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A.I. Rouban

THE SENSITIVITY FUNCTIONALS IN THE BOLTS PROBLEM FOR MULTIVARIATE DYNAMIC SYSTEMS DESCRIBED BY INTEGRAL EQUATIONS WITH DELAY TIME

The variational method of calculation of sensitivity functionals (connecting first variation of quality functionals with variations of variable parameters) and sensitivity coefficients (components of vector gradient from the quality functional to constant parameters) for multivariate non-linear dynamic systems described by continuous vectorial Volterra's integral equations of the second-kind with delay time is developed. The presence of a discontinuity in an initial value of coordinates and dependence the initial and final instants and magnitude of delay time from parameters are taken into account also. The base of calculation is the decision of corresponding integral conjugate equations for Lagrange's multipliers in the opposite direction of time.
Keywords: variational method; sensitivity functional; sensitivity coefficient; integral equation; conjugate equation; delay time.

The sensitivity functional (SF) connect the first variation of quality functional with variations of variable and constant parameters and the sensitivity coefficients (SC) are components of vector gradient from quality functional according to constant parameters. Sensitivity coefficients are components of SF.

The problem of calculation of SF and SC of dynamic systems is principal in the analysis and syntheses of control laws, identification, optimization, stability [1–25]. The first-order sensitivity characteristics are mostly used. Later on we shall examine only SC and SF of the first-order. The most difficult are the distributed objects which are described by the dynamic equations with delays and in partial derivatives [2, 10, 11, 13, 17, 18, 20, 23–25].

Consider a vector output $y(t)$ of dynamic object model under continuous time $t \in [t_0, t^1]$, implicitly depending on vectors parameters $\tilde{\alpha}(t), \overline{\alpha}$ and functional I constructed on $y(t)$ under $t \in [t_0, t^1]$. The first variation δI of functional I and variations $\delta\tilde{\alpha}(t)$ are connected with each other with the help of a single-line functional – SF with respect to variable parameters $\tilde{\alpha}(t)$: $\delta_{\tilde{\alpha}(t)} I = \int_{t_0}^{t^1} V(t)\delta\tilde{\alpha}(t)dt$. SC with respect to constant parameters $\overline{\alpha}$ are called a gradient of I on $\overline{\alpha}$: $(dI/d\overline{\alpha})^T \equiv \nabla_{\overline{\alpha}} I$. SC are a coefficients of single-line relationship between the first variation of functional δI and the variations $\delta\overline{\alpha}$ of constant parameters $\overline{\alpha}$: $\delta_{\overline{\alpha}} I = (\nabla_{\overline{\alpha}} I)^T \delta\overline{\alpha} = (dI/d\overline{\alpha})\delta\overline{\alpha} \equiv \sum_{j=1}^{m} \frac{\partial I}{\partial \overline{\alpha}_j}\delta\overline{\alpha}_j$.

The direct method of SC calculation (by means of the differentiation of quality functional with respect to constant parameters) inevitably requires a solution of cumbersome sensitivity equations to sensitivity functions $W(t)$. $W(t)$ is the matrix of single-line relationship of the first variation of dynamic model output with parameter variations $\delta y(t) = W(t)\delta\overline{\alpha}$. For instance, for functional $I = \int_{t_0}^{t^1} f_0(y(t), \overline{\alpha}, t)dt$ we have following SC vector (row vector): $dI/d\overline{\alpha} = \int_{t_0}^{t^1} [(\partial f_0/\partial y)W(t) + \partial f_0/\partial\overline{\alpha}]dt$. For obtaining the matrix $W(t)$ it is necessary to decide a bulky system equations – sensitivity equations. The j-th column of matrix $W(t)$ is made of the sensitivity functions $dy(t)/d\overline{\alpha}_j$ with respect to component $\overline{\alpha}_j$ of vector $\overline{\alpha}$. They satisfy a vector equation (if y is a vector) resulting from dynamic model (for y) by derivation on a parameter $\overline{\alpha}_j$.

To variable parameters such a method is inapplicable because the sensitivity functions exist with respect to constant parameters.

For relatively simply classes of dynamic systems it is shown that in the SC calculation it is possible to get rid of deciding the bulky sensitivity equations due to the passage of deciding the conjugate equations – conjugate with respect to dynamic equations of object. Method of receipt of conjugate equations (it was offered in 1962) is cumbersome, because it is based on the analysis of sensitivity equations, and it does not get its developments.

Variational method [7], ascending to Lagrange's, Hamilton's, Euler's memoirs, makes possible to simplify the process of determination of conjugate equations and formulas of account of SF and SC. On the basis of this method it is an extension of quality functional by means of inclusion into it object dynamic equations by means of Lagrange's multipliers and obtaining the first variation of extended functional on phase coordinates of object and on interesting parameters. Dynamic equations for Lagrange's multipliers are obtained due to set equal to a zero (in the first variation of extended functional) the functions before the first variations of phase coordinates. Given simplification first variation of extended functional brings at presence in the right part only parameter variations, i.e. it is got the SF. If all parameters are constant that the parameters variations are carried out from corresponding integrals and at the final result in obtained functional variation the coefficients before parameters variations are the required SC. Given method was used in [21] for dynamic systems described by ordinary continuous Volterra's of the second-kind integral and integro-differential equations (the Lagrange problem) and in [22] for dynamic systems described by ordinary continuous general Volterra's of the second-kind integral equations (the Bolts problem). In this article the variational method of account of SC and of SF develops more general (on a comparison with papers [23–25]) continuous many-dimensional non-linear dynamic systems circumscribed by the vectorial non-linear continuous more common Volterra's of the second-kind integral equations with delay time. The more common quality functional (the Bolts problem) is used also.

1. Problem statement

We suppose that the dynamic object is described by system of non-linear continuous Volterra's of the second-kind integral equations (IE) with delay time τ [17. P. 75]

$$y(t) = r(\tilde{\alpha}(t),\overline{\alpha},t_0,t) + \int_{t_0}^{t} K(t, y(s), y(s-\tau), \tilde{\alpha}(s), \overline{\alpha}, s)\, ds,\ t_0 \le t \le t^1 , \tag{1}$$

$$y(t) = \psi(\tilde{\alpha}(t),\overline{\alpha},\, t),\ t_0 - \tau \le t < t_0,\ \ 0 < \tau ,$$

$$t_0 = t_0(\overline{\alpha}),\ t^1 = t^1(\overline{\alpha}), \tau = \tau(\overline{\alpha}) .$$

Here: initial t_0 and final t^1 instants and also the delay time τ are known functions of constant parameters $\overline{\alpha}$. $\tilde{\alpha}(t), \overline{\alpha}$ are a vector-columns of interesting variable and constant parameters; y is a vector-column of phase coordinates; $r(\cdot)$, $K(\cdot)$, $\psi(\cdot)$ are known continuously differentiated limited vector-functions. The phase coordinate y in an index point t_0 makes a discontinuity, if certainly $y^{+}(t_0) \equiv y(t_0+0) \equiv r(\tilde{\alpha}(t_0),\overline{\alpha},t_0,t_0) \ne$ $\ne y(t_0 - 0) \equiv \psi(\tilde{\alpha}(t_0),\overline{\alpha},\, t_0)$. At the expense of it the right member of the IE (1) (the magnitude of a phase coordinates y) is continuous in points $t_0 + n\tau,\ n = 1, 2, \cdots$.

