where $H^{(k)}$ denotes the k-th member of the derived series of H. Then the right side of (3) belongs to the $(k+1)$-th member $H^{(k+1)} = [H^{(k)}, H^{(k)}]$ of the derived series of H. By continuing in this way, we get $g \in H^{(r)} = \{1\}$. This contradicts the our assumption.

The following example shows that there is not only regular even unimodular equation that is not solvable over some torsion-free metabelian group in the class $\mathcal{M}$.

Example 1. Let G be subgroup of group $\mathrm{GL}_2(\mathbb{Q})$ generated by two matrices:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } \mathrm{B} = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}.$$

Then the following unimodular equation is not solvable in every metabelian group $H \geqslant G$:

$$x^2Bx^{-1}B^{-1} = [A, B]. \tag{4}$$

Proof. Note that $[A, B] = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$. Let $x = h$ is a solution of the equation (4) in a metabelian group $H \geqslant G$. Then $h = [A, B][h, B]^{-1} \in H'$. We consider H' as a module over group ring $\mathbb{Z}[H]$. Group H acts on H' by conjugation. As H' acts trivially, we get a module H' over group ring $\mathbb{Z}[H/H']$. We denote the image of $f \in H$ in H/H' by $\bar{f}$. For each $\bar{f} \in H/H'$ and each $u \in H'$, we write $u^{\bar{f}}$ as a result of action of $\bar{f}$ on u. The equality corresponding to (4) can be written in the form

$$h^{2-\bar{B}} = [A, B].$$

Then

$$h^{(2-\bar{B})(2+\bar{B})} = h^{4-\bar{B}^2} = 1, \tag{5}$$

because $B^2 = 4E$ (E is identical matrix). Hence, the action of $2 + \bar{B}$ on the left-hand side of (5) gives trivial element as a result. But direct computation gives the following result of the action of $2 + \bar{B}$ on the right-hand side of equality (4):

$$[A, B]^{2+\bar{B}} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = [A, B]. \tag{6}$$

In view of (5), the nontriviality of the right-hand side of (6) contradicts our assumption about solvability of (4) in H. ■

The group G in Example 1 is obviosly torsion-free. Moreover, it is locally indicable. The equation (4) is unimodular. Thus, the statement of Example 1 implies that both Theorems 2 and 3 are not true in the class $\mathcal{M}$ of metabelian groups. Moreover, it is easy to modify Example 1 considering subgroup G_p of $\mathrm{GL}_2(\mathbb{Z}_p)$ generated by matrices A and B over the modular ring (finite prime field) $\mathbb{Z}_p$ (p is an odd prime) and to prove that the equation (4) is unsolvable in the class $\mathcal{M}_{\mathrm{fin}}$ of all finite metabelian groups. The argument is practically the same as in the proof above. This shows that Theorem 1 is wrong in $\mathcal{M}_{\mathrm{fin}}$.

3. Solvability of equations over a group in the variety $\mathcal{M}$ of metabelian groups: positive results

Let M_n be the free metabelian group of rank n with basis $\{f_1, \dots, f_n\}$ and $A_n = M_n/M_n'$. Then A_n is a free abelian group with the basis $\{a_1, \dots, a_n\}$ corresponding to the basis $\{f_1, \dots, f_n\}$, that is, $f_i \mapsto a_i$ under the standard homomorphism $\mu : M_n \twoheadrightarrow A_n$, $i = 1, \dots, n$. For brevity, we denote $\bar{g} = \mu(g)$ for each $g \in M_n$. Thus, $a_i = \bar{f}_i$, $i = 1, \dots, n$.

Now, we apply the partial Fox derivatives and the Magnus embedding. For the basic notions and properties of these tools see [13].

