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Application of Monte Carlo methods for solving the regular and degenerate problem of two-phase filtration

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Abstract. One of popular mathematical models of filtration is the classical elastic regime model describing the nonstationary equilibrium filtration. It is also called the Muskat–Leverett model. Solving filtration problems by Monte Carlo methods makes it possible to find the solution of the problem at an individual point of the domain and to estimate derivatives of the solution. This paper is devoted to applying algorithms of the Monte Carlo method to problems of filtration. The Monte Carlo algorithms of random walk by spheres and on boundaries are used for solving the stationary problem of filtration of two immiscible inhomogeneous incompressible fluids in a porous medium and for estimating the solution and the derivatives of the solution of this problem.
Keywords: Monte Carlo method, continuity equation, Dirichlet problem, Markov chains, estimate of the solution and its derivatives

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Научная статья

Применение методов Монте-Карло для решения регулярной и вырожденной задачи двухфазной фильтрации

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Аннотация. Одной из популярных математических моделей фильтрации является классическая модель упругого режима, описывающая фильтрацию с нестационарным равновесием. Ее также называют моделью Маскета-Леверетта. Решение задач фильтрации методами Монте-Карло позволяет найти решение задачи в отдельной точке области и оценить производные решения. Данная статья посвящена применению алгоритмов Монте-Карло к задачам фильтрации. Алгоритмы случайного блуждания по сферам и по границам методом Монте-Карло используются для решения стационарной задачи фильтрации двух несмешивающихся неоднородных несжимаемых жидкостей в пористой среде и для оценки решения и производных от решения этой задачи.

Ключевые слова: метод Монте-Карло, уравнение неразрывности, задача Дирихле, цепи Маркова, оценка решения и его производных

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Formulation of the problem

Let us consider the initially boundary problem for saturation and reduced pressure (s, p) in a given finite domain $\Omega \in \text{Rn}$ ($n \geq 2$) with the boundary $\partial\Omega$, $Q = \Omega \times [0, T]$, $G = \partial\Omega \times [0, T]$:

$$m\frac{\partial s}{\partial t} = \operatorname{div}(K_0 a\nabla s + K_1\nabla p + \vec{f}_0),\ (x,t)\in Q, \tag{1}$$

$$\operatorname{div}(K\nabla p + \vec{f}) = 0,\ (x,t)\in Q, \tag{2}$$

$$s(x,t) = s_0(x,t),\ (x,t)\in G, \tag{3}$$

$$p(x,t) = p_0(x,t),\ (x,t)\in G, \tag{4}$$

$$s(x,0) = s^0(x,0),\ x\in\Omega, \tag{5}$$

where the coefficients K_0, a, K, $\vec{f}_0$, K, and $\vec{f}$, as well as the boundary and initial conditions, are given [1].

For the approximate solution of problem (1)–(5), two methods were proposed in [2]:

Method 1.

$$L_1 s_{i+1} \equiv -m\frac{\partial s_{i+1}}{\partial t} + \operatorname{div}(\overline{K}(x,s_i)\nabla s_{i+1}) + \vec{B}(x,s_i)\nabla s_{i+1} + \\ +D(x,s_i)\nabla p_{i+1}\nabla s_{i+1} = 0,\ (x,t)\in Q, \tag{6}$$

$$s_{i+1}(x,t) = s_0(x,t),\ (x,t)\in G, \tag{7}$$

$$s_{i+1}(x,0) = s^0(x,0),\ x\in\Omega, \tag{8}$$

$$L_2 p_{i+1} \equiv \operatorname{div}(K(x,s_i)\nabla p_{i+1} + \vec{f}(x,s_i)) = 0,\ (x,t)\in Q, \tag{9}$$

$$p_{i+1}(x,t) = p_0(x,t),\ (x,t)\in G \tag{10}$$

where $m\overline{K} = K_0 a$, $m\overline{B} = \overline{K}\nabla m + \overrightarrow{f'}_0 - bf'$, $mD = kb'$, $b(s) = K_1K^{-1}$, $k = k_{01} + k_{02}$, k_{0i} are the phase permeabilities for a homogeneous isotropic soil ($i = 1, 2$).

Method 2.