Variables $\eta(t)$ at each current moment of time t are connected with phase coordinates $y(t)$ by known transformation

$$\eta(t) = \eta(y(t),\tilde{\alpha}(t),\overline{\alpha},t),\ \ \ t \in [t_0,t^1], \tag{2}$$

where $\eta(\cdot)$ – also continuous, continuously differentiable, limited (together with the first derivatives) vector-function. Equation (2) is often known as model of a measuring apparatus. The required parameters $\tilde{\alpha}(t),\overline{\alpha}$ are inserted also in it. A dimensionalities of vectors y and η can be various.

The quality of functioning of system it is characterised of functional

$$I(\alpha)=\int_{t_0}^{t^1} f_0(\eta(t),\tilde{\alpha}(t),\overline{\alpha},t)\,dt+I_1(\eta(t^1),\overline{\alpha},t^1) \tag{3}$$

depending on $\tilde{\alpha}(t)$ and $\overline{\alpha}$. The conditions for function $f_0(\cdot)$, $I_1(\cdot)$ are the same as for $K(\cdot)$, $r(\cdot)$, $\psi(\cdot)$. With use of a functional (3) the optimization problem (in the theory of optimal control) are named as the Bolts problem. From it as the individual variants follow: Lagrange's problem (when there is only integral component) and Mayer's problem (when there is only second component – function from phase coordinates at a finishing point).

With the purpose of simplification of appropriate deductions with preservation of a generality in all transformations (1)–(3) there are two vectors of parameters $\tilde{\alpha}(t),\overline{\alpha}$. If in the equations (1)–(3) parameters are different then it is possible formally to unit them in two vectors $\tilde{\alpha}(t),\overline{\alpha}$, to use obtained outcomes and then to make appropriate simplifications, taking into account a structure of a vectors $\tilde{\alpha}(t),\overline{\alpha}$.

By obtaining of results the obvious designations:

$$r(t)\equiv r(\tilde{\alpha}(t),\overline{\alpha},t_0,t),\ K(t,s)\equiv K(t,y(s),y(s-\tau),\tilde{\alpha}(s),\overline{\alpha},s),$$

$$\psi(t)\equiv\psi(\tilde{\alpha}(t),\overline{\alpha},t),\ \eta(t)\equiv\eta(y(t),\tilde{\alpha}(t),\overline{\alpha},t),\quad f_0(t)\equiv f_0(\eta(t),\tilde{\alpha}(t),\overline{\alpha},t),\ I_1(t^1)\equiv I_1(\eta(t^1),\overline{\alpha},t^1)$$

are used.

It is shown also that the variation method without basic modifications allows to receive SF $\delta I(\alpha)=\int_{t_0-\tau}^{t^1} V(t)\delta\tilde{\alpha}(t)dt+(dI(\alpha)/d\tilde{\alpha}(t^1))\delta\tilde{\alpha}(t^1)+(dI(\alpha)/d\overline{\alpha})\,\delta\overline{\alpha}$ in relation to variable and constant parameters.

2. Variational method at use of models (1)–(3)

Complement a quality functional (2) by restrictions-equalities (1) by means of Lagrange's multipliers $\gamma(t)$, $t\in[t_0,t^1]$; $\overline{\gamma}(t),\ t\in[t_0-\tau,t_0]$ (column vectors) and get the extended functional

$$I=I(\alpha)+\int_{t_0}^{t^1}\gamma^{\mathrm{T}}(t)\ [r(t)+\int_{t_0}^{t}K(t,s)\ ds-y(t)]dt\ +\int_{t_0-\tau}^{t_0}\overline{\gamma}^{\mathrm{T}}(t)[\psi(t)-y(t)]\ dt\ , \tag{4}$$

which complies with $I(\alpha)$ when (1) is fulfilled. Take into account the form of functional I, change an order of integrating in double integral inside of triangular area (see fig. 1) $\left(\text{i.e.} \int_{t_0}^{t^1}\int_{t_0}^{t} A(t,s)\,ds\,dt=\int_{t_0}^{t^1}\int_{t}^{t^1} A(s,t)\,ds\,dt\right)$

$$\int_{t_0}^{t^1}\gamma^T(t)\int_{t_0}^{t}K(t,s)\,ds\,dt=\int_{t_0}^{t^1}\int_{t}^{t^1}\gamma^T(s)K(s,t)\,ds\,dt \tag{5}$$

and then extended functional (4) accepts a form:

$$\begin{gathered}I=I_1(t^1)+\int_{t_0}^{t^1}\{f_0(t)+\gamma^T(t)[r(t)-y(t)]+\\+\int_{t}^{t^1}\gamma^T(s)K(s,t)\ ds\}dt+\int_{t_0-\tau}^{t_0}\overline{\gamma}^T(t)[\psi(t)-y(t)]\ dt\end{gathered}. \tag{6}$$

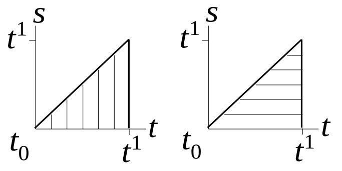

Fig. 1. Triangular area and order of an integration

Find the first variation for I with respect to $\delta y(t)$ and to $\delta\tilde{\alpha}(t)$ ($t\in[t_0,t^1)$), $\delta\tilde{\alpha}(t^1)$, $\delta\bar{\alpha}$ taking account: 1) continuity solution of IE (1) in singular points: $y(t_0+n\tau+0)=y(t_0+n\tau-0),\ n=1,2,\ldots$, 2) dependence the right member of IE (1) on $y(t)$ and on $y(t-\tau)$, 3) interconnection (3) between $\eta(t)$ and $y(t)$, $\tilde{\alpha}(t)$, $\bar{\alpha}$, 4) dependence t_0, t^1, τ, $I_1(t^1)$ on $\bar{\alpha}$ (i.e. $t_0=t_0(\bar{\alpha})$, $t^1=t^1(\bar{\alpha})$, $\tau=\tau(\bar{\alpha})$, $I_1(t^1)\equiv I_1(\eta(t^1),\bar{\alpha},t^1)$):