The partial Fox derivatives on the group ring $\mathbb{Z}[M_n]$ may be defined as the mappings

$$\partial/\partial f_j : \mathbb{Z}[M_n] \to \mathbb{Z}[A_n] \quad \text{for} \quad 1 \leqslant j \leqslant n,$$

satisfying the following conditions for all $\alpha, \beta \in \mathbb{Z}$, $u, v \in \mathbb{Z}[M_n]$, and $g, h \in M_n$:

$$\partial f_i/\partial f_j = \delta_{ij} \text{ (where } \delta_{ij} \text{ is the Kronecker delta)},$$
$$\partial(\alpha u + \beta v)/\partial f_j = \alpha \partial u/\partial f_j + \beta \partial v/\partial f_j,$$
$$\partial(gh)/\partial f_j = \partial g/\partial f_j + \bar{g} \cdot \partial h/\partial f_j.$$

For any $n \in \mathbb{N}$, $n \geqslant 2$, the free metabelian group M_n can be treated via the Magnus embedding as a subgroup of the wreath product $W = A_n \mathrm{Wr} A_n$, where A_n is the free abelian group of rank n. In linear representation, the group W is the group of matrices of the form

$$\begin{pmatrix} a & \sum\limits_{i=1}^{n} u_i t_i \\ 0 & 1 \end{pmatrix},$$

where $\{t_1, \dots, t_n\}$ is a basis of the free module T over $\mathbb{Z}[A_n]$, $a \in A_n$, $u_j \in \mathbb{Z}[A_n]$ for $j = 1, \dots, n$. Hence,

$$W = \begin{pmatrix} A_n & \sum\limits_{i=1}^{n} \mathbb{Z}[A_n] \cdot t_i \\ 0 & 1 \end{pmatrix}.$$

The group M_n is embedded into W by the following map, where $g \in M_n$ and $\mu : M_n \to A_n$ is the standard homomorphism,

$$\nu : g \mapsto \begin{pmatrix} \mu(g) & \sum_{j=1}^{n} \partial g/\partial f_j \cdot t_j \\ 0 & 1 \end{pmatrix}.$$

In particular, $\nu(f_i) = \begin{pmatrix} a_i & t_i \\ 0 & 1 \end{pmatrix}$ for $i = 1, \ldots, n$.

The group ring $\mathbb{Z}[A_n]$ is the ring of Laurent polynomials $\mathbb{Z}[a^{\pm 1}, \ldots, a_n^{\pm 1}]$.

This ring is an integral domain; so, we can consider the field of fractions $\mathrm{Frac}(\mathbb{Z}[A_n])$ of $\mathbb{Z}[A_n]$. Recall that the fraction u/v denotes the equivalence class of pairs (u, v), where (u, v) is equivalent to (r, s) if and only if $us = rv$. The field of fractions $\mathrm{Frac}(\mathbb{Z}[A_n])$ is defined as the set of all such fractions u/v. The sum of u/v and r/s is defined as $us + rv/vs$, and the product of u/v and r/s is defined as ur/vs. The embedding of $\mathbb{Z}[A_n]$ into $\mathrm{Frac}(\mathbb{Z}[A_n])$ maps each $u \in \mathbb{Z}[A_n]$ to the fraction $u/1$.

The wreath product W (thus, the free metabelian group M_n) can be considered as subgroup of the following metabelian group:

$$\tilde{W} = \begin{pmatrix} A_n & \sum_{i=1}^{n} \mathrm{Frac}(\mathbb{Z}[A_n]) t_i \\ 0 & 1 \end{pmatrix}.$$

Now, we are ready to prove the following theorem.

Theorem 4. For each $n \in \mathbb{N}$, $n \geqslant 2$, every regular equation over the free metabelian group M_n is solvable in the class $\mathcal{M}$ of all metabelian groups. Moreover, there is a metabelian group $\tilde{M}_n$ that contains a solution of every unimodular equation over M_n.

Proof. Let

$$g_0 \cdot x^{\varepsilon_1} \cdot g_1 \cdot \ldots \cdot g_{n-1} \cdot x^{\varepsilon_n} = 1 \tag{7}$$

be a regular equation over M_n with exponent $\varepsilon \neq 0$. This equation induces equation of the form $x^{\varepsilon}\bar{g} = 1$ over the free abelian group $A_n = M_n/M_n'$. Here, $\bar{g}$ is the image of $g = g_0 g_1 \ldots g_{n-1}$ in A_n.