After dividing the time interval $[0, T]$ into N parts ($\tau = T/N$) for each time layer $t\in[l\tau, (l + 1)\tau]$, $l = 0, \ldots, N - 1$, the initially boundary value problem is solved for the functions $s_{l+1}(x,t),\ p_{l+1}(x),\ (s_l(x,l\tau) = s^l(x),\ s_0(x,0) = s^0(x) = s^0(x,0))$:

$$L_3 s_{l+1} \equiv m\frac{\partial s_{l+1}}{\partial t} + \mathrm{div}(\overline{K}(x,s^l)\nabla s_{l+1}) + \vec{B}(x,s^l)\nabla s_{l+1} + \\ + D(x,s^l)\nabla p_{l+1}\nabla s_{l+1} = 0,\ \ (x,t)\in Q, \tag{11}$$

$$s_{l+1}(x,t) = s_0(x,t),\ \ x\in G, \tag{12}$$

$$s_{l+1}(x,l\tau) = s^l,\ \ x\in\Omega, \tag{13}$$

$$L_4 p_{l+1} \equiv \mathrm{div}(\overline{K}(x,s^l))\nabla p_{l+1} + \vec{f}(x,s^l)) = 0,\ \ (x,t)\in Q, \tag{14}$$

$$p_{l+1}(x) = p_0(x,l\tau),\ \ x\in\partial\Omega, \tag{15}$$

Let us describe the general scheme of using Monte Carlo algorithms for methods 1 and 2 [3].

For method 1, the Dirichlet problem is first solved for the linear elliptic equation (9), (10) for $p_{i+1}(x, t)$ at a given saturation value s_i and fixed $t = t_0$, in particular, $t_0 = 0$. Then, equation (6) is split only with respect to the time variable, i.e., for the iteration index $i + 1$ the time interval $[t_0, T]$ is divided into M parts ($\tau = (T–t_0)/M$) and for each time layer $t_j = \tau j + t_0$, $j = 0, \ldots, M – 1$, by use of the implicit difference scheme for (6), the Dirichlet problem for an elliptical equation for the variable $s_{i+1}^{j+1}(x)$ is obtained. The corresponding boundary and initial conditions (7) and (8) are written in the form

$$s_{i+1}^{j+1}(x) = s_0^{j+1}(x),\ x\in\partial\Omega,\ j = \overline{0, M-1},$$

$$s_{i+1}^{0}(x,0) = s^0(x,0),\ x\in\Omega.$$

Now, omitting the subscript, we obtain

$$-m\frac{s^{j+1}-s^j}{\tau} + \mathrm{div}(\overline{K}(x,s^j)\nabla s^{j+1}) + \vec{B}(x,s^j)\nabla s^{j+1} + \\ + D(x,s^j)\nabla p^{j+1}\cdot\nabla s^{j+1} = 0,\ j = 0,1,\ldots,M-1, \tag{16}$$

$$s^{j+1}(x) = s_0^{j+1}(x),\ x\in\partial\Omega,\ j = 0,1,\ldots M-1, \tag{17}$$

$$s^0(x,0) = s^0(x),\ x\in\Omega. \tag{18}$$

For method 2, similarly to method 1, the Dirichlet problem for the linear elliptical equation (14), (15) is first solved for $p_{l+1}(x)$ at a given saturation s^l; in particular, at $l = 0$ from (13) we obtain $s_1(x, 0) = s^0(x)$, $x\in\Omega$. Now, using the purely implicit scheme, approximating only with respect to the time variable, for the initially boundary problem (11)–(13) for the variable $s_{l+1}(x) = s^{j+1}(x)$, ($l = j$), we obtain the Dirichlet problem for the elliptical equation, i.e., problem (16)–(18).

Thus, to determine p and s, one has to solve the Dirichlet problem for the elliptical type [4].