$$
\begin{aligned}
\delta I = {} & \Phi(t^1)\,\delta y(t^1)+\int_{t_0}^{t^1}\Big[\frac{\partial f_0(t)}{\partial\eta(t)}\frac{\partial\eta(t)}{\partial y(t)}+\int_t^{t^1}\gamma^{\mathrm{T}}(s)\frac{\partial K(s,t)}{\partial y(t)}ds-\gamma^{\mathrm{T}}(t)\Big]\,\delta y(t)\,dt+\\
&+\int_{t_0}^{t^1}\int_t^{t^1}\gamma^{\mathrm{T}}(s)\frac{\partial K(s,t)}{\partial y(t-\tau)}ds\;\delta y(t-\tau)dt-\int_{t_0-\tau}^{t_0}\bar{\gamma}^{\mathrm{T}}(t)\delta y(t)dt+\\
&+\int_{t_0}^{t^1}\Big[\frac{\partial f_0(t)}{\partial\eta(t)}\frac{\partial\eta(t)}{\partial\tilde{\alpha}(t)}+\frac{\partial f_0(t)}{\partial\tilde{\alpha}(t)}+\gamma^{\mathrm{T}}(t)\frac{\partial r(t)}{\partial\tilde{\alpha}(t)}+\int_t^{t^1}\gamma^{\mathrm{T}}(s)\frac{\partial K(s,t)}{\partial\tilde{\alpha}(t)}ds\Big]\delta\tilde{\alpha}(t)\,dt+\\
&+\int_{t_0-\tau}^{t_0}\bar{\gamma}^{\mathrm{T}}(t)\frac{\partial\psi(t)}{\partial\tilde{\alpha}(t)}\delta\tilde{\alpha}(t)\,dt+\frac{\partial I_1(t^1)}{\partial\eta(t^1)}\frac{\partial\eta(t^1)}{\partial\tilde{\alpha}(t^1)}\delta\tilde{\alpha}(t^1)+\\
&+\Big\{\frac{\partial I_1(t^1)}{\partial\eta(t^1)}\frac{\partial\eta(t^1)}{\partial\bar{\alpha}}+\frac{\partial I_1(t^1)}{\partial\bar{\alpha}}+\int_{t_0}^{t^1}\Big[\frac{\partial f_0(t)}{\partial\eta(t)}\frac{\partial\eta(t)}{\partial\bar{\alpha}}+\frac{\partial f_0(t)}{\partial\bar{\alpha}}+\gamma^{\mathrm{T}}(t)\frac{\partial r(t)}{\partial\bar{\alpha}}+\int_t^{t^1}\gamma^{\mathrm{T}}(s)\frac{\partial K(s,t)}{\partial\bar{\alpha}}ds\Big]\,dt+\\
&+\int_{t_0-\tau}^{t_0}\bar{\gamma}^{\mathrm{T}}(t)\frac{\partial\psi(t)}{\partial\bar{\alpha}}\,dt+\Big[-f_0(t_0)+\int_{t_0}^{t^1}\gamma^{\mathrm{T}}(t)\Big[\frac{\partial r(t)}{\partial t_0}-K(t,t_0)\Big]dt+\\
&+1(t^1-t_0-\tau)\int_{t_0+\tau}^{t^1}\gamma^{\mathrm{T}}(t)[K(t,t_0+\tau-0)-K(t,t_0+\tau+0)]dt\Big]\frac{dt_0}{d\bar{\alpha}}+\\
&+\Big[\frac{\partial I_1(t^1)}{\partial\eta(t^1)}\frac{\partial\eta(t^1)}{\partial t^1}+\frac{\partial I_1(t^1)}{\partial t^1}+f_0(t^1)\Big]\frac{dt^1}{d\bar{\alpha}}+\Big[1(t^1-t_0-\tau)\int_{t_0+\tau}^{t^1}\gamma^{\mathrm{T}}(t)[K(t,t_0+\tau-0)-K(t,t_0+\tau+0)]dt-\\
&-\int_{t_0}^{t^1}\gamma^{\mathrm{T}}(t)\int_{t_0}^{t}\frac{\partial K(t,s)}{\partial y(s-\tau)}\frac{dy(s-\tau)}{d(s-\tau)}ds\,dt\Big]\frac{d\tau}{d\bar{\alpha}}\Big\}\delta\bar{\alpha}. \qquad (7)
\end{aligned}
$$

Here $\dfrac{\partial I_1(t^1)}{\partial\eta(t^1)}\dfrac{\partial\eta(t^1)}{\partial y(t^1)}\equiv\Phi(t^1)$. $1(z)$ – single function: it is equal to zero under negative values of argument and is equal to unit under positive values z. The appropriate addends with single function are absent in (7) if the singular point $t_0+\tau$ is outside of an interval $[t_0,t^1]$. The argument $(t_0+\tau-0)$ in appropriate functions designates that the function undertakes to the left of a point $t_0+\tau$ and the argument $(t_0+\tau+0)$ is similar specifies that the function undertakes to the right of a point $t_0+\tau$.

Out of object equation (1) we calculate the first variation $\delta y(t^1)$ (variation, included in the first addend of (7))

$$
\delta y(t^1)=\int_{t_0}^{t^1}\frac{\partial K(t^1,s)}{\partial y(s)}\delta y(s)\,ds+\int_{t_0}^{t^1}\frac{\partial K(t^1,s)}{\partial y(s-\tau)}\delta y(s-\tau)\,ds+\int_{t_0}^{t^1}\frac{\partial K(t^1,s)}{\partial\tilde{\alpha}(s)}\delta\tilde{\alpha}(s)\,ds+
$$

$$+\frac{\partial r(t^1)}{\partial\tilde{\alpha}(t^1)}\delta\tilde{\alpha}(t^1)+\left\{\frac{\partial r(t^1)}{\partial\bar{\alpha}}+\int_{t_0}^{t^1}\frac{\partial K(t^1,s)}{\partial\bar{\alpha}}ds+\right.$$

$$+[\frac{\partial r(t^1)}{\partial t_0}-K(t^1,t_0)+1(t^1-t_0-\tau)(K(t^1,t_0+\tau-0)-K(t^1,t_0+\tau+0))]\frac{dt_0}{d\bar{\alpha}}+$$

$$+[\frac{\partial r(t^1)}{\partial t^1}+K(t^1,t^1)+\int_{t_0}^{t^1}\frac{\partial K(t^1,s)}{\partial t^1}ds]\frac{dt^1}{d\bar{\alpha}}+$$

$$\left.+[1(t^1-t_0-\tau)(K(t^1,t_0+\tau-0)-K(t^1,t_0+\tau+0))-\int_{t_0}^{t^1}\frac{\partial K(t^1,s)}{\partial y(s-\tau)}\frac{dy(s-\tau)}{d(s-\tau)}ds]\frac{d\tau}{d\bar{\alpha}}\right\}\delta\bar{\alpha}\,. \tag{8}$$

