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On endo-commutative algebraic structures on two-dimensional vector spaces over an arbitrary field

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Abstract. In this paper, we describe the class of all two-dimensional endo-commutative algebras over any base field. Thereby, we extend recent results of Takahasi, Shirayanagi, and Tsukada on description of the class of two-dimensional endo-commutative algebras to the case of an arbitrary field. The concept of an endo-commutative algebra was first introduced by aforementioned authors; in the same works, the motivations to study this class of algebras also were presented. In this paper, we present the canonical representatives of the isomorphism classes of two-dimensional endo-commutative algebras over an arbitrary field.
Keywords: endo-commutative algebra, isomorphism, matrix of structure constants, classification

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Научная статья

Об эндо-коммутативных алгебраических структурах на двумерных векторных пространствах над произвольным полем

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Аннотация. Описывается класс всех двумерных эндо-коммутативных алгебр над любым полем. Обобщены недавние результаты Такахаси, Шираянаги и Цукады о классификации двумерных эндо-коммутативных алгебр до произвольного поля. Понятие эндо-коммутативной алгебры впервые было введено в работах вышеупомянутых авторов, там же приведены доводы для изучения этого класса алгебр. В данной работе будут представлены канонические представители классов изоморфизмов эндо-коммутативных алгебр размерности два над произвольным полем.
Ключевые слова: эндо-коммутативная алгебра, изоморфизм, матрица структурных констант, классификация

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1. Introduction

The classification problem for a given class of algebras, up to isomorphism, is one of the important and difficult problems of algebra. So far, two approaches are known to the solution of the problem. One of them is the structural (basis free, invariant) approach. For instance, the classifications of finite-dimensional simple and semi-simple associative algebras by Wedderburn and simple and semi-simple Lie algebras by Cartan are examples of such approach. But it is observed that this approach becomes more difficult when considering more general types of algebras. Another approach to the solution of the problem is coordinate-based. Many researchers have used this type of approach to classify various, mainly finite-dimensional, classes of algebras: associative [1–2], Lie [3–7], Jordan [8], and Leibniz [9–12]. These two approaches are somehow complementary to each other.

There were attempts to classify all fixed-dimensional algebras, for example, in [13] a classification of all 2-dimensional algebras, by the basis-free approach, was stated over any basic field. One disadvantage of the basis-free approach is that an application of the obtained classification result to classification of a given class of algebras is hardly possible. In this respect, the coordinate-based classification has advantage over it. For the coordinate-based approach in classification of all two-dimensional algebras over fields with some restrictions, one can see [14–16] and, in [17, 18], its different applications.

The concept of endo-commutative algebra was first introduced in [19] (also see [20, 21]), where the authors gave a complete classification of two-dimensional endo-commutative algebras over certain fields. In addition, these authors give a justification to study the class of endo-commutative algebras.

In this paper we provide a complete classification of all endo-commutative algebras structures on a two-dimensional vector space over any base field. The result of the paper is based on a result obtained recently in [22] on complete classification of all two-dimensional algebras over any base field. The result of the paper generalizes those obtained in [19–21].

The organization of the paper is as follows. The next section is Preliminaries, where we include the necessary definitions and results to be used throughout the paper. The main results of the paper are in Section 3 and onward. Subsections 3.1, 3., and 3.3 contain the description of all two-dimensional endo-commutative algebras over a field $\mathbb{F}$ whose characteristic is neither 2 nor 3, characteristic 2 and characteristic 3, respectively. In Corollary 3.1 we recover the result obtained in [19]. Section 4 contains the classification of all two-dimensional curled algebras over any base field, and Section 5 is devoted to the description of two-dimensional endo-commutative curled algebras, where we show how to recover the result of [19] on the endo-commutative curled algebras.

2. Preliminaries

We begin with shortly recalling some concepts that are used in the paper.

Let A be an n-dimensional algebra over a field $\mathbb{F}$ and e = (e_1, e_2, …, e_n) be a basis of the underlying vector space of A. Then on the basis e the algebra A is represented by an $n \times n^2$ matrix (called the matrix of structure constants, shortly MSC)

$$A=\begin{pmatrix} a_{11}^{1} & a_{12}^{1} & \dots & a_{1n}^{1} & a_{21}^{1} & a_{22}^{1} & \dots & a_{2n}^{1} & \dots & a_{n1}^{1} & a_{n2}^{1} & \dots & a_{nn}^{1} \\ a_{11}^{2} & a_{12}^{2} & \dots & a_{1n}^{2} & a_{1n}^{2} & a_{22}^{2} & \dots & a_{2n}^{2} & \dots & a_{n1}^{2} & a_{n2}^{2} & \dots & a_{nn}^{2} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{11}^{n} & a_{12}^{n} & \dots & a_{1n}^{n} & a_{1n}^{n} & a_{22}^{n} & \dots & a_{2n}^{n} & \dots & a_{n1}^{n} & a_{n2}^{n} & \dots & a_{nn}^{n} \end{pmatrix}$$

as follows

$$e_i \cdot e_j = \sum_{k=1}^{n} a_{ij}^{k} \cdot e_k, \text{ where } i, j = 1, 2, \dots, n.$$