Suppose that $\varepsilon = 1$, i. e., the equation (7) is unimodular. In this case, we set $x = g^{-1}y$, where y is a new variable. The new induced equation over A_n is expressed as $y = 1$. Then (7) is equivalent to an equation of the form

$$g_0 \cdot (g^{-1}y)^{\varepsilon_1} \cdot g_1 \cdot \ldots \cdot g_{n-1} \cdot (g^{-1}y)^{\varepsilon_n} = 1. \tag{8}$$

The equation (8) has a solution y in the group $\tilde{W}$ such that y belongs to the abelian normal subgroup $\tilde{W}_1 = \begin{pmatrix} 1 & \sum_{i=1}^{n} \mathrm{Frac}(\mathbb{Z}[A_n]) t_i \\ 0 & 1 \end{pmatrix}$. Indeed, we consider this abelian normal subgroup as a module via conjugations over group ring $\mathbb{Z}[A_n]$ as we did before. Then (8) can be rewritten in the form

$$y^{\bar{u}} = v, \text{ where } \bar{u} \in \mathbb{Z}[A_n], \ v \in M_n'.$$

Since $M_n' \leqslant \tilde{W}_1$ and $\tilde{W}_1$ is a vector space, it will be enough to prove that $\bar{u} \neq 0$. We can take $y = v^{\bar{u}^{-1}}$ and succeed. Since (8) is unimodular, $\bar{v} = \sum_{i=1}^{k+1} f_i - \sum_{j=1}^{k} h_j$ for some $k \in \mathbb{N}$

and $f_i, h_j \in A_n$. Let $\delta : \mathbb{Z}[A_n] \to \mathbb{Z}$ be a trivialization homomorphism that sends every group element $a \in A_n$ to 1. Then $\delta(\bar{v}) = 1$ and $\bar{v} \neq 0$. We proved that every unimodular equation (7) is solvable in metabelian group $\tilde{M}_n$.

Suppose that $\varepsilon > 1$, i. e., the equation (7) is regular, but not unimodular. We deal with M_n as a subgroup of other group $H_n \simeq M_n$ with the basis $\{h_1, \dots, h_n\}$ via embedding corresponding to map $f_i \mapsto h_i^\varepsilon$ for $i = 1, \dots, n$. This embedding is well-defined. Indeed, a subgroup of a free metabelian group is free if and only if it is generated by a set of elements, which are independent modulo the derived group [14]. Thus, a subgroup of M_n generated by some k independent modulo M_n' elements is isomorphic to M_k with the basis consisting of k given elements. Denote $A_n = M_n/M_n'$ and $B_n = H_n/H_n'$. Then A_n is a free abelian subgroup of B_n, $A_n \simeq B_n$, with the basis $\{b_1, \dots, b_n\}$ corresponding to $\{h_1, \dots, h_n\}$. As above, (7) induces an equation of the form $x^\varepsilon \bar{g} = 1$ over the free abelian group A_n, or, more generally, over B_n. There is a unique solution $x = g$ of this induced equation. The rest of the proof is practically the same as the proof above in the case of unimodular equation. The group $\tilde{W}$ in this proof is constructed for bigger free metabelian group M_n. There is a small difference in the last step in these two proofs. Now, $\bar{v} = \sum_{i=1}^{k+\varepsilon} f_i - \sum_{j=1}^{k} h_j$. Then $\delta(\bar{v}) = \varepsilon$. But we have $\bar{v} \neq 0$, and succeed again. ■

Remark 1. We see from the proof that in the case of regular and not unimodular equation (7), this equation is solvable in metabelian overgroup isomorphic to $\tilde{W}$. The difference with the unimodular case is in embedding of M_n into $\tilde{W}$. This embedding depends on ε.

We can now give a generalization of Theorem 4 to the class of rigid metabelian groups. Recall that a group G is called *m-rigid* if it contains a finite series of normal subgroups

$$G > G_1 > \ldots > G_{m+1} = 1,$$

with abelian quotients G_i/G_{i+1} (thus G is solvable) such that each quotient G_i/G_{i+1} considered as a module over $\mathbb{Z}[G/G_i]$ via conjugation has no module torsion, i. e., for each nontrivial $g \in G_i/G_{i+1}$ and for every nontrivial $v \in \mathbb{Z}[G/G_i]$, it follows $g^v \neq 1$. Such a series (if it exists) is uniquely defined by the group G that is solvable of the derived length exactly m [15]. A group G is called *rigid* if it is m-rigid for some m. If G is metabelian, then $m = 2$.