Then the first variation (7) obtains the following form:

$$\delta I=\delta_{y(t)}I+\delta_{\tilde{\alpha}(t)}I+\delta_{\alpha(t^1)}I+\delta_{\bar{\alpha}}I\,; \tag{9}$$

$$\delta_{y(t)}I=\int_{t_0}^{t^1}[\Phi(t^1)\frac{\partial K(t^1,t)}{\partial y(t)}+\frac{\partial f_0(t)}{\partial\eta(t)}\frac{\partial\eta(t)}{\partial y(t)}+\int_{t}^{t^1}\gamma^T(s)\frac{\partial K(s,t)}{\partial y(t)}ds-\gamma^T(t)]\delta y(t)\,dt+$$

$$+\int_{t_0}^{t^1}[\Phi(t^1)\frac{\partial K(t^1,t)}{\partial y(t-\tau)}+\int_{t}^{t^1}\gamma^{\mathrm{T}}(s)\frac{\partial K(s,t)}{\partial y(t-\tau)}ds]\delta y(t-\tau)dt-\int_{t_0-\tau}^{t_0}\bar{\gamma}^{\mathrm{T}}(t)\delta y(t)dt\,; \tag{10}$$

$$\delta_{\tilde{\alpha}(t)}I=\int_{t_0}^{t^1}[\frac{\partial f_0(t)}{\partial\eta(t)}\frac{\partial\eta(t)}{\partial\tilde{\alpha}(t)}+\frac{\partial f_0(t)}{\partial\tilde{\alpha}(t)}+\gamma^{\mathrm{T}}(t)\frac{\partial r(t)}{\partial\tilde{\alpha}(t)}+\Phi(t^1)\frac{\partial K(t^1,t)}{\partial\tilde{\alpha}(t)}+$$

$$+\int_{t}^{t^1}\gamma^{\mathrm{T}}(s)\frac{\partial K(s,t)}{\partial\tilde{\alpha}(t)}ds]\delta\tilde{\alpha}(t)\,dt+\int_{t_0-\tau}^{t_0}\bar{\gamma}^{\mathrm{T}}(t)\frac{\partial\psi(t)}{\partial\tilde{\alpha}(t)}\delta\tilde{\alpha}(t)\,dt\,; \tag{11}$$

$$\delta_{\tilde{\alpha}(t^1)}I=[\frac{\partial I_1(t^1)}{\partial\eta(t^1)}\frac{\partial\eta(t^1)}{\partial\tilde{\alpha}(t^1)}+\Phi(t^1)\frac{\partial r(t^1)}{\partial\tilde{\alpha}(t^1)}]\delta\tilde{\alpha}(t^1)\,; \tag{12}$$

$$\delta_{\bar{\alpha}}I=\left\{\frac{\partial I_1(t^1)}{\partial\eta(t^1)}\frac{\partial\eta(t^1)}{\partial\bar{\alpha}}+\frac{\partial I_1(t^1)}{\partial\bar{\alpha}}+\Phi(t^1)[\frac{\partial r(t^1)}{\partial\bar{\alpha}}+\int_{t_0}^{t^1}\frac{\partial K(t^1,s)}{\partial\bar{\alpha}}ds]+\right.$$

$$+\int_{t_0}^{t^1}[\frac{\partial f_0(t)}{\partial\eta(t)}\frac{\partial\eta(t)}{\partial\bar{\alpha}}+\frac{\partial f_0(t)}{\partial\bar{\alpha}}+\gamma^{\mathrm{T}}(t)\frac{\partial r(t)}{\partial\bar{\alpha}}+\int_{t}^{t^1}\gamma^{\mathrm{T}}(s)\frac{\partial K(s,t)}{\partial\bar{\alpha}}ds]\,dt+$$

$$+\int_{t_0-\tau}^{t_0}\bar{\gamma}^{\mathrm{T}}(t)\frac{\partial\psi(t)}{\partial\bar{\alpha}}\,dt+\left[\Phi(t^1)[\frac{\partial r(t^1)}{\partial t_0}-K(t^1,t_0)+1(t^1-t_0-\tau)(K(t^1,t_0+\tau-0)-K(t^1,t_0+\tau+0))]-\right.$$

$$\left.-f_0(t_0)+\int_{t_0}^{t^1}\gamma^{\mathrm{T}}(t)(\frac{\partial r(t)}{\partial t_0}-K(t,t_0))dt+1(t^1-t_0-\tau)\int_{t_0+\tau}^{t^1}\gamma^{\mathrm{T}}(t)[K(t,t_0+\tau-0)-K(t,t_0+\tau+0)]dt\right]\frac{dt_0}{d\bar{\alpha}}+$$

$$+\left[\Phi(t^1)[\frac{\partial r(t^1)}{\partial t^1}+K(t^1,t^1)+\int_{t_0}^{t^1}\frac{\partial K(t^1,s)}{\partial t^1}ds]+\frac{\partial I_1(t^1)}{\partial\eta(t^1)}\frac{\partial\eta(t^1)}{\partial t^1}+\frac{\partial I_1(t^1)}{\partial t^1}+f_0(t^1)\right]\frac{dt^1}{d\bar{\alpha}}+$$

$$+\left[\Phi(t^1)[1(t^1-t_0-\tau)(K(t^1,t_0+\tau-0)-K(t^1,t_0+\tau+0))-\int_{t_0}^{t^1}\frac{\partial K(t^1,s)}{\partial y(s-\tau)}\frac{dy(s-\tau)}{d(s-\tau)}ds]+\right.$$

$$+1(t^1-t_0-\tau)\int_{t_0+\tau}^{t^1}\gamma^{\mathrm{T}}(t)(K(t,t_0+\tau-0)-K(t,t_0+\tau+0))dt-$$

$$\left.\left.-\int_{t_0}^{t^1}\gamma^{\mathrm{T}}(t)\int_{t_0}^{t}\frac{\partial K(t,s)}{\partial y(s-\tau)}\frac{dy(s-\tau)}{d(s-\tau)}ds\,dt\right]\frac{d\tau}{d\bar{\alpha}}\right\}\delta\bar{\alpha}\,. \tag{13}$$