Therefore, the product on A with respect to the basis e is written as follows:

$$\mathrm{x}\cdot\mathrm{y} = \mathrm{e}\,A(x \otimes y) \tag{2.1}$$

for any x = ex, y = ey, where $x = (x_1, x_2, \dots, x_n)^T$ and $y = (y_1, y_2, \dots, y_n)^T$ are column coordinate vectors of x and y, respectively, $x \otimes y$ is the tensor (Kronecker) product of the vectors x and y. Now and onward for the product "x·y" on A we use the juxtaposition "xy". Since in this paper we work with a basis, we do not distinguish an algebra A and its MSC A in a fixed basis.

If A is a two-dimensional algebra over a field $\mathbb{F}$ and e = (e_1, e_2) is a basis, then we take MSC of A as follows:

$$A=\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{pmatrix}.$$

In the paper, we make use of the following results on classification, up to isomorphism, for all two-dimensional algebras over any base field $\mathbb{F}$. The results were obtained in [22] in terms of their MSC.

Theorem 1. Any nontrivial two-dimensional algebra over a field $\mathbb{F}$ with $\mathrm{Char}(\mathbb{F}) \neq 2,3$ is isomorphic to only one of the following algebras listed by their matrices of structure constants:

- $A_1(c)=\begin{pmatrix}\alpha_1 & \alpha_2 & 1+\alpha_2 & \alpha_4\\ \beta_1 & -\alpha_1 & 1-\alpha_1 & -\alpha_2\end{pmatrix}$, where $c=(\alpha_1,\alpha_2,\alpha_4,\beta_1)\in\mathbb{F}^4$;
- $A_2(c)=\begin{pmatrix}\alpha_1 & 0 & 0 & \alpha_4\\ 1 & \beta_2 & 1-\alpha_1 & 0\end{pmatrix}$, where $c=(\alpha_1,\alpha_4,\beta_2)\in\mathbb{F}^3$ and $\alpha_4\neq 0$;
- $A_3(c)=\begin{pmatrix}\alpha_1 & 0 & 0 & \alpha_4\\ 0 & \beta_2 & 1-\alpha_1 & 0\end{pmatrix}\simeq\begin{pmatrix}\alpha_1 & 0 & 0 & a^2\alpha_4\\ 0 & \beta_2 & 1-\alpha_1 & 0\end{pmatrix}$, where

$c=(\alpha_1,\alpha_4,\beta_2)\in\mathbb{F}^3$, $a\in\mathbb{F}$ and $a\neq 0$;

- $A_4(c)=\begin{pmatrix}0 & 1 & 1 & 0\\ \beta_1 & \beta_2 & 1 & -1\end{pmatrix}$, where $c=(\beta_1,\beta_2)\in\mathbb{F}^2$;
- $A_5(c)=\begin{pmatrix}\alpha_1 & 0 & 0 & 0\\ 1 & 2\alpha_1-1 & 1-\alpha_1 & 0\end{pmatrix}$, where $c=\alpha_1\in\mathbb{F}$;
- $A_6(c)=\begin{pmatrix}\alpha_1 & 0 & 0 & \alpha_4\\ 1 & 1-\alpha_1 & -\alpha_1 & 0\end{pmatrix}$, where $c=(\alpha_1,\alpha_4)\in\mathbb{F}^2$ and $\alpha_4\neq 0$;
- $A_7(c)=\begin{pmatrix}\alpha_1 & 0 & 0 & \alpha_4\\ 0 & 1-\alpha_1 & -\alpha_1 & 0\end{pmatrix}\simeq\begin{pmatrix}\alpha_1 & 0 & 0 & a^2\alpha_4\\ 0 & 1-\alpha_1 & -\alpha_1 & 0\end{pmatrix}$, where

$c=(\alpha_1,\alpha_4)\in\mathbb{F}^2$, $a\in\mathbb{F}$ and $a\neq 0$;

- $A_8(c)=\begin{pmatrix}0 & 1 & 1 & 0\\ \beta_1 & 1 & 0 & -1\end{pmatrix}$, where $c=\beta_1\in\mathbb{F}$;
- $A_9=\begin{pmatrix}\frac{1}{3} & 0 & 0 & 0\\ 1 & \frac{2}{3} & -\frac{1}{3} & 0\end{pmatrix}$;
- $A_{10}(c)=\begin{pmatrix}0 & 1 & 1 & 1\\ \beta_1 & 0 & 0 & -1\end{pmatrix}\simeq\begin{pmatrix}0 & 1 & 1 & 1\\ \beta_1'(a) & 0 & 0 & -1\end{pmatrix}$, where $c=\beta_1\in\mathbb{F}$, $a\in\mathbb{F}$, the

polynomial $(\beta_1t^3-3t-1)(\beta_1t^2+\beta_1t+1)(\beta_1t^3+6\beta_1t^2+3\beta_1t+\beta_1-2)$ has no root in $\mathbb{F}$ and $\beta_1'(t)=\dfrac{(\beta_1t^3+6\beta_1t^2+3\beta_1t+\beta_1-2)^2}{(\beta_1t^2+\beta_1t+1)^3}$;

