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On a class of homeomorphisms of function spaces preserving the Lindelöf number of domains

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Abstract. We consider the class of all homeomorphisms between the function spaces of the form $C_p(X)$, $C_p(Y)$ such that the images of Y and X under their dual and, respectively, inverse dual mappings consist of finitely supported functionals. We prove that if a homeomorphism belongs to this class, then Lindelöf numbers $l(X)$ and $l(Y)$ are equal. This result generalizes the known theorem of A. Bouziad for linear homeomorphisms of function spaces.
Keywords: Lindelöf number, function space, pointwise convergence topology, finite support property

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Научная статья

Об одном классе гомеоморфизмов пространств функций, сохраняющем число Линделёфа областей определения

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Аннотация. А. Бузиад доказал, что если пространства непрерывных функций $C_p(X)$, $C_p(Y)$ линейно гомеоморфны, то числа Линделёфа пространств X, Y равны. В данной статье этот результат распространяется на более широкий класс гомеоморфизмов пространств функций. Для этого вводятся в рассмотрение специальные подпространства в пространствах функционалов с конечным носителем, которые тем не менее строго шире пространств линейных непрерывных функционалов. Далее рассматривается класс таких гомеоморфизмов h пространств $C_p(X)$, $C_p(Y)$, что образ Y при сопряженном к h отображении и образ X при отображении, сопряженном к отображению h^{-1}, содержатся в рассмотренных подпространствах функционалов. Учитывая, что эти подпространства строго шире пространства линейных непрерывных функционалов, приходим к заключению, что введенный класс гомеоморфизмов строго шире класса линейных гомеоморфизмов. Доказано, что техника А. Бузиада может быть применена к этому классу гомеоморфизмов. Таким образом, установле-

но, что если пространства $C_p(X)$, $C_p(Y)$ гомеоморфны, и гомеоморфизм принадлежит к рассматриваемому классу, то числа Линделёфа пространств X, Y равны.
Ключевые слова: число Линделёфа, пространство функций, топология поточечной сходимости, свойство конечного носителя

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Introduction

We assume all topological spaces under consideration to be Tykhonoff and call them simply "spaces." For each space X, let $C_p(X)$ be the set of continuous real-valued functions on X with the topology of pointwise convergence. It means that a basic neighborhood $W(\varphi, K, \varepsilon)$ of any function $\varphi \in C_p(X)$ consists of functions $\psi \in C_p(X)$ such that $|\varphi(x) - \psi(x)| < \varepsilon$ for each point x of a finite subset $K \subset X$.

A. Bouziad proved [1] that if two function spaces $C_p(X)$, $C_p(Y)$ are linearly homeomorphic, then the Lindelöf numbers $l(X)$, $l(Y)$ of X, Y are equal. For the prehistory of this result, the reader may refer to the rather complete survey in the same article [1]. In addition, we just note the interesting partial results of A.V. Arbit [2, 3], concerning uniform homeomorphisms of function spaces.

In this paper, we describe some class $\mathcal{H}$ of homeomorphisms $h : C_p(X) \to C_p(Y)$ such that $l(X) = l(Y)$ whenever $h \in \mathcal{H}$. This class $\mathcal{H}$, by its definition, is wider than the class of linear homeomorphisms. Hence we obtain a generalization of the above-mentioned result of A. Bouziad.

We denote by $C_p^0 C_p(X)$ the subspace in $C_p\big(C_p(X)\big)$ consisting of all continuous functions $f : C_p(X) \to \mathbb{R}$ such that $f\left(0^X\right) = 0$, where 0^X is zero-function on X. In what follows, we identify each space X with its image under natural homeomorphic embedding $\theta : X \to C_p^0 C_p(X)$ defined by the rule $\theta(x)(\varphi) = \varphi(x)$, where $x \in X$, $\varphi \in C_p(X)$. Recall that for each continuous mapping $h : C_p(X) \to C_p(Y)$ such that $h\left(0^X\right) = 0^Y$ its dual mapping $h^* : C_p^0 C_p(Y) \to C_p^0 C_p(X)$ is defined by the rule $h^*(f)(\varphi) = \left(f \circ h\right)(\varphi) = f(h(\varphi))$.