For union of integrals with identical variations δy we shift back interval of an integration on magnitude τ in integral with $\delta y(t-\tau)$ (in this connection the argument in integrand thus will increase on τ) and obtain a following result:

$$\int_{t_0}^{t^1}[\Phi(t^1)\frac{\partial K(t^1,t)}{\partial y(t-\tau)}+\int_t^{t^1}\gamma^{\mathrm{T}}(s)\frac{\partial K(s,t)}{\partial y(t-\tau)}ds]\delta y(t-\tau)dt=$$

$$=\int_{t_0}^{t^1}1(t^1-\tau-t)[\Phi(t^1)\frac{\partial K(t^1,t+\tau)}{\partial y(t)}+\int_{t+\tau}^{t^1}\gamma^{\mathrm{T}}(s)\frac{\partial K(s,t+\tau)}{\partial y(t)}ds]\delta y(t)dt+$$

$$+\int_{t_0-\tau}^{t_0}1(t^1-\tau-t)[\Phi(t^1)\frac{\partial K(t^1,t+\tau)}{\partial y(t)}+\int_{t+\tau}^{t^1}\gamma^{\mathrm{T}}(s)\frac{\partial K(s,t+\tau)}{\partial y(t)}ds]\delta y(t)dt\,.$$

Here for compact writing the single function $1(z)$ (which equals to zero by negative value z) is introduced. In this connection such variants are taken into account when instant $t^1-\tau$ is found inside and outside of interval of system operating period $[t_0,t^1]$.

We substitute this formula in the first variation (10), join components with identical variations and obtain that

$$\delta_{y(t)}I=\int_{t_0}^{t^1}\Bigg[\Phi(t^1)\frac{\partial K(t^1,t)}{\partial y(t)}+\frac{\partial f_0(t)}{\partial\eta(t)}\frac{\partial\eta(t)}{\partial y(t)}+\int_t^{t^1}\gamma^{\mathrm{T}}(s)\frac{\partial K(s,t)}{\partial y(t)}ds+$$

$$+1(t^1-\tau-t)[\Phi(t^1)\frac{\partial K(t^1,t+\tau)}{\partial y(t)}+\int_{t+\tau}^{t^1}\gamma^{\mathrm{T}}(s)\frac{\partial K(s,t+\tau)}{\partial y(t)}ds]-\gamma^{\mathrm{T}}(t)\Bigg]\delta y(t)dt+$$

$$+\int_{t_0-\tau}^{t_0}\Bigg[1(t^1-\tau-t)[\Phi(t^1)\frac{\partial K(t^1,t+\tau)}{\partial y(t)}+\int_{t+\tau}^{t^1}\gamma^{\mathrm{T}}(s)\frac{\partial K(s,t+\tau)}{\partial y(t)}ds]-\bar{\gamma}^{\mathrm{T}}(t)\Bigg]\delta y(t)dt\,. \quad (14)$$

In a variation (14) we equate with zero factors before variations of phase coordinates δy and discover: the conjugate equations for basic Lagrange's multipliers $\gamma(t)$

$$\gamma^{\mathrm{T}}(t)=\Phi(t^1)\frac{\partial K(t^1,t)}{\partial y(t)}+\frac{\partial f_0(t)}{\partial\eta(t)}\frac{\partial\eta(t)}{\partial y(t)}+\int_t^{t^1}\gamma^{\mathrm{T}}(s)\frac{\partial K(s,t)}{\partial y(t)}ds+$$

$$+1(t^1-\tau-t)[\Phi(t^1)\frac{\partial K(t^1,t+\tau)}{\partial y(t)}+\int_{t+\tau}^{t^1}\gamma^{\mathrm{T}}(s)\frac{\partial K(s,t+\tau)}{\partial y(t)}ds]\,,\ t_0\le t\le t^1\,, \quad (15)$$

and equation of account of Lagrange's multipliers $\bar{\gamma}(t)$ appropriate to initial function of integral equations with delay time (1)

$$\bar{\gamma}^{\mathrm{T}}(t)=1(t^1-\tau-t)[\Phi(t^1)\frac{\partial K(t^1,t+\tau)}{\partial y(t)}+\int_{t+\tau}^{t^1}\gamma^{\mathrm{T}}(s)\frac{\partial K(s,t+\tau)}{\partial y(t)}ds]\,,\ t_0-\tau\le t\le t_0\,. \quad (16)$$

These equations are decided in the opposite direction of time (from t^1).

From the conjugate equations (15), (16) it is possible to remove single function and to add them a customary aspect.

If $t_0\le t^1-\tau\le t^1$, i.e. length of an interval $[t_0,t^1]$ transcends magnitude of a delay time τ, then:

$$\gamma^{\mathrm{T}}(t)=\Phi(t^1)\frac{\partial K(t^1,t)}{\partial y(t)}+\frac{\partial f_0(t)}{\partial\eta(t)}\frac{\partial\eta(t)}{\partial y(t)}+\int_t^{t^1}\gamma^{\mathrm{T}}(s)\frac{\partial K(s,t)}{\partial y(t)}ds \quad \text{for } t^1-\tau\le t\le t^1\,,$$

$$\gamma^{\mathrm{T}}(t)=\Phi(t^1)\frac{\partial K(t^1,t)}{\partial y(t)}+\frac{\partial f_0(t)}{\partial\eta(t)}\frac{\partial\eta(t)}{\partial y(t)}+\int_t^{t^1}\gamma^{\mathrm{T}}(s)\frac{\partial K(s,t)}{\partial y(t)}ds+$$

$$+\Phi(t^1)\frac{\partial K(t^1,t+\tau)}{\partial y(t)}+\int_{t+\tau}^{t^1}\gamma^{\mathrm{T}}(s)\frac{\partial K(s,t+\tau)}{\partial y(t)}ds \quad \text{for } t_0\le t\le t^1-\tau\,,$$

$$\overline{\gamma}^{\mathrm{T}}(t)=\Phi(t^1)\frac{\partial K(t^1,t+\tau)}{\partial y(t)}+\int\limits_{t+\tau}^{t^1}\gamma^{\mathrm{T}}(s)\frac{\partial K(s,t+\tau)}{\partial y(t)}ds \text{ for } t_0-\tau\le t\le t_0 .$$

If $t^1-\tau\le t_0$, i.e. the magnitude of delay τ transcends length of an interval $[t_0,t^1]$, (in this case magnitude $t_0+\tau$ transcends t^1 – goes out for an interval of object work):

$$\gamma^{\mathrm{T}}(t)=\Phi(t^1)\frac{\partial K(t^1,t)}{\partial y(t)}+\frac{\partial f_0(t)}{\partial\eta(t)}\frac{\partial\eta(t)}{\partial y(t)}+\int\limits_{t}^{t^1}\gamma^{\mathrm{T}}(s)\frac{\partial K(s,t)}{\partial y(t)}ds \quad \text{for } t_0\le t\le t^1 ,$$

$$\overline{\gamma}^{\mathrm{T}}(t)=0 \quad \text{for } t^1-\tau\le t\le t_0 ,$$

$$\overline{\gamma}^{\mathrm{T}}(t)=\Phi(t^1)\frac{\partial K(t^1,t+\tau)}{\partial y(t)}+\int\limits_{t+\tau}^{t^1}\gamma^{\mathrm{T}}(s)\frac{\partial K(s,t+\tau)}{\partial y(t)}ds \quad \text{for } t_0-\tau\le t\le t^1-\tau .$$

As a result three components of the first variation $\delta I=\delta_{\tilde{\alpha}(t)}I+\delta_{\tilde{\alpha}(t^1)}I+\delta_{\overline{\alpha}}I$ of functional (3) in relation to variables $\tilde{\alpha}(t)$ and constant parameters $\tilde{\alpha}(t^1)$, $\overline{\alpha}$, are presented accordingly by formulas (11), (12) and (13).