- $A_{11}(c)=\begin{pmatrix}0 & 0 & 0 & 1\\ \beta_1 & 0 & 0 & 0\end{pmatrix}\simeq\begin{pmatrix}0 & 0 & 0 & 1\\ a^3\beta_1^{\pm1} & 0 & 0 & 0\end{pmatrix}$, where the polynomial β_1-t^3

has no roots in $\mathbb{F}$, $a,c=\beta_1\in\mathbb{F}$ and a, $\beta_1\neq 0$;

- $A_{12}(c)=\begin{pmatrix}0 & 1 & 1 & 0\\ \beta_1 & 0 & 0 & -1\end{pmatrix}\simeq\begin{pmatrix}0 & 1 & 1 & 0\\ a^2\beta_1 & 0 & 0 & -1\end{pmatrix}$, where $a,c=\beta_1\in\mathbb{F}$ and $a\neq 0$;
- $A_{13}=\begin{pmatrix}0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0\end{pmatrix}$.

Theorem 2. Any nontrivial two-dimensional algebra over a field $\mathbb{F}$ with $\mathrm{Char}(\mathbb{F})=2$ is isomorphic to only one of the following algebras listed by their matrices of structure constants:

- $A_{1,2}(c)=\begin{pmatrix}\alpha_1 & \alpha_2 & 1+\alpha_2 & \alpha_4\\ \beta_1 & \alpha_1 & 1+\alpha_1 & \alpha_2\end{pmatrix}$, where $c=(\alpha_1,\alpha_2,\alpha_4,\beta_1)\in\mathbb{F}^4$;
- $A_{2,2}(c)=\begin{pmatrix}\alpha_1 & 0 & 0 & \alpha_4\\ 1 & \beta_2 & 1+\alpha_1 & 0\end{pmatrix}$, where $c=(\alpha_1,\alpha_4,\beta_2)\in\mathbb{F}^3$ and $\alpha_4\neq 0$;
- $A_{2,2}(\alpha_1,0,1)=\begin{pmatrix}\alpha_1 & 0 & 0 & 0\\ 1 & 1 & 1+\alpha_1 & 0\end{pmatrix}$, where $\alpha_1\in\mathbb{F}$;
- $A_{3,2}(c)=\begin{pmatrix}\alpha_1 & 0 & 0 & \alpha_4\\ 0 & \beta_2 & 1+\alpha_1 & 0\end{pmatrix}\simeq\begin{pmatrix}\alpha_1 & 0 & 0 & a^2\alpha_4\\ 0 & \beta_2 & 1+\alpha_1 & 0\end{pmatrix}$, where

$c=(\alpha_1,\alpha_4,\beta_2)\in\mathbb{F}^3$, $a\in\mathbb{F}$ and $a\neq 0$;
- $A_{4,2}(c)=\begin{pmatrix}\alpha_1 & 1 & 1 & 0\\ \beta_1 & \beta_2 & 1+\alpha_1 & 1\end{pmatrix}\simeq\begin{pmatrix}\alpha_1 & 1 & 1 & 0\\ \beta_1+(1+\beta_2)a+a^2 & \beta_2 & 1+\alpha_1 & 1\end{pmatrix}$, where

$c=(\alpha_1,\beta_1,\beta_2)\in\mathbb{F}^3$;
- $A_{5,2}(c)=\begin{pmatrix}\alpha_1 & 0 & 0 & \alpha_4\\ 1 & 1+\alpha_1 & \alpha_1 & 0\end{pmatrix}$, where $c=(\alpha_1,\alpha_4)\in\mathbb{F}^2$ and $\alpha_4\neq 0$;
- $A_{5,2}(1,0)=\begin{pmatrix}1 & 0 & 0 & 0\\ 1 & 0 & 1 & 0\end{pmatrix}$;
- $A_{6,2}(c)=\begin{pmatrix}\alpha_1 & 0 & 0 & \alpha_4\\ 0 & 1+\alpha_1 & \alpha_1 & 0\end{pmatrix}\simeq\begin{pmatrix}\alpha_1 & 0 & 0 & a^2\alpha_4\\ 0 & 1+\alpha_1 & \alpha_1 & 0\end{pmatrix}$, where

$c=(\alpha_1,\alpha_4)\in\mathbb{F}^2$, $a\in\mathbb{F}$ and $a\neq 0$;
- $A_{7,2}(c)=\begin{pmatrix}\alpha_1 & 1 & 1 & 0\\ \beta_1 & 1+\alpha_1 & \alpha_1 & 1\end{pmatrix}\simeq\begin{pmatrix}\alpha_1 & 1 & 1 & 0\\ \beta_1+a\alpha_1+a+a^2 & 1+\alpha_1 & \alpha_1 & 1\end{pmatrix}$, where