Omitting indices of the time layer j, we obtain for the determination of pressure $\tilde{p}(x)$ ($\tilde{p}(x) \equiv p^{j+1}(x)$) the problem

$$\mathrm{div}(K(x,s))\nabla\tilde{p} + \vec{f}(x,s)) = 0,\ x\in\Omega,\ \tilde{p}(x) = p_0(x),\ x\in\partial\Omega$$

or

$$K(x,s))\Delta\tilde{p}(x) + \sum_{i=1}^{n} C_i(x,s)\cdot\frac{\partial\tilde{p}(x)}{\partial x_i} + g(x,s) = 0,\ x\in\Omega, \tag{19}$$

$$\tilde{p}(x) = p_0(x),\ x\in\partial\Omega, \tag{20}$$

where $C_i(x, s) = \frac{\partial}{\partial x_i}K(x, s)$, $g(x, s) = \operatorname{div} \vec{f}(x, s)$; when determining the saturation $\tilde{s}(x)$ ($\tilde{s}(x) = s^{j+1}(x)$), we obtain

$$\operatorname{div}(\overline{K}(x,s))\nabla\tilde{s})x)) + (\vec{B}(x,s) + D(x,s)\nabla\tilde{p}(x))\nabla\tilde{s}(x) - \tilde{m}\tilde{s}(x) = -\tilde{m}s(x),\ x \in \Omega,$$

$$\tilde{s}(x) = \tilde{s}_0(x),\ x \in \partial\Omega,\ s(x) = s^0(x),\ x \in \Omega$$

or

$$K(x,s)\Delta\tilde{s}(x) + \sum_{i=1}^{n} E_i(x,s)\frac{\partial s(x)}{\partial x_i} + \sum_{i=1}^{n} B_i(x,s)\frac{\partial\tilde{s}(x)}{\partial x_i} + \tag{21}$$
$$+ D(x,s)\cdot\sum_{i=1}^{n}\frac{\partial\tilde{p}(x)}{\partial x_i}\frac{\partial\tilde{s}(x)}{\partial x_i} - \tilde{m}(x)\tilde{s}(x) = -\tilde{m}(x)\tilde{s}(x),\ x \in \Omega,$$

$$\tilde{s}(x) = \tilde{s}_0(x),\ \ x \in \partial\Omega \tag{22}$$

where $s(x)$ is a known function by virtue of the initial data, $\tilde{m}(x) = \frac{m(x)}{\tau}$, $\tilde{m}(x) = \frac{m(x)}{\tau}$, $E_i(a) = \frac{\partial}{\partial x_i}\overline{K}(x,s)$, $B_i(x,s)$ are components of the vector $\vec{B}(x,s)$.

Let us construct a random process and algorithm for solving problem (19), (20). Consider the Dirichlet problem for a function φ continuous at the boundary $\partial\Omega$, a measurable function f, and an elliptical operator L

$$Lu(x) = -f(x),\ x \in \Omega, \tag{23}$$

$$u(x) = \phi(x),\ x \in \partial\Omega. \tag{24}$$

Let us construct random processes for numerical finding of the solution u. We suppose that the domain Ω and operator L are such that problem (23), (24) has a unique solution, continuous in $\overline{\Omega}$ and regular in Ω, for any sufficiently smooth f and φ [5].

It is known that the integral representation

$$u(x) = \int_{V(x)} k(x,y)u(y)dy + \int_{V(x)} \Lambda(y,x)f(y)dy, \tag{25}$$

where $k(x,y) = N_y\Lambda(y,x) \geq 0$, $\Lambda(y,x)$ – is a Levy function, N_y is an operator adjoint to the operator L, $V(x)$ is an ellipsoid,

$$V(x) = V_R(x) = \left\{ y : \sigma(y,x) = \left(A^{-1}(x)(y-x),(y-x)\right)^{\frac{1}{2}} \leq R(x) \right\}.$$

is valid for the solution $u(x)$ of the boundary problem (23), (24).

Here, $R(x)$ is a maximum radius ball with a center at the point x lying in Ω and A is the matrix of higher coefficients of the operator L; the matrix is symmetric [6].