This result is more common in relation to appropriate results of works [17, 21–25]. An additional important summand $I_1(\eta(t^1),\overline{\alpha},t^1)$ in a quality functional I and dependence t_0,t^1,τ from $\overline{\alpha}$ are taken into account.

Example 2.1. (*The integral equations without delay time* [22]). We shall consider an object model as **ordinary** non-linear continuous vector of Volterra's of the second-kind integral equations with variable and constant parameters $\tilde{\alpha}(t)$, $\overline{\alpha}$:

$$y(t)=r(\tilde{\alpha}(t),\overline{\alpha},t_0,t)+\int\limits_{t_0}^{t}K(t,y(s),\tilde{\alpha}(s),\overline{\alpha},s)\,ds,\ t_0\le t\le t^1,\ t_0=t_0(\overline{\alpha}),\ t^1=t^1(\overline{\alpha}) .$$

The model of measuring apparatus and quality functional are the same as before:

$$\eta(t)=\eta(y(t),\tilde{\alpha}(t),\overline{\alpha},t),\ \ t\in[t_0,t^1],\ I(\alpha)=\int\limits_{t_0}^{t^1}f_0(\eta(t),\tilde{\alpha}(t),\overline{\alpha},t)\,dt+I_1(\eta(t^1),\overline{\alpha},t^1) .$$

From (15) we have the conjugate equations for Lagrange's multipliers $\gamma(t)$:

$$\gamma^{\mathrm{T}}(t)=\Phi(t^1)\frac{\partial K(t^1,t)}{\partial y(t)}+\frac{\partial f_0(t)}{\partial\eta(t)}\frac{\partial\eta(t)}{\partial y(t)}+\int\limits_{t}^{t^1}\gamma^{\mathrm{T}}(s)\frac{\partial K(s,t)}{\partial y(t)}ds,\ t_0\le t\le t^1,$$

and from (11), (12), (13) – SF:

$$\delta I=\delta_{\tilde{\alpha}(t)}I+\delta_{\alpha(t^1)}I+\delta_{\overline{\alpha}}I ,$$

$$\delta_{\tilde{\alpha}(t)}I=\int\limits_{t_0}^{t^1}[\frac{\partial f_0(t)}{\partial\eta(t)}\frac{\partial\eta(t)}{\partial\tilde{\alpha}(t)}+\frac{\partial f_0(t)}{\partial\tilde{\alpha}(t)}+\gamma^{\mathrm{T}}(t)\frac{\partial r(t)}{\partial\tilde{\alpha}(t)}+\Phi(t^1)\frac{\partial K(t^1,t)}{\partial\tilde{\alpha}(t)}+\int\limits_{t}^{t^1}\gamma^{\mathrm{T}}(s)\frac{\partial K(s,t)}{\partial\tilde{\alpha}(t)}ds]\delta\tilde{\alpha}(t)\,dt ,$$

$$\delta_{\tilde{\alpha}(t^1)}I=[\frac{\partial I_1(t^1)}{\partial\eta(t^1)}\frac{\partial\eta(t^1)}{\partial\tilde{\alpha}(t^1)}+\Phi(t^1)\frac{\partial r(t^1)}{\partial\tilde{\alpha}(t^1)}]\delta\tilde{\alpha}(t^1),$$

$$\delta_{\overline{\alpha}}I=\left\{\frac{\partial I_1(t^1)}{\partial\eta(t^1)}\frac{\partial\eta(t^1)}{\partial\overline{\alpha}}+\frac{\partial I_1(t^1)}{\partial\overline{\alpha}}+\Phi(t^1)[\frac{\partial r(t^1)}{\partial\overline{\alpha}}+\int\limits_{t_0}^{t^1}\frac{\partial K(t^1,s)}{\partial\overline{\alpha}}ds]+\right.$$

$$+\int\limits_{t_0}^{t^1}[\frac{\partial f_0(t)}{\partial\eta(t)}\frac{\partial\eta(t)}{\partial\overline{\alpha}}+\frac{\partial f_0(t)}{\partial\overline{\alpha}}+\gamma^{\mathrm{T}}(t)\frac{\partial r(t)}{\partial\overline{\alpha}}+\int\limits_{t}^{t^1}\gamma^{\mathrm{T}}(s)\frac{\partial K(s,t)}{\partial\overline{\alpha}}ds]\,dt+$$

$$+\left[\Phi(t^1)[\frac{\partial r(t^1)}{\partial t_0}-K(t^1,t_0)-f_0(t_0)+\int\limits_{t_0}^{t^1}\gamma^{\mathrm{T}}(t)(\frac{\partial r(t)}{\partial t_0}-K(t,t_0))dt\right]\frac{dt_0}{d\overline{\alpha}}+$$

$$+\left[\Phi(t^1)[\frac{\partial r(t^1)}{\partial t^1}+K(t^1,t^1)+\int_{t_0}^{t^1}\frac{\partial K(t^1,s)}{\partial t^1}ds]+\frac{\partial I_1(t^1)}{\partial\eta(t^1)}\frac{\partial\eta(t^1)}{\partial t^1}+\frac{\partial I_1(t^1)}{\partial t^1}+f_0(t^1)\right]\frac{dt^1}{d\bar\alpha}\Bigg\}d\bar\alpha .$$

This result is more common in relation to appropriate results of works [7–9] and certainly agrees with result in [10]. An additional important summand $I_1(\eta(t^1),\bar\alpha,t^1)$ in a quality functional I and dependence t_0,t^1,τ from $\bar\alpha$ are taken into account.