$c=(\alpha_1,\beta_1)\in\mathbb{F}^2$ and $a\in\mathbb{F}$;
- $A_{8,2}(c)=\begin{pmatrix}0 & 1 & 1 & 1\\ \beta_1 & 0 & 0 & 1\end{pmatrix}\simeq\begin{pmatrix}0 & 1 & 1 & 1\\ \beta_1'(a) & 0 & 0 & 1\end{pmatrix}$, where the polynomial

$(\beta_1t^3+t+1)(\beta_1t^2+\beta_1t+1)$ has no root in $\mathbb{F}$, $a\in\mathbb{F}$ and $\beta_1'(t)=\dfrac{\left(\beta_1{}^2t^3+\beta_1t+\beta_1\right)^2}{\left(\beta_1t^2+\beta_1t+1\right)^3}$;
- $A_{9,2}(c)=\begin{pmatrix}0 & 0 & 0 & 1\\ \beta_1 & 0 & 0 & 0\end{pmatrix}\simeq\begin{pmatrix}0 & 0 & 0 & 1\\ a^3\beta_1^{\pm1} & 0 & 0 & 0\end{pmatrix}$, where $a,c=\beta_1\in\mathbb{F}$ and $a\neq 0$, the

polynomial β_1+t^3 has no root in $\mathbb{F}$;
- $A_{10,2}(c)=\begin{pmatrix}1 & 1 & 1 & 0\\ \beta_1 & 1 & 1 & 1\end{pmatrix}\simeq\begin{pmatrix}1 & 1 & 1 & 0\\ \beta_1+a+a^2 & 1 & 1 & 1\end{pmatrix}$, where $a,c=\beta_1\in\mathbb{F}$;

- $A_{11,2}(c)=\begin{pmatrix}0 & 1 & 1 & 0\\ \beta_1 & 0 & 0 & 1\end{pmatrix}\simeq\begin{pmatrix}0 & 1 & 1 & 0\\ b^2(\beta_1+a^2) & 0 & 0 & 1\end{pmatrix}$, where $a,b,c=\beta_1\in\mathbb{F}$ and $b\neq 0$;
- $A_{12,2}=\begin{pmatrix}0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0\end{pmatrix}$.

Theorem 3. Any non-trivial two-dimensional algebra over a field $\mathbb{F}$ with $\mathrm{Char}(\mathbb{F})=3$ is isomorphic to only one of the following algebras listed by their matrices of structure constants:

- $A_{1,3}(c)=\begin{pmatrix}\alpha_1 & \alpha_2 & \alpha_2+1 & \alpha_4\\ \beta_1 & -\alpha_1 & 1-\alpha_1 & -\alpha_2\end{pmatrix}$, where $c=(\alpha_1,\alpha_2,\alpha_4,\beta_1)\in\mathbb{F}^4$;
- $A_{2,3}(c)=\begin{pmatrix}\alpha_1 & 0 & 0 & \alpha_4\\ 1 & \beta_2 & 1-\alpha_1 & 0\end{pmatrix}$, where $c=(\alpha_1,\alpha_4,\beta_2)\in\mathbb{F}^3$ and $\alpha_4\neq 0$;
- $A_{3,3}(c)=\begin{pmatrix}\alpha_1 & 0 & 0 & \alpha_4\\ 0 & \beta_2 & 1-\alpha_1 & 0\end{pmatrix}\simeq\begin{pmatrix}\alpha_1 & 0 & 0 & a^2\alpha_4\\ 0 & \beta_2 & 1-\alpha_1 & 0\end{pmatrix}$, where

$c=(\alpha_1,\alpha_4,\beta_2)\in\mathbb{F}^3$, $a\in\mathbb{F}$ and $a\neq 0$;

- $A_{4,3}(c)=\begin{pmatrix}0 & 1 & 1 & 0\\ \beta_1 & \beta_2 & 1 & -1\end{pmatrix}$, where $c=(\beta_1,\beta_2)\in\mathbb{F}^2$;
- $A_{5,3}(c)=\begin{pmatrix}\alpha_1 & 0 & 0 & 0\\ 1 & 2\alpha_1-1 & 1-\alpha_1 & 0\end{pmatrix}$, where $c=\alpha_1\in\mathbb{F}$;
- $A_{6,3}(c)=\begin{pmatrix}\alpha_1 & 0 & 0 & \alpha_4\\ 1 & 1-\alpha_1 & -\alpha_1 & 0\end{pmatrix}$, where $c=(\alpha_1,\alpha_4)\in\mathbb{F}^2$ and $\alpha_4\neq 0$;
- $A_{7,3}(c)=\begin{pmatrix}\alpha_1 & 0 & 0 & \alpha_4\\ 0 & 1-\alpha_1 & -\alpha_1 & 0\end{pmatrix}\simeq\begin{pmatrix}\alpha_1 & 0 & 0 & a^2\alpha_4\\ 0 & 1-\alpha_1 & -\alpha_1 & 0\end{pmatrix}$, where