1. Finitely supported functionals on $C_p(X)$

Definition 1.1. A function $f \in C_p^0 C_p(X)$ is said to be a finitely supported functional (briefly, FSF) if there exists a finite (may be empty) subset $K \subset X$ such that the pair (f, K) satisfies the following two conditions:

(i) For each $\varepsilon > 0$ and each $\varphi \in C_p(X)$, there exists $\delta > 0$ such that

$$f\left(W(\varphi, K, \delta)\right) \subset \left(f(\varphi) - \varepsilon; f(\varphi) + \varepsilon\right);$$

(ii) There exists $\varepsilon_0 > 0$ such that for each $x \in K$ and for each its open neighborhood U_x one can find functions $\varphi_x, \psi_x \in C_p(X)$ which coincide out of U_x but $\left|f(\varphi_x) - f(\psi_x)\right| > \varepsilon_0$.

If conditions (i), (ii) hold then we say that K is the (finite) support of f and we write $K = \operatorname{supp} f$.

Definition 1.2. We write $f \in \hat{L}_p(X)$ if f is an FSF with the following additional properties:

(iii) If $f(\varphi) \neq 0$, then there exists $n_0 \in \mathbb{N}$ such that for all integer $n \geq n_0$ holds $\left|f(n \cdot \varphi)\right| \geq 1$;

(iv) If $\left|f(n \cdot \varphi)\right| \geq 1$ for some $n \in \mathbb{N}$ then $f(\varphi) \neq 0$.

Remark 1.3. The denotation $\hat{L}_p(X)$ is motivated by the fact that each linear continuous functional $f = \lambda_1 x_1 + ... + \lambda_{n(f)} x_{n(f)} \in L_p(X)$ satisfies mentioned conditions (i) – (iv) with $K = \{x_1, ..., x_{n(f)}\}$.

Proposition 1.4. (a) $\operatorname{supp} f = \varnothing$ iff $f = 0^{C_p(X)}$;

(b) The set $\operatorname{supp} f$ is unique for each FSF f;

(c) If $\varphi, \psi \in C_p(X)$ and $\varphi(x) = \psi(x)$ for each $x \in \operatorname{supp} f$ then $f(\varphi) = f(\psi)$;

(d) The mapping $s: \hat{L}_p(X) \to X$, $s(f) = \operatorname{supp} f$ is a well-defined finite-valued lower semicontinuous function.

Proof. We obviously have (i) $\Rightarrow$ (a), (i) $\Rightarrow$ (c).

(b) Let $f \in C_p^0 C_p(X)$, $f \neq 0^{C_p(X)}$ and there exist two different finite subsets K, M in X satisfying conditions (i) and (ii). Let, for example, $x_0 \in K \setminus M$. Take any neighborhood U_0 of x_0 with $U_0 \cap \left((K \cup M) \setminus \{x_0\}\right) = \varnothing$. Since $K = \operatorname{supp} f$, by (ii) there exist two functions $\varphi_0, \psi_0 \in C_p(X)$ coinciding out of U_0 such that $\left|f(\varphi_0) - f(\psi_0)\right| > \varepsilon_0 > 0$. At the same time, $M = \operatorname{supp} f$ as well, φ_0 coincides with ψ_0 on M, and now (c) implies $f(\varphi_0) = f(\psi_0)$, a contradiction.

(d) Evidently only the lower semicontinuity needs to be proved. Take an arbitrary open set $G \subset X$ and let $\operatorname{supp} f \cap G \neq \varnothing$ for some FSF f. Choose a disjoin family of neighborhoods U_x of points $x \in \operatorname{supp} f$ such that $U_x \subset G$ for each $x \in \operatorname{supp} f$. Fix the functions φ_x, ψ_x existing by (ii) for each point $x \in \operatorname{supp} f$ and its neighborhood U_x. Put

$$W = \bigcap_{x \in \operatorname{supp} f} \left\{g \in \hat{L}_p(X) : \left|g\left(\varphi_x\right) - g\left(\psi_x\right)\right| > \varepsilon_0\right\}. \quad (1)$$

It is easy to prove that W is open set in $\hat{L}_p(X)$, containing *f*. Moreover, if g is FSF and $\operatorname{supp} g \cap G = \varnothing$, then for any $x \in \operatorname{supp} f$ and for all $z \in \operatorname{supp} g$ we have $\varphi_x(z) = \psi_x(z) = 0$. Hence $g(\varphi_x) = g(\psi_x) = 0$ and $g \notin W$. Thus $f \in W \subset \subset \{g : \operatorname{supp} g \cap G \neq \varnothing\}$. ■

Remark 1.5. Formula (2.2) in [4] is incorrect. It must be in form (1).