Example 2.2. (***The differential equations with delay time***)**.** Consider that the dynamic object is described by system of non-linear continuous differential equations with delay time and with variable and constant parameters $\tilde\alpha(t)$, $\bar\alpha$:

$$\dot y(t)=f(y(t),y(t-\tau),\tilde\alpha(t),\bar\alpha,t),\ \ t_0\le t\le t^1,$$
$$y(t)=\psi(\tilde\alpha(t),\bar\alpha,\ t), t\in[t_0-\tau,t_0),\ y(t_0)=y_0(\bar\alpha,t_0)\ . \tag{17}$$

In an index point t_0 the phase coordinates y can have a break: $y_0(\bar\alpha,t_0)\ne\psi(\tilde\alpha(t),\bar\alpha,t_0)$.

We transform model (17) in Volterra's integral equation of the second genus with delay time (1)

$$y(t)=y_0(\bar\alpha,t_0)+\int_{t_0}^{t}f(y(s),y(s-\tau),\tilde\alpha(s),\bar\alpha,s)ds,\ t_0\le t\le t^1,$$
$$y(t)=\psi(\tilde\alpha(t),\bar\alpha,\ t), t\in[t_0-\tau,t_0)\ . \tag{18}$$

Now

$$r(t)=y_0(\bar\alpha,t_0),\ K(t,s)=f(y(s),y(s-\tau),\tilde\alpha(s),\bar\alpha,s)\equiv f(s)\ .$$

We write the conjugate equations (15), (16) for Lagrange's multipliers

$$\gamma^{\mathrm T}(t)=\frac{\partial f_0(t)}{\partial\eta(t)}\frac{\partial\eta(t)}{\partial y(t)}+[\Phi(t^1)+\int_t^{t^1}\gamma^{\mathrm T}(s)ds]\frac{\partial f(t)}{\partial y(t)}+$$
$$+1(t^1-\tau-t)[\Phi(t^1)+\int_{t+\tau}^{t^1}\gamma^{\mathrm T}(s)ds]\frac{\partial f(t+\tau)}{\partial y(t)},\ \ t_0\le t\le t^1,$$
$$\bar\gamma^{\mathrm T}(t)=1(t^1-\tau-t)[\Phi(t^1)+\int_{t+\tau}^{t^1}\gamma^{\mathrm T}(s)ds]\frac{\partial f(t+\tau)}{\partial y(t)},\ \ t_0-\tau\le t\le t_0$$

and SF (11)–(13)

$$\delta I=\delta_{\tilde\alpha(t)}I+\delta_{\alpha(t^1)}I+\delta_{\bar\alpha}I\ ,\ \delta_{\tilde\alpha(t)}I=\int_{t_0}^{t^1}[\frac{\partial f_0(t)}{\partial\eta(t)}\frac{\partial\eta(t)}{\partial\tilde\alpha(t)}+\frac{\partial f_0(t)}{\partial\tilde\alpha(t)}+\Phi(t^1)\frac{\partial f(t)}{\partial\tilde\alpha(t)}+$$
$$+\int_t^{t^1}\gamma^{\mathrm T}(s)\frac{\partial f(s)}{\partial\tilde\alpha(t)}ds]\delta\tilde\alpha(t)\ dt+\int_{t_0-\tau}^{t_0}\bar\gamma^{\mathrm T}(t)\frac{\partial\psi(t)}{\partial\tilde\alpha(t)}\delta\tilde\alpha(t)\ dt\ ,\ \delta_{\tilde\alpha(t^1)}I=\frac{\partial I_1(t^1)}{\partial\eta(t^1)}\frac{\partial\eta(t^1)}{\partial\tilde\alpha(t^1)}\delta\tilde\alpha(t^1),$$
$$\delta_{\bar\alpha}I=\Bigg\{\frac{\partial I_1(t^1)}{\partial\eta(t^1)}\frac{\partial\eta(t^1)}{\partial\bar\alpha}+\frac{\partial I_1(t^1)}{\partial\bar\alpha}+\Phi(t^1)[\frac{\partial y_0(\bar\alpha,t_0)}{\partial\bar\alpha}+\int_{t_0}^{t^1}\frac{\partial f(s)}{\partial\bar\alpha}ds]+$$
$$+\int_{t_0}^{t^1}[\frac{\partial f_0(t)}{\partial\eta(t)}\frac{\partial\eta(t)}{\partial\bar\alpha}+\frac{\partial f_0(t)}{\partial\bar\alpha}+\gamma^{\mathrm T}(t)\frac{\partial y_0(\bar\alpha,t_0)}{\partial\bar\alpha}+\int_t^{t^1}\gamma^{\mathrm T}(s)ds]\frac{\partial f(t)}{\partial\bar\alpha}dt+\int_{t_0-\tau}^{t_0}\bar\gamma^T(t)\ dt\frac{\partial y_0(\bar\alpha,t_0)}{\partial\bar\alpha}+$$
$$+\Bigg[\Phi(t^1)[\frac{\partial y_0(\bar\alpha,t_0)}{\partial t_0}-f(t_0)+1(t^1-t_0-\tau)(f(t_0+\tau-0)-f(t_0+\tau+0))]-$$
$$-f_0(t_0)+\int_{t_0}^{t^1}\gamma^T(t)dt(\frac{\partial y_0(\bar\alpha,t_0)}{\partial t_0}-f(t_0))+1(t^1-t_0-\tau)\int_{t_0+\tau}^{t^1}\gamma^T(t)dt[f(t_0+\tau-0)-f(t_0+\tau+0)]\Bigg]\frac{dt_0}{d\bar\alpha}+$$
$$+\left[\Phi(t^1)f(t^1)+\frac{\partial I_1(t^1)}{\partial\eta(t^1)}\frac{\partial\eta(t^1)}{\partial t^1}+\frac{\partial I_1(t^1)}{\partial t^1}+f_0(t^1)\right]\frac{dt^1}{d\bar\alpha}+$$

$$+\Big[\Phi(t^1)\,[1(t^1-t_0-\tau)(f(t_0+\tau-0)-f(t_0+\tau+0))-\int_{t_0}^{t^1}\frac{\partial f(s)}{\partial y(s-\tau)}\frac{dy(s-\tau)}{d(s-\tau)}ds]+$$

$$+1(t^1-t_0-\tau)\int_{t_0+\tau}^{t^1}\gamma^{\mathrm{T}}(t)dt[f(t_0+\tau-0)-f(t_0+\tau+0)]-\int_{t_0}^{t^1}\gamma^{\mathrm{T}}(t)dt\int_{t_0}^{t}\frac{\partial f(s)}{\partial y(s-\tau)}\frac{dy(s-\tau)}{d(s-\tau)}ds\Bigg]\frac{d\tau}{d\bar{\alpha}}\Bigg\}\delta\bar{\alpha}\,.$$