In the class of all rigid groups, we distinguish *divisible* groups G with quotients G_i/G_{i+1}, whose elements are divisible by any nontrivial element of the respective group ring $\mathbb{Z}[G/G_i]$, i. e., for each $g \in G_i/G_{i+1}$ and for every nontrivial $v \in \mathbb{Z}[G/G_i]$, there exists $h \in G_i/G_{i+1}$ such that $h^v = g$. It has been shown in [16] that every m-rigid group can be embedded into divisible m-rigid group. In particular, every metabelian rigid group can be embedded into metabelian divisible rigid group. It has been proved in [16] that there exists a minimal divisible m-rigid completion $\hat{G}$ of the given m-rigid group G preserving linear independence, and $\hat{G}$ is uniquely defined up to G-isomorphism.

Theorem 5. Every regular equation over a rigid metabelian group M is solvable in the class $\mathcal{M}$ of all metabelian groups. Moreover, every regular equation over a rigid metabelian group M has a solution in the divisible rigid completion $\hat{M}$.

A proof of Theorem 5 is practically such as the proof of Theorem 4.

REFERENCES

1. *Noskov G. A., Remeslennikov V. N., and Roman'kov V. A.* Infinite groups. J. Sov. Math., 1982, vol. 18, pp. 669–735.
2. *Remeslennikov V. N. and Roman'kov V. A.* Model-theoretic and algorithmic questions in group theory. J. Sov. Math., 1985, vol. 31, pp. 2887–2939.
3. *Kharlampovich O. and Myasnikov A.* Equations and Algorithmic Problems in Groups. Publicacoes Math., IMPA, Brazil, 2008. 79 p.
4. *Roman'kov V.* Equations over groups. Groups, Complexity, Cryptology, 2012, vol. 4, pp. 191–239.
5. *Gerstenhaber V. and Rothaus O. S.* The solutions of sets of equations in groups. Proc. Nat. Acad. Sci.U.S., 1962, vol. 48, pp. 1951–1953.
6. *Rothaus O. S.* On the non-triviality of some group extensions given by generators and relators. Ann. Math., 1977, vol. 106, pp. 599–612.
7. *Klyachko A.* Funny property of sphere and equations over groups. Comm. Algebra, 1993, vol. 21, pp. 2555–2575.
8. *Brodskiĭ S. D.* Equations over groups and groups with a single defining relation. Russian Math. Surveys, 1980, vol. 35, pp. 165.
9. *Brodskiĭ S. D.* Equations over groups and groups with a single defining relator. Siberian Math. J., 1984, vol. 25, pp. 84–103.
10. *Howie J.* On locally indicable groups. Math. Z., 1982, vol. 180, pp. 445–461.
11. *Gersten S. M.* Reducible diagrams and equations over groups. In: Gersten, S. M., ed. Essays in Group Theory. MSRI Publ., Springer Verlag, 1985, vol. 8, pp. 15–73.
12. *Krstić S.* Systems of equations over locally $p-$indicable groups. Invent. Math., 1985, vol. 81, pp. 373–378.
13. *Timoshenko E. I.* Endomorfizmy i universal'nye teorii razreshimykh grupp [Endomorphisms and universal theories of solvable groups]. Novosibirsk, NSTU Publishers, 2011. 327 p. (NSTU Monographs Ser.) (in Russian).
14. *Baumslag G.* Some theorems on the free groups of certain product varieties. J. Combinatorial Theory, 1967, vol. 2, pp. 77–99.
15. *Myasnikov A. and Romanovskiy N. S.* Krull dimension of solvable groups. J. Algebra, 2010, vol. 324, pp. 2814–2831.
16. *Romanovskiy N. S.* Divisible rigid groups. Algebra and Logic, 2008, vol. 47, pp. 426–434.