2. Main result

At first let us define the class $\mathcal{H}$ of homeomorphisms, mentioned in the introduction.

Definition 2.1. Define the class $\mathcal{H}$ to be consisting of all homeomorphisms $h : C_p(X) \to C_p(Y)$ such that $h\left(0^X\right) = 0^Y$ and for all $x \in X$ and $y \in Y$ their images $\left(h^{-1}\right)^*(x)$ and $h^*(y)$ are in $\hat{L}_p(Y)$ and $\hat{L}_p(X)$, respectively.

Remark 2.2. It follows from Theorem 3.1 in [5] that, in particular, there exists a homeomorphism $h : C_p([1,\omega]) \to C_p([1,\omega^\omega])$, $h \in \mathcal{H}$, but these function spaces are not linearly homeomorphic (see [6]).

Our main result is

Theorem 2.3. Let $h : C_p(X) \to C_p(Y)$, $h \in \mathcal{H}$. Then $l(X) = l(Y)$.

A.V. Osipov in [7] gave such a characterization of the Lindelöf property.

Theorem 2.4. ([7], Theorem 3.7) A space X is Lindelöf iff the function space $C_p(X)$ has the following property:

Each 1-dense set in $C_p(X)$ contains a countable 1-dense subset. (2)

Recall that a set $A \subset C_p(X)$ is said to be 1-dense if $A \cap W(f,\{x\},\varepsilon) \neq \varnothing$ for each $f \in C_p(X)$, each $x \in X$, and $\varepsilon > 0$. By Theorem 2.3 we have such

Corollary 2.5. Let $h : C_p(X) \to C_p(Y)$, $h \in \mathcal{H}$ and $C_p(X)$ satisfies (2). Then $C_p(Y)$ satisfies (2) as well.

To prove the Theorem 2.3 it certainly suffices to establish only the following

Lemma 2.6. Let τ be an (infinite) cardinal, a homeomorphism $h : C_p(X) \to C_p(Y)$ belongs to $\mathcal{H}$ and $l(Y) \le \tau$. Then $l(X) \le \tau$.

We prove this Lemma following the same pattern as in [1], but we shall need a new definition of extractor.

For an arbitrary homeomorphism $h : C_p(X) \to C_p(Y)$, $h \in \mathcal{H}$, define the mappings $s : Y \to 2^X$, $s' : X \to 2^Y$ by the rules $s(y) = \operatorname{supp}\left(h^*(y)\right)$, $s'(x) = \operatorname{supp}\left(\left(h^{-1}\right)^*(x)\right)$.

First, we prove the surjectivity of *s*.

Proposition 2.7. The mapping *s* is a well-defined finite-valued lower semicontinuous surjective function.

Proof. All the statements are evident but surjectivity. Assume that there is a point $x_0 \in X \setminus s(Y)$. Consequently, $x_0 \notin S = \cup\left\{\operatorname{supp}\left(h^*(y)\right) : y \in \operatorname{supp}\left(\left(h^{-1}\right)^*(x_0)\right)\right\}$. Choose a function $\varphi_0 \in C_p(X)$ with $\varphi_0\big|_S \equiv 0$, $\varphi_0(x_0) = 1$. By item (c) of Proposition 1.4 we have $h^*(y)(\varphi_0) = h(\varphi_0)(y) = 0$ for each $y \in \operatorname{supp}\left(\left(h^{-1}\right)^*(x_0)\right)$. Applying (c) again, we obtain

$$\left(\left(h^{-1}\right)^*(x_0)\right)\left(h(\varphi_0)\right) = x_0\left(h^{-1}\left(h(\varphi_0)\right)\right) = x_0(\varphi_0) = \varphi_0(x_0) = 0,$$

a contradiction. ■

Let the symbol τ_X denote the topology of the space *X*. For each $y \in Y$ and each $V \in \tau_X$ put $r_V(y) = \left|h^*(y)\left(\varphi_y^V\right)\right|$, where $\varphi_y^V \in C_p(X)$, $\varphi_y^V(x) = \begin{cases} 0, & x \in V \cap s(y) \\ 1, & x \notin V \end{cases}$.