$c=(\alpha_1,\alpha_4)\in\mathbb{F}^2$, $a\in\mathbb{F}$ and $a\neq 0$;

- $A_{8,3}(c)=\begin{pmatrix}0 & 1 & 1 & 0\\ \beta_1 & 1 & 0 & -1\end{pmatrix}$, where $c=\beta_1\in\mathbb{F}$;
- $A_{9,3}(\beta_1)=\begin{pmatrix}0 & 1 & 1 & 1\\ \beta_1 & 0 & 0 & -1\end{pmatrix}\simeq\begin{pmatrix}0 & 1 & 1 & 1\\ \beta_1'(a) & 0 & 0 & -1\end{pmatrix}$, where the polynomial

$(1-t^3)(\beta_1t^2+\beta_1t+1)(\beta_1^2t^3+\beta_1-2)$ has no roots in $\mathbb{F}$, $a\in\mathbb{F}$ and $\beta_1'(t)=\frac{(\beta_1^2t^3+\beta_1-2)^2}{(\beta_1t^2+\beta_1t+1)^3}$;

- $A_{10,3}(c)=\begin{pmatrix}0 & 0 & 0 & 1\\ \beta_1 & 0 & 0 & 0\end{pmatrix}\simeq\begin{pmatrix}0 & 0 & 0 & 1\\ a^3\beta_1^{\pm 1} & 0 & 0 & 0\end{pmatrix}$, where the polynomial β_1-t^3 has

no roots $a,c=\beta_1\in\mathbb{F}$ and $a,\beta_1\neq 0$;

- $A_{11,3}(c)=\begin{pmatrix}0 & 1 & 1 & 0\\ \beta_1 & 0 & 0 & -1\end{pmatrix}\simeq\begin{pmatrix}0 & 1 & 1 & 0\\ a^2\beta_1 & 0 & 0 & -1\end{pmatrix}$, where $a,c=\beta_1\in\mathbb{F}$, $a\neq 0$;

- $A_{12,3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 \end{pmatrix}$;
- $A_{13,3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

Definition 1. An algebra A is said to be endo-commutative if $x^2y^2 = (xy)^2$, for any $x, y \in A$.

Lemma 1. An algebra A is endo-commutative if and only if

$$A(A\otimes A)\left(x^{\otimes 2}\otimes y^{\otimes 2}-(x\otimes y)^{\otimes 2}\right)=0, \quad (2.2)$$

where A is MSC of A.

Proof. According to (2.1) we write

$$x^2 = eAx^{\otimes 2},\ y^2 = eAy^{\otimes 2},\ x^2y^2 = eA\left(Ax^{\otimes 2}\otimes Ay^{\otimes 2}\right) = eA(A\otimes A)\left(x^{\otimes 2}\otimes y^{\otimes 2}\right)$$

and $(xy)^2 = eA\left(A(x\otimes y)\otimes A(x\otimes y)\right) = eA(A\otimes A)(x\otimes y)^{\otimes 2}$. Therefore, the equality $x^2y^2 = (xy)^2$ in terms of MSC and the coordinate vectors are written as follows

$$eA(A\otimes A)\left(x^{\otimes 2}\otimes y^{\otimes 2}\right) = eA(A\otimes A)(x\otimes y)^{\otimes 2} \text{ i.e.,}$$

$$A(A\otimes A)\left(x^{\otimes 2}\otimes y^{\otimes 2}-(x\otimes y)^{\otimes 2}\right)=0$$

what is required to get.

In [19], the class of endo-commutative algebras was split into two classes: those of curled and straight algebras. One can find there a list of such curled algebras, up to isomorphism, over the field $\mathbb{F}=\mathbb{Z}_2$ as well. The definition was given as follows:

Definition 2. An algebra A is said to be curled if $x^2 = \lambda(x)x$ for any $x \in A$, where $\lambda(x)\in\mathbb{F}$.

Let $e = (e_1, e_2, \dots, e_n)$ be a basis of A. According to (2.1) one has $x^2 = eAx^{\otimes 2}$, $\lambda(x)x = e\lambda(x)x$ and, therefore, the equality $x^2 = \lambda(x)x$ in terms of MSC of A and the coordinate vector of x can be written as follows:

$$Ax^{\otimes 2}-\lambda(x)x=0. \quad (2.3)$$

In Section 4 we give a complete classification of all two-dimensional curled algebras over any base field and then split them into curled and straight algebras. The lists of these algebras given in [19] come as a particular case.

First, we start with the classification of all two-dimensional endo-commutative algebras based on the result of Theorems 1–3.