Representation (25) is called the mean value theorem. Note that if the coefficient $C \leq 0$ at $u(x)$ in equation (23) of the operator L, then the kernel $k(x, y)$ is substochastic, i.e., $\int_{V(x)} k(x,y)dy \leq 1$ [7]. Representation (22) allows one to construct unbiased estimates for the solution of problem (23), (24). Any regular solution of problem (23), (24) satisfies equation (25) and boundary condition (24). In connection with this, we define operator K acting on functions from $C(\overline{\Omega})$ by the formula

$$(Ku)(x)=\begin{cases}\int\limits_{V(x)} k(x,y)u(y)dy, & x\in\Omega,\\ u(x), & x\in\partial\Omega.\end{cases} \tag{26}$$

Consider the following problem: for $\phi\in C(\partial\Omega)$ and $F\in C(\Omega)$, find $u\in C(\overline{\Omega})$ such that

$$\begin{cases}u(x)=(Ku)(x)+F(x),\ x\in\Omega,\\ \quad u(x)=\phi(x),\ x\in\partial\Omega.\end{cases} \tag{27}$$

If $F(x)=\int\limits_{V(x)}\Lambda(y,x)f(y)dy,$ then the solution of problem (27) is a solution for (23), (24).

In [8], the uniqueness theorem was proved for the class of kernels $k(x,y)=N_y\Lambda(y,x)$ of the operator K acting by formula (26) and having the properties

1) $\mathrm{mes}(V(x_1)\Delta V(x_2))\to 0$ as $x_1\to x_2$,
2) $\mathrm{diam}(V(x))\to 0$ for $x\in\partial\Omega,$
3) kernel $k(x,y)$ is substochastic and weakly polar, $k(x,y)=W(x,y)/|x-y|^{n-1}$, $n\geq 3,$ where the function $W(x,y)$ can be continuously extended from $\Omega\times V(x)$ to $\Omega\times\overline{\Omega},$
4) $\|K\|_{L^\infty}=1;$ then the integral operator acting in $L^\infty(\Omega)$ by the formula $(\mathrm{K}u)(x)=\int\limits_{V(x)}k(x,y)u(y)dy$ and satisfying properties 1), 2), and 3) maps functions bounded in Ω into continuous ones. It is easy to establish that for an operator satisfying properties 1), 2), 3) the following conditions hold:

a) for any $u\in C(\overline{\Omega}),$

$\int\limits_{V(x)}k(x,y)u(y)dy\to u(x_0)$ as $x\to x_0\in\partial\Omega,$

b) $\int\limits_{V(x)}k(x,y)dy\to 1$ as $x\to x_0\in\partial\Omega.$

From the above, it follows that the operator defined by formula (26) is bounded in $C(\overline{\Omega})$, i.e., problem (27) can be solved in $C(\overline{\Omega})$ and problem (27) has not more than one solution in $C(\overline{\Omega})$.

Construction of unbiased estimates of solution (27)

Let $V(x)$ and kernel $k(x, y)$ satisfy properties 1), 2), 3) and additional requirement b). By virtue of the last property, the spectral radius of the operator K is equal to unity; therefore, one cannot use the standard estimates for solving integral equations of the second kind by Monte Carlo methods. Here, the estimation scheme is based on the martingale theory. In this case, it is easy to analyze the variance of the estimates.

A terminating Markovian chain is determined with a transition density $P(x,y)=k(x,y),\ y\in V(x).$ The probability of termination $q(x)=1-\int\limits_{V(x)}k(x,y)dy$

tends to zero at $x \in \partial\Omega$; therefore, the trajectory of the chain can have an infinite length. For the set of trajectories $\tilde{B}$ having an infinite length, the following Lemma is valid:

Lemma. Let problem (27) be solvable for $F \equiv 0, \varphi \equiv 1$. If there exists a function $0 \le H \in C(\Omega)$ such that problem (27) for $F = H, \varphi \equiv 0$ is solvable, then for a chain $\{x_m\}_{m=0}^{\infty}$ with transition density $P(x, y)$ almost all trajectories of infinite length approach the boundary $\partial\Omega$:

$$P_x\left\{\operatorname{dist}(x_n, \partial\Omega) \xrightarrow[n\to\infty]{} 0 \middle| \tilde{B}\right\} = 1, \text{ and } P_x\left\{\tilde{B}\right\} > 0. \qquad (28)$$

Proof. From Lemma 2.3.3 (see Lemma 2.3.3, [4] it follows that for the solution $u(x)$ of problem (27) with $F \equiv H, \varphi \equiv 1$ the maximum principle is valid: $u(x)$ reaches the smallest value on the boundary of the domain. Hence, $u(x) \ge 0$. Let χ_i be the indicator of an event (the moment of chain termination > i), $\{A_n\}_{n=0}^{\infty}$ be a sequence of σ-algebras generated by the chain up to the time moment n.