Also put $G : \tau_X \to 2^Y$, $G(V) = \left\{y \in Y : r_V(y) = \left|h^*(y)\left(\varphi_y^V\right)\right| = 0\right\}$.

Recall the basic definition from [1]:

Definition 2.8. Given multivalued lower semicontinuous function $\eta : Y \to 2^X$, any mapping $G : \tau_X \to 2^Y$ is said to be η-extractor if the following three conditions hold:

(e1) For each open $U \subset X$ we have $\eta^*(U) = \{y \in Y : \eta(y) \subset U\} \subset G(U)$;

(e2) If $U, V \in \tau_X$, $U \subset V$, and $y \in G(V) \setminus G(U)$ then $\eta(y) \cap (V \setminus U) \neq \varnothing$;

(e3) If a sequence $(U_n)_{n\in\mathbb{N}} \subset \tau_X$, $U_n \subset U_{n+1}$ is such that $Y \subset \cup_{n\in\mathbb{N}}\left(\cap_{m\geq n} G(U_m)\right)$ then $X \subset \cup_{n\in\mathbb{N}} U_n$.

So, we now must check the conditions (e1), (e2), (e3) for $\eta = s$.

Proposition 2.9. The function *s* and mapping *G* satisfy conditions (e1), (e2), (e3).

Proof. Let *V* be an open subset in *X* and $y \in Y$ is such that $s(y) \subset V$ (i.e., $y \in s^*(V)$). Then $\varphi_y^V(s(y)) = \{0\}$. Consequently, by 1.4 (c), $h^*(y)\left(\varphi_y^V\right) = 0$, i.e., $y \in G(V)$ and (e1) holds.

Now take any $U, V \in \tau_X$, such that $U \subset V$ and $y \in G(V) \setminus G(U)$. Since $y \notin G(U)$ then $h^*(y)\left(\varphi_y^U\right) \neq 0$ and we have $s(y) \not\subset U$ by definition of the function φ_y^U. The assumption $s(y) \cap V = \varnothing$ implies $\varphi_y^V\big|_{s(y)} \equiv 1 \equiv \varphi_y^U\big|_{s(y)}$. Therefore, by 1.4 (c) again, $h^*(y)\left(\varphi_y^V\right) = h^*(y)\left(\varphi_y^U\right) \neq 0$, a contradiction with $y \in G(V)$. The item (e2) is proved.

Let us verify (e3). Let us suppose that (e3) is not true and $x_0 \in X \setminus \left(\cup_{n\in\mathbb{N}} U_n\right)$. Inclusion $Y \subset \cup_{n\in\mathbb{N}}\left(\cap_{m\geq n} G(U_m)\right)$ implies that there exists some $k \in \mathbb{N}$ such that

$s'(x_0) \subset G(U_k)$. Hence, $h^*(y)\left(\varphi_y^{U_k}\right) = 0$ for each $y \in s'(x_0)$. Consider the function $\varphi \in C_p(X)$, $\varphi(x) = \begin{cases} 1, & x \in X \setminus U_k \\ 0, & x \in s(s'(x_0)) \cap U_k \end{cases}$. Evidently we have $\varphi(x_0) = 1$ and $\varphi\big|_{s(y)} \equiv \varphi_y^{U_k}\big|_{s(y)}$ for any $y \in s'(x_0)$. It follows from this that $h^*(y)\left(\varphi_y^{U_k}\right) = h^*(y)(\varphi) = h(\varphi)(y) = 0$. It means that $h(\varphi)\big|_{s'(x_0)} \equiv 0$ and therefore $\left(h^{-1}\right)^*(x_0)(h(\varphi)) = \varphi(x_0) = 0$. This contradiction finishes the proof. ■

We need the next two lemmas to show, using the terminology of [1], that the s-extractor G is synchronized with the Lindelöf number of Y. The Lemmas 10 and 11 below are an adaptation of the lemmas 6 and 7 from [1] to our nonlinear situation.