These results it is possible to represent in more customary (for differential equations) form. After change of variables:

$$\Phi(t^1)+\int_{t}^{t^1}\gamma^{\mathrm{T}}(s)ds=\lambda^{\mathrm{T}}(t)\,;\quad \text{ore}\quad -\dot{\lambda}^{\mathrm{T}}(t)=\gamma^{\mathrm{T}}(t),\,\lambda^{\mathrm{T}}(t^1)=\Phi(t^1)\,;$$

we obtain the conjugate equations in differential form

$$-\dot{\lambda}^{\mathrm{T}}(t)=\frac{\partial f_0(t)}{\partial\eta(t)}\frac{\partial\eta(t)}{\partial y(t)}+\lambda^{\mathrm{T}}(t)\frac{\partial f(t)}{\partial y(t)}+1(t^1-\tau-t)\lambda^{\mathrm{T}}(t+\tau)\frac{\partial f(t+\tau)}{\partial y(t)},\ \lambda^{\mathrm{T}}(t^1)=\Phi(t^1),\ t_0\le t\le t^1,$$

$$\bar{\gamma}^{\mathrm{T}}(t)=1(t^1-\tau-t)\lambda^{\mathrm{T}}(t+\tau)\frac{\partial f(t+\tau)}{\partial y(t)},\ t_0-\tau\le t\le t_0\,.$$

and than SF have the form

$$\delta I=\delta_{\tilde{\alpha}(t)}I+\delta_{\alpha(t^1)}I+\delta_{\bar{\alpha}}I\,,\ \delta_{\tilde{\alpha}(t)}I=\int_{t_0}^{t^1}[\frac{\partial f_0(t)}{\partial\eta(t)}\frac{\partial\eta(t)}{\partial\tilde{\alpha}(t)}+\frac{\partial f_0(t)}{\partial\tilde{\alpha}(t)}+\lambda^{\mathrm{T}}(t)\frac{\partial f(t)}{\partial\tilde{\alpha}(t)}]\delta\tilde{\alpha}(t)\,dt\,+$$

$$+\int_{t_0-\tau}^{t_0}1(t^1-\tau-t)\lambda^{\mathrm{T}}(t+\tau)\frac{\partial f(t+\tau)}{\partial y(t)}\frac{\partial\psi(t)}{\partial\tilde{\alpha}(t)}\delta\tilde{\alpha}(t)\,dt\,,\ \delta_{\tilde{\alpha}(t^1)}I=\frac{\partial I_1(t^1)}{\partial\eta(t^1)}\frac{\partial\eta(t^1)}{\partial\tilde{\alpha}(t^1)}\delta\tilde{\alpha}(t^1)\,,$$

$$\delta_{\bar{\alpha}}I=\Bigg\{\frac{\partial I_1(t^1)}{\partial\eta(t^1)}\frac{\partial\eta(t^1)}{\partial\bar{\alpha}}+\frac{\partial I_1(t^1)}{\partial\bar{\alpha}}+\lambda^{\mathrm{T}}(t_0)\frac{\partial y_0(\bar{\alpha},t_0)}{\partial\bar{\alpha}}+$$

$$+\int_{t_0}^{t^1}[\frac{\partial f_0(t)}{\partial\eta(t)}\frac{\partial\eta(t)}{\partial\bar{\alpha}}+\frac{\partial f_0(t)}{\partial\bar{\alpha}}+\lambda^{\mathrm{T}}(t)\frac{\partial f(t)}{\partial\bar{\alpha}}]dt+\int_{t_0-\tau}^{t_0}1(t^1-\tau-t)\lambda^{\mathrm{T}}(t+\tau)\frac{\partial f(t+\tau)}{\partial y(t)}dt+\frac{\partial y_0(\bar{\alpha},t_0)}{\partial\bar{\alpha}}+$$

$$+\Bigg[\lambda^{\mathrm{T}}(t_0)[\frac{\partial y_0(\bar{\alpha},t_0)}{\partial t_0}-f(t_0)+1(t^1-t_0-\tau)(f(t_0+\tau-0)-f(t_0+\tau+0))]-f_0(t_0)\Bigg]\frac{dt_0}{d\bar{\alpha}}+$$

$$+\Bigg[\Phi(t^1)\,f(t^1)+\frac{\partial I_1(t^1)}{\partial\eta(t^1)}\frac{\partial\eta(t^1)}{\partial t^1}+\frac{\partial I_1(t^1)}{\partial t^1}+f_0(t^1)\Bigg]\frac{dt^1}{d\bar{\alpha}}+$$

$$+\Big[\lambda^{\mathrm{T}}(t^1)\,[1(t^1-t_0-\tau)(f(t_0+\tau-0)-f(t_0+\tau+0))-\int_{t_0}^{t^1}\frac{\partial f(s)}{\partial y(s-\tau)}\frac{dy(s-\tau)}{d(s-\tau)}ds\Bigg]\frac{d\tau}{d\bar{\alpha}}\Bigg\}\delta\bar{\alpha}\,.$$

Conclusion

The merit of variational method is applicability of its both for calculation of SF and SC. Besides the equations for Lagrange's multipliers remain without change.

Variables and constant parameters are present also at model of the measuring device and at generalized quality functional for system (the Bolts problem). In a basis of calculation of sensitivity functionals the decision of the integrated equations of model in a forward direction of time and obtained integrated equations for Lagrange's multipliers in the opposite direction of time lays.

Variation method of calculation of SF and SC allows a generalization on objects described by vectorial Volterra's second-kind integro-differential equations with delay time.

Results are applicable at design of high-precision systems and devices.

This paper continues research in [17, 21–25].

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Рубан А.И. ФУНКЦИОНАЛЫ ЧУВСТВИТЕЛЬНОСТИ В ЗАДАЧЕ БОЛЬЦА ДЛЯ МНОГОМЕРНЫХ ДИНАМИЧЕСКИХ СИСТЕМ, ОПИСЫВАЕМЫХ ИНТЕГРАЛЬНЫМИ УРАВНЕНИЯМИ С ЗАПАЗДЫВАЮЩИМ АРГУМЕНТОМ. *Вестник Томского государственного университета. Управление, вычислительная техника и информатика.* 2019. № 46. С. 83–92

Вариационный метод применен для расчета функционалов чувствительности, которые связывают первую вариацию функционалов качества работы систем с вариациями переменных и постоянных параметров, для многомерных нелинейных динамических систем, описываемых обобщенными интегральными уравнениями Вольтерра второго рода с запаздывающим аргументом и обобщенным функционалом качества работы системы (функционалом Больца).

Ключевые слова: вариационный метод; функционал чувствительности; интегральное уравнение с запаздывающим аргументом; функционал качества работы системы; задача Больца; сопряженное уравнение.

ROUBAN Anatoly Ivanovich (Doktor of Technical Sciences, Professor of Computer Science Department of Siberian Federal University, Krasnoyarsk, Russian Federation).
E-mail: ai-rouban@mail.ru