Consider a sequence of random variables

$$\eta_n = \sum_{i=0}^{n-1} H(x_i)\chi_i + \chi_n u(x_n).$$

The sequence $\{\eta_n\}_{n=0}^{\infty}$ is a positive martingale with respect to $\{A_n\}_{n=0}^{\infty}$. Indeed,

$$M_x\left\{\eta_n \middle| A_{n-1}\right\} = \sum_{i=0}^{n-1} H(x_i)\chi_i + M_{x_{n-1}}\left\{\chi_n u(x_n)\right\} =$$

$$= \sum_{i=0}^{n-1} H(x_i)\chi_i + x_{n-1} \cdot \int_{T(x_{n-1})} k(x_{n-1}, x_n) u(x_n) dx_n = \eta_{n-1}.$$

Then, by the martingale convergence theorem [9], there exists a random value η_∞ such that $M_x\eta_\infty < +\infty$ and $\eta_n \to \eta_\infty$ as $n \to \infty$ with probability 1. Consequently, almost everywhere on the set $\tilde{B}$ $H(x_n) \to 0$ as $n \to \infty$. Let

$$\partial\Omega_\varepsilon = \left\{x \in \Omega' : \operatorname{dist}(x, \partial\Omega) < \varepsilon\right\}, \Omega_\varepsilon = \Omega \setminus \partial\Omega_\varepsilon.$$

It is evident that $H(x) \ge \text{const} = c(\varepsilon) > 0$ on Ω_ε. If $\tilde{B}_1$ is a subset of trajectories from $\tilde{B}$ such that $H(x_n) \to 0$ as $n \to \infty$, then $P_n\left(\tilde{B} \backslash \tilde{B}_1\right) = 0$. Hence, if $P_x\left(\tilde{B}_1\right) > 0$, then

$$P_x\left(\operatorname{dist}(x_n, \partial\Omega) \xrightarrow[n\to\infty]{} 0 \middle| \tilde{B}\right) = P_x\left(\operatorname{dist}(x_n, \partial\Omega) \xrightarrow[n\to\infty]{} 0 \middle| \tilde{B}_1\right).$$

Let $X = (x_0, x_1, ..., x_n,) \in \tilde{B}_1$, but $\operatorname{dist}(x_n, \partial\Omega) \xrightarrow[n\to\infty]{not} 0$. Then there also exists ε_0 and an increasing sequence $\{n_k\}_{k=0}^{\infty}$ such that $\operatorname{dist}(x_{n_k}, \partial\Omega) > \varepsilon_0$. Hence, $H(x_{n_k}) > c(\varepsilon) > 0$, so $X \notin \tilde{B}_1$. It follows that $\left\{X : \operatorname{dist}(x_n, \partial\Omega) \xrightarrow[n\to\infty]{} 0\right\} \supset \tilde{B}_1$. Therefore, if $P_x(\tilde{B}_1) > 0$, then $P_x\left\{\operatorname{dist}(x_n, \partial\Omega) \xrightarrow[n\to\infty]{} 0 \middle| \tilde{B}_1\right\} = 1$.

Now let us prove that $P_x(\tilde{B}_1) > 0.$ By condition of the lemma, there exists a solution to problem (27) with $F \equiv H$ and $\varphi \equiv 1.$ Let us denote it as $\nu(x).$ It can be shown that $\inf \nu(x) > 0,\ x \in \Omega'.$ A martingale $\eta_n = \chi_n \nu(x_n)$ is uniformly integrable; therefore, by the martingale convergence theorem, $\nu(x) = M_x \lim_{n\to\infty} \eta_n = 0,$ if $P_x(\tilde{B}) \equiv P_x(\tilde{B}_1) = 0.$ And this contradiction proves the inequality $P_x(\tilde{B}_1) > 0.$ The lemma is proved.