Recall that an open subset $V \subset X$ is said to be adequate (see [1]) if some its decomposition $V = \cup\{F_k : k \in \mathbb{N}\}$ in increasing sequence of zero-sets F_k has the property that for each $k \in \mathbb{N}$ there exist $y_k \in Y$ with $s(y_k) \subset V$ and $s(y_k) \setminus F_k \neq \varnothing$.

Lemma 2.10. Let I be an infinite set of cardinality $|I| = \tau \geq \aleph_0$, and let $\gamma = \{V_i : i \in I\}$ be some family of adequate open subsets in X, which is stable under taking finite unions. Put $V = \cup\gamma$. Then $F(V)$ is a F_τ-set in Y.

Proof. For each $V_i \in \gamma$ fix its decomposition $\left(F_k^i\right)_{k \in \mathbb{N}}$ and for each $k \in \mathbb{N}$ fix a function $\eta_k^i \in C_p(X)$, which is equal to zero on F_k^i and equal to k out of V_i. If $y \in Y$, $s(y) \subset V_i$, and $k \in \mathbb{N}$ then put

$$U_k^i(y) = \bigcap_{j \leq k} \left\{ y' \in Y : \left| h\left(\eta^i_{j+k_y^i}\right)(y') \right| < 1 \right\},$$

where $k_y^i = \min\left\{k : s(y) \subset F_k^i\right\}$. Of course, $U_k^i(y)$ is open neighborhood of y because the functions $h\left(\eta^i_{j+k_y^i}\right)$ are continuous on Y and are equal to zero at y $(j = 1, 2, \ldots k)$.

Put

$$A_i = \bigcap_{k \in \mathbb{N}} \bigcup \left\{U_k^i(y) : s(y) \subset V_i\right\},\ B_i = \left\{y \in Y : s(y) \cap (V \setminus V_i) \neq \varnothing\right\},\ A = \bigcap_{i \in I}\left(A_i \cup B_i\right).$$

All sets A_i clearly are G_δ-sets. All sets B_i are G_δ-sets as well. Indeed, since the mapping s is finite-valued and lower semicontinuous, then we have $B_i = \bigcap_{k \in \mathbb{N}} \left\{ y \in Y : s(y) \cap \left(V \setminus F_k^i\right) \neq \varnothing \right\}$. Let us show that $F(V) = Y \setminus A$.

Take any $y \in F(V)$. It means that $h^*(y)\left(\varphi_y^V\right) \neq 0$. In addition, since the set $s(y)$ is finite, there exists $i \in I$ such that $s(y) \cap V \subset V_i$. Therefore, $y \notin B_i$. Moreover, applying the item (iii) from Definition 1.2, one can find a $k \in \mathbb{N}$ such that $s(y) \cap V \subset F_n^i \subset V_i$ and $\left|h^*(y)\left(n \cdot \varphi_y^V\right)\right| > 1$ for all $n \geq k$. Let us note that in this case we have $\left.\left(n \cdot \varphi_y^V\right)\right|_{s(y)} \equiv \left.\eta_n^i\right|_{s(y)}$ for all $n \geq k$. By item (c) of Proposition 1.4 we conclude that $h^*(y)\left(n \cdot \varphi_y^V\right) = h^*(y)\left(\eta_n^i\right)$ for all $n \geq k$. We are going to show that $y \notin \bigcup\left\{U_k^i(z) : s(z) \subset V_i\right\}$.

Let $s(z) \subset V_i$. Then we have by item (iii) from Definition 1.2 that $h^*(y)\left(\eta_{k+n_z^i}^i\right) = h^*(y)\left(\left(k + n_z^i\right) \cdot \varphi_y^V\right) \geq 1$. This inequality means that $y \notin U_k^i(z)$. Therefore, $y \notin A_i$ and, consequently, $y \notin A$. The inclusion $F(V) \subset Y \setminus A$ is true.