Let now A be a two-dimensional algebra over a field $\mathbb{F}$ and let

$$A=\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{pmatrix}$$

be its MSC on a basis $e = (e_1, e_2)$, $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Then, (2.2) is nothing else than

$$\begin{cases} x_1^2y_1y_2(A_3-A_5)+x_1^2y_2^2(A_4-A_6)+x_1x_2y_1y_2(A_6-A_4+A_{11}-A_{13}) \\ +x_1x_2y_1^2(A_5-A_3)+x_1x_2y_2^2(A_{12}-A_{14})+x_2^2y_1^2(A_{13}-A_{11})+x_2^2y_1y_2(A_{14}-A_{12})=0, \\ x_1^2y_1y_2(B_3-B_5)+x_1^2y_2^2(B_4-B_6)+x_1x_2y_1y_2(B_6-B_4+B_{11}-B_{13}) \\ +x_1x_2y_1^2(B_5-B_3)+x_1x_2y_2^2(B_{12}-B_{14})+x_2^2y_1^2(B_{13}-B_{11})+x_2^2y_1y_2(B_{14}-B_{12})=0, \end{cases} \quad (2.4)$$

where

$A_3=\alpha_1^2\alpha_3+\alpha_1\alpha_2\beta_3+\alpha_3^2\beta_1+\alpha_4\beta_1\beta_3$, $A_4=\alpha_1^2\alpha_4+\alpha_1\alpha_2\beta_4+\alpha_3\alpha_4\beta_1+\alpha_4\beta_1\beta_4$,

$A_5=\alpha_1^2\alpha_2+\alpha_2{}^2\beta_1+\alpha_1\alpha_3\beta_2+\alpha_4\beta_1\beta_2$, $A_6=\alpha_1\alpha_2{}^2+\alpha_2{}^2\beta_2+\alpha_2\alpha_3\beta_2+\alpha_4\beta_2{}^2$,

$A_{11}=\alpha_1\alpha_3{}^2+\alpha_2\alpha_3\beta_3+\alpha_3^2\beta_3+\alpha_4\beta_3{}^2$, $A_{12}=\alpha_1\alpha_3\alpha_4+\alpha_2\alpha_3\beta_4+\alpha_3\alpha_4\beta_3+\alpha_4\beta_3\beta_4$,

$A_{13}=\alpha_1^2\alpha_4+\alpha_2\alpha_4\beta_1+\alpha_1\alpha_3\beta_4+\alpha_4\beta_1\beta_4$, $A_{14}=\alpha_1\alpha_2\alpha_4+\alpha_2\alpha_4\beta_2+\alpha_2\alpha_3\beta_4+\alpha_4\beta_2\beta_4$,

$B_3=\alpha_1\alpha_3\beta_1+\alpha_1\beta_2\beta_3+\alpha_3\beta_1\beta_3+\beta_1\beta_3\beta_4$, $B_4=\alpha_1\alpha_4\beta_1+\alpha_1\beta_2\beta_4+\alpha_4\beta_1\beta_3+\beta_1\beta_4{}^2$,

$B_5=\alpha_1\alpha_2\beta_1+\alpha_2\beta_1\beta_2+\alpha_1\beta_2\beta_3+\beta_1\beta_2\beta_4$, $B_6=\alpha_2^2\beta_1+\alpha_2\beta_2{}^2+\alpha_2\beta_2\beta_3+\beta_2{}^2\beta_4$,

$B_{11}=\alpha_3^2\beta_1+\alpha_3\beta_2\beta_3+\alpha_3\beta_3{}^2+\beta_3{}^2\beta_4$, $B_{12}=\alpha_3\alpha_4\beta_1+\alpha_3\beta_2\beta_4+\alpha_4\beta_3{}^2+\beta_3\beta_4{}^2$,

$B_{13}=\alpha_1\alpha_4\beta_1+\alpha_4\beta_1\beta_2+\alpha_1\beta_3\beta_4+\beta_1\beta_4{}^2$, $B_{14}=\alpha_2\alpha_4\beta_1+\alpha_4\beta_2{}^2+\alpha_2\beta_3\beta_4+\beta_2\beta_4{}^2$.

Note that the set of functions

$$\{x_1^2y_1y_2,\ x_1^2y_2^2,\ x_1x_2y_1^2,\ x_1x_2y_1y_2,\ x_1x_2y_1^2,\ x_1x_2y_2^2,\ x_2^2y_1^2,\ x_2^2y_1y_2\}$$

is linearly independent. Therefore, system (2.4) in terms of A_i and B_j (i, j = 3, 4, …, 14) can be rewritten as follows:

$$\begin{cases} A_3-A_5=0 & A_4-A_6=0 & A_{12}-A_{14}=0 & A_{13}-A_{11}=0 \\ B_3-B_5=0 & B_4-B_6=0 & B_{12}-B_{14}=0 & B_{13}-B_{11}=0 \end{cases} \tag{2.5}$$