If we take $H(x) = \int\limits_{V(x)} \Lambda(x,y)dy$ as $H(x)$, then we can construct estimates for the solution $u(x)$ of problem (27) on trajectories of the chain $\{x_m\}_{m=0}^{\infty}$ with a transition density

$$P(x,y) = \begin{cases} k(x,y), & y \in V(x), \\ 0, & y \notin V(x). \end{cases}$$

The sequence of estimates $\{\eta_m\}_{m=0}^{\infty}$ is determined by the equality $\eta_m = \sum_{i=0}^{m-1} F(x_i)\chi_i + \chi_i u(x_i),$ where χ_i is the event indicator {the moment of the chain termination > i}. Obviously, $M_x \eta_{m_\infty} = u(x),$ i.e. estimates η_m are unbiased. The sequence $\{\eta_m\}_{m=0}^{\infty}$ forms a martingale with respect to $\{A_m\}_{m-0}^{\infty}$ – a sequence of σ-algebras. A_m is generated by the chain up to the time instant m. The last statement is proved in the same way as the Lemma. From this we have

Corollary. For a Markovian chain $\{x_n\}_{n=0}^{\infty}$ determined by the transition density $P(x,y) = k(x,y) = N_y \Lambda(y,x),$ (28) is fulfilled.

Let τ_1 be the moment of chain termination, τ_2 be the moment when the chain enters the δ-neighborhood of the boundary $\tau_\delta = \min(\tau_1, \tau_2).$ A sequence $\{\xi_m\}_{m=0}^{\infty}$ of unbiased estimates for the solution $u(x)$ of problem (23), (24) is called admissible if there exists a sequence of σ-algebras $\{\Upsilon_m\}_{m=0}^{\infty}$ such that $A_m \subset \Upsilon_m$ and $A_m \subset \Upsilon_{m+1}$, and ξ_m has the form $\xi_m = \xi_m + \chi_m u(x_m),$ where ξ_m Υ_m are measurable. For an admissible sequence of estimates $\{\xi_m\}_{m=0}^{\infty}$, we define a random variable ξ_δ by the equality

$$\xi_\sigma = \xi_{\tau_\delta} + \phi(x_{\tau_\delta}), \tag{29}$$

where x_{τ_δ} denoted a border point closest to x_{τ_δ}. The definition is correct since $\tau_\delta < +\infty$ by virtue of the above Lemma.

We finally obtain

Theorem. If an admissible sequence of estimates $\{\xi_m\}_{m=0}^{\infty}$ forms a square integrable martingale with respect to a family of σ-algebras, $\{\Upsilon_m\}_{m=0}^{\infty}$, then the random variable ξ_δ is a ε(δ)-biased estimate for $u(x)$, its variance is a bounded function of the parameter δ (ε(δ) is the modulus of continuity of the function).

Proof. Let χ denote the indicator of the event $\{\tau_1 = \tau_2\}$. By the theorem about the free choice transformation [6], $M_x \xi_{\tau_\delta} = u(x)$, therefore, $\left|u(x) - M_x \xi_\delta\right| = \left|M_x \xi_{\tau_\delta} - M_x \xi_\delta\right| \le$ $\le M_x \chi \left|u(x_{\tau_\delta}) - \varphi(x_{\tau_\delta})\right| \le \varepsilon(\delta)$, i.e. $\varepsilon(\delta)$ and the bias of the estimate is proven. Now let us prove that the variance is bounded. We introduce an abbreviation ξ for ξ_{τ_δ}, then we obtain

$$D\xi_\delta = M_x\left(\xi_\delta - M_x\xi_\delta\right)^2 = M_x\left(\xi_\delta - \xi + \xi - u(x) + u(x) - M_x\xi_\delta\right)^2 \le$$
$$\le 4\left[M_x\left(\xi_\delta - \xi\right)^2 + M_x\left(\xi - u(x)\right)^2 + \left(u(x) - M_x\xi_\delta\right)^2\right] \le 8\varepsilon^2(\delta) + 4D\xi$$

By virtue of square integrability of the martingale $\{\xi_n\}_{n=1}^{\infty}$, $\sup\limits_{n} M_x \xi_n^2 < +\infty$, therefore, the $D\xi_\delta < +\infty$. The theorem is proved.

Consider now the sequence of estimates $\{\eta_n\}_{n=1}^{\infty}$.

Lemma. If problem (27) is solvable when $F(x)$ is replaced by $|F(x)|$ and $F^2(x)$, $\phi \equiv 0$, then the martingale $\{\eta_n\}_{n=0}^{\infty}$ is square integrable.