Let us check the inverse inclusion. Take $y \notin A$ and let $i \in I$ be such that $y \notin A_i \cup B_i$. We can suppose by adequateness of V_i that for each $k \in \mathbb{N}$ there exists $y_k \in Y$ such that $s(y_k) \subset V_i$ and $s(y_k) \setminus F_k^i \neq \varnothing$. Since $y \notin B_i$, then $s(y) \cap V \subset V_i$. Fix any $p \in \mathbb{N}$ such that $s(y) \cap V \subset F_p^i \subset V_i$. Since $y \notin A_i$, there exists $m \in \mathbb{N}$ such that $y \notin \bigcup\left\{U_m^i(z) : s(z) \subset V_i\right\}$. Choose z such that $s(z) \setminus F_p^i \neq \varnothing$. Then $n_z^i > p$ and there exists $l \leq m$ such that $\left|h\left(\eta_{l+n_z^i}^i\right)(y)\right| = \left|h^*(y)\left(\eta_{l+n_z^i}^i\right)\right| \geq 1$.

It follows from the definition of the functions $\eta_{l+n_z^i}^i$, φ_y^V and from inclusions $s(y) \cap V \subset F_p^i \subset F_{l+n_z^i}^i \subset V_i$ that $\left.\left(\left(l + n_z^i\right) \cdot \varphi_y^V\right)\right|_{s(y)} \equiv \left.\eta_{l+n_z^i}^i\right|_{s(y)}$. Therefore, using again the item (c) of Proposition 1.4, we obtain $1 \leq \left|h^*(y)\left(\eta_{l+n_z^i}^i\right)\right| = \left|h^*(y)\left(\left(l + n_z^i\right) \cdot \varphi_y^V\right)\right|$. Now we can conclude by item (iv) from Definition 1.2 that $h^*(y)\left(\varphi_y^V\right) \neq 0$ and $y \in F(V)$. ■

For an arbitrary family $\mathcal{U}$ of sets, we denote by $\mathcal{U}'$ the family of unions of all at most countable subfamilies of $\mathcal{U}$. Denote by $\mathcal{L}$ the family of all F_τ-subsets of *Y*, and let $\mathcal{B}$ be a base of topology of *X* consisting of cozero-sets.

The proof of the next lemma is the same as in [1], Lemma 7, and by this reason we omit it.

Lemma 2.11. Let τ be an infinite cardinal, $\mathcal{U} \subset \mathcal{B}$ be an open non τ-trivial cover of *X*. Then, for any subfamily $\gamma \subset \mathcal{U}$ with $|\gamma| \leq \tau$, there exists a subfamily $\gamma' \subset \mathcal{U}'$ which is stable under finite unions, consisting of adequate sets, has a cardinality $|\gamma'| \leq \tau$, and satisfies $\cup\gamma \subset \cup\gamma'$.

Proof of Lemma 2.6. If $l(Y) \le \tau$, then, certainly, $l(Z) \le \tau$ for each $Z = Z_1 \cap ... \cap Z_n$, where $n \in \mathbb{N}$, $Z_1, ..., Z_n \in \mathcal{L}$. Now, combining Lemma 2.11 and Lemma 2.10, we can conclude that for each open non τ-trivial cover $\mathcal{U} \subset \mathcal{B}$ of X and for each subfamily $\gamma \subset \mathcal{U}$ with $|\gamma| \le \tau$ there exists a subfamily $\mu \subset \mathcal{U}$ with $|\mu| \le \tau$ such that $\cup\gamma \subset \cup\mu$ and $F(\cup\mu) \in \mathcal{L}$. Indeed, apply Lemma 2.10 to the family μ of elements of $\mathcal{U}$, which belong to at most countable subfamilies of $\mathcal{U}$ forming elements of the family γ' from Lemma 2.11.

By Proposition 2.7, we have a finite-valued lower semicontinuous mapping $s : Y \to 2^X$ with nonempty values $s(y)$ (see 1.4, (a)). Thus, all conditions of Proposition 3 from [1] holds. Consequently, $l(X) \le \tau$. ■

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