In terms of the structure constants α_i,β_j (i, j = 1, 2, …, 4), system (2.5) is written as follows:

$$\begin{cases} \left(\alpha_1^2\alpha_3+\alpha_1\alpha_2\beta_3+\alpha_3^2\beta_1+\alpha_4\beta_1\beta_3\right)-\left(\alpha_1^2\alpha_2+\alpha_2{}^2\beta_1+\alpha_1\alpha_3\beta_2+\alpha_4\beta_1\beta_2\right)=0 \\ \left(\alpha_1^2\alpha_4+\alpha_1\alpha_2\beta_4+\alpha_3\alpha_4\beta_1+\alpha_4\beta_1\beta_4\right)-\left(\alpha_1\alpha_2{}^2+\alpha_2{}^2\beta_2+\alpha_2\alpha_3\beta_2+\alpha_4\beta_2{}^2\right)=0 \\ \left(\alpha_1\alpha_3\alpha_4+\alpha_2\alpha_3\beta_4+\alpha_3\alpha_4\beta_3+\alpha_4\beta_3\beta_4\right)-\left(\alpha_1\alpha_2\alpha_4+\alpha_2\alpha_4\beta_2+\alpha_2\alpha_3\beta_4+\alpha_4\beta_2\beta_4\right)=0 \\ \left(\alpha_1^2\alpha_4+\alpha_2\alpha_4\beta_1+\alpha_1\alpha_3\beta_4+\alpha_4\beta_1\beta_4\right)-\left(\alpha_1\alpha_3{}^2+\alpha_2\alpha_3\beta_3+\alpha_3^2\beta_3+\alpha_4\beta_3{}^2\right)=0 \\ \left(\alpha_1\alpha_3\beta_1+\alpha_1\beta_2\beta_3+\alpha_3\beta_1\beta_3+\beta_1\beta_3\beta_4\right)-\left(\alpha_1\alpha_2\beta_1+\alpha_2\beta_1\beta_2+\alpha_1\beta_2\beta_3+\beta_1\beta_2\beta_4\right)=0 \\ \left(\alpha_1\alpha_4\beta_1+\alpha_1\beta_2\beta_4+\alpha_4\beta_1\beta_3+\beta_1\beta_4{}^2\right)-\left(\alpha_2^2\beta_1+\alpha_2\beta_2{}^2+\alpha_2\beta_2\beta_3+\beta_2{}^2\beta_4\right)=0 \\ \left(\alpha_3\alpha_4\beta_1+\alpha_3\beta_2\beta_4+\alpha_4\beta_3{}^2+\beta_3\beta_4{}^2\right)-\left(\alpha_2\alpha_4\beta_1+\alpha_4\beta_2{}^2+\alpha_2\beta_3\beta_4+\beta_2\beta_4{}^2\right)=0 \\ \left(\alpha_1\alpha_4\beta_1+\alpha_4\beta_1\beta_2+\alpha_1\beta_3\beta_4+\beta_1\beta_4{}^2\right)-\left(\alpha_3^2\beta_1+\alpha_3\beta_2\beta_3+\alpha_3\beta_3{}^2+\beta_3{}^2\beta_4\right)=0 \end{cases} \tag{2.6}$$

Further we refer to (2.6) as a general system of equations for two-dimensional endo-commutative algebras.

3. A complete classification of two-dimensional endo-commutative algebras

In this section and the following ones, we give canonical representatives of the isomorphism classes of two-dimensional endo-commutative algebras over a field $\mathbb{F}$ whose characteristic is neither 2 nor 3, characteristic 2, and characteristic 3, respectively.

3.1. Classification over $\mathbb{F}$ *with* $\mathrm{Char}(\mathbb{F})\neq 2,3$.

The list of canonical representatives of the isomorphism classes of two-dimensional endo-commutative algebras over a field $\mathbb{F}$ with $\mathrm{Char}(\mathbb{F})\neq 2,3$ is given as follows.

Theorem 4. Any nontrivial two-dimensional endo-commutative algebra over a field $\mathbb{F}$ with $\mathrm{Char}(\mathbb{F}) \neq 2,3$ is isomorphic to only one of the following listed by their matrices of structure constants, such algebras:

- $A_3(\alpha_1, 0, \beta_2) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \beta_2 & 1-\alpha_1 & 0 \end{pmatrix}$, where $\alpha_1, \beta_2 \in \mathbb{F}^2$;
- $A_3\left(\frac{1}{2}, \alpha_4, \frac{1}{2}\right) = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \alpha_4 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$, where $\alpha_4 \in \mathbb{F}$ and $\alpha_4 \neq 0$;
- $A_3\left(\frac{1}{2}, \alpha_4, -\frac{1}{2}\right) = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \alpha_4 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$, where $\alpha_4 \in \mathbb{F}$ and $\alpha_4 \neq 0$;
- $A_5(\alpha_1) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 1 & 2\alpha_1 - 1 & 1-\alpha_1 & 0 \end{pmatrix}$, where $\alpha_1 \in \mathbb{F}$;
- $A_7(\alpha_1, 0) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & 1-\alpha_1 & -\alpha_1 & 0 \end{pmatrix}$, where $\alpha_1 \in \mathbb{F}$;
- $A_7\left(\frac{1}{2}, \alpha_4\right) = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \alpha_4 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix}$, where $\alpha_4 \in \mathbb{F}$ and $\alpha_4 \neq 0$;
- $A_9 = \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 \\ 1 & \frac{2}{3} & -\frac{1}{3} & 0 \end{pmatrix}$;
- $A_{10}(\beta_1) = \begin{pmatrix} 0 & 1 & 1 & 1 \\ \beta_1 & 0 & 0 & -1 \end{pmatrix} \simeq \begin{pmatrix} 0 & 1 & 1 & 1 \\ \beta_1'(a) & 0 & 0 & -1 \end{pmatrix}$, where $a, \beta_1 \in \mathbb{F}$, the polynomial $(\beta_1 t^3 - 3t - 1)(\beta_1 t^2 + \beta_1 t + 1)(\beta_1 t^3 + 6\beta_1 t^2 + 3\beta_1 t + \beta_1 - 2)$ has no root in $\mathbb{F}$ and $\beta_1'(t) = \dfrac{\left(\beta_1 t^3 + 6\beta_1 t^2 + 3\beta_1 t + \beta_1 - 2\right)^2}{\left(\beta_1 t^2 + \beta_1 t + 1\right)^3}$;
- $A_{11}(\beta_1) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ \beta_1 & 0 & 0 & 0 \end{pmatrix} \simeq \begin{pmatrix} 0 & 0 & 0 & 1 \\ a^3\beta_1^{\pm 1} & 0 & 0 & 0 \end{pmatrix}$, where the polynomial $\beta_1 - t^3$ has no root in $\mathbb{F}$, $a, \beta_1 \in \mathbb{F}$ and $a \neq 0$;
- $A_{12}(\beta_1) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_1 & 0 & 0 & -1 \end{pmatrix} \simeq \begin{pmatrix} 0 & 1 & 1 & 0 \\ a^2\beta_1 & 0 & 0 & -1 \end{pmatrix}$, where $a, \beta_1 \in \mathbb{F}$ and $a \neq 0$;
- $A_{13} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

Proof. To classify the two-dimensional endo-commutative algebras, it suffices to solve the general system of equations (2.6) with respect to MSC of each A_i (i = 1, 2, …, 13) given in Theorem 1.

For $A = A_1(c) = \begin{pmatrix} \alpha_1 & \alpha_2 & 1+\alpha_2 & \alpha_4 \\ \beta_1 & -\alpha_1 & 1-\alpha_1 & -\alpha_2 \end{pmatrix}$, where $c = (\alpha_1, \alpha_2, \alpha_4, \beta_1) \in \mathbb{F}^4$, the system of equations (2.6) looks like

$$\begin{cases} \left(\alpha_1^2(1+\alpha_2)+\alpha_1\alpha_2(1-\alpha_1)+(1+\alpha_2)^2\beta_1+\alpha_4\beta_1(1-\alpha_1)\right)- \\ -\left(\alpha_1{}^2\alpha_2+\alpha_2{}^2\beta_1-\alpha_1{}^2(1+\alpha_2)-\alpha_1\alpha_4\beta_1\right)=0 \\ \left(\alpha_1^2\alpha_4-\alpha_1\alpha_2{}^2+(1+\alpha_2)\alpha_4\beta_1-\alpha_2\alpha_4\beta_1\right)-\left(\alpha_1\alpha_2{}^2-\alpha_1\alpha_2{}^2-\alpha_1\alpha_2(1+\alpha_2)-\alpha_4\alpha_1{}^2\right)=0 \\ \left(\alpha_1(1+\alpha_2)\alpha_4+(1+\alpha_2)\alpha_4(1-\alpha_1)-\alpha_2\alpha_4(1-\alpha_1)\right)-\left(\alpha_1\alpha_2\alpha_4-\alpha_1\alpha_2\alpha_4+\alpha_4\alpha_1\alpha_2\right)=0 \\ \left(\alpha_1^2\alpha_4+\alpha_2\alpha_4\beta_1+\alpha_1(1+\alpha_2)(-\alpha_2)-\alpha_4\beta_1\alpha_2\right)= \\ -\left(\alpha_1(1+\alpha_2)^2+\alpha_2(1+\alpha_2)(1-\alpha_1)+(1+\alpha_2)^2(1-\alpha_1)-\alpha_4(1-\alpha_1)^2\right)=0 \\ \left(\alpha_1(1+\alpha_2)\beta_1+(1+\alpha_2)\beta_1(1-\alpha_1)+\beta_1(1-\alpha_1)(-\alpha_2)\right)- \\ -\left(\alpha_1\alpha_2\beta_1-\alpha_1\alpha_2\beta_1+\beta_1(-\alpha_1)(-\alpha_2)\right)=0 \\ \left(\alpha_1\alpha_4\beta_1-\alpha_1\alpha_2\beta_1+\alpha_4\beta_1(1-\alpha_1)+\beta_1(-\alpha_2)^2\right)- \\ -\left(\alpha_1{}^2\beta_1+\alpha_2(-\alpha_1)^2+\alpha_2(-\alpha_1)(1-\alpha_1)+(-\alpha_1)^2(-\alpha_2)\right)=0 \\ \left((1+\alpha_2)\alpha_4