Proof. Let us put $s_i = \chi_i F(x_i)$; and let $v_{F,\phi}$ denote a solution of problem (27). Under conditions of the Lemma, $F(x) \to 0$ as $x \to x_0 \in \partial\Omega$; therefore, $F(x)$ is bounded in Ω. Let $F = \|F\| = \max\limits_{x\in\Omega}|F(x)|$; then, for the function $H(x) = |F(x)|/F$, equation (27) is solvable at $\varphi = 0$ разрешимо. Its solution $v_{G,0}$ can be represented as a series $v_{G,0} = \sum\limits_{m=0}^{\infty} K^n G$. Since $0 \le G \le 1$, the series $\sum\limits_{m=0}^{\infty} K^n G^2$ also converges and yields a solution $v_{G^2,0}$ of problem (27). Therefore, there exists $v_{F^2,0}$. Then,

$$\eta_n^2 = \left(\sum_{i=0}^{n-1} s_i + \chi_n u(x_n)\right)^2 \le 2\left(\sum_{i=0}^{n-1} s_i\right)^2 + 2\chi_n u^2(x_n),\ M_x \chi_n u^2(x_n) \le \|u\|^2_{C(\Omega')},$$

$$M_x\left(\sum_{i=0}^{n-1} s_i\right)^2 \le M_x \sum_{i=0}^{n-1} s_i^{\,2} + 2M_x \sum_{i=0}^{n-2} s_i \sum_{k=i+1}^{n-1} s_k = I_1 + 2I_2, I_1 \le v_{F^2,0}.$$

To estimate the second summand, we use the Markov property:

$$I_2 \le \sum_{i=0}^{n-2} M_x\left(|s_i| M_x\left\{\sum_{k=i+1}^{n-1} |s_k| \|Y_i|\right\}\right) \le M_x \sum_{i=0}^{n-2} |s_i| v_{|F|,0}(x_i).$$

Therefore, $I_2 \le \left\|v_{|F|,0}\right\|^2_{C(\Omega')}$. Thus, $\sup\limits_n M_x \eta_n^2 \le 2\|u\|^2 + v_{F^2,0} + \left\|v_{|F|,0}\right\|^2$. Lemma is proved.

Now, it remains to determine for problem (23), (24) a sequence of unbiased estimates which are obtained from η_m according to the estimate $F(x_i)$ by a single random node with the density $\Lambda(y, x_i)/h(x_i)$, $h(x) = \int\limits_{V(x)} \Lambda(y,x)dy$. In this case, the sequence of unbiased estimates $\xi_m + \sum\limits_{i=0}^{m-1} h(x_i) f(y_i) \chi_i + \chi_m u(x_m)$ forms a martingale with respect to $\{Y_m\}_{m=0}^{\infty}$. Here, $y_0, y_1, ..., y_{m-1}, ...$ are independent random vectors with the densities $\Lambda(y, x_i)/h(x_i), Y_m$ is a σ-algebra generated by $x_0, x_1, ..., x_m, y_0, y_1, ..., y_{m-1}$. The Lemma below follows from the previous one.

Lemma. The martingale $\{\xi_m\}_{m=0}^{\infty}$ is square integrable.

Finally, applying the theorem to the martingale $\{\xi_m\}_{m=0}^{\infty}$, we obtain

Theorem. Let ε(δ) be the modulus of continuity of the solution $u(x)$ of problem (23), (24); then, the estimate ξ_δ, determined according to $\{\xi_m\}_{m=0}^{\infty}$ by formula (29), is ε(δ)-biased for $u(x)$. $D\xi_\delta$ is a bounded function of the parameter δ.

Conclusions

To implement the algorithm in practice, it is necessary to learn to model the chain $\{x_m\}_{m=0}^{\infty}$ on trajectories of which estimates of the solution are constructed, and algorithms of simulation of the Markovian chain $\Lambda(y, x)/h(x)$ are based on the von Neumann selection method. Modeling distributions requires special investigations, especially in cases where it is necessary to model the distributions regularly and repeatedly. Such investigations are usually carried out for each specific equation if it is solved several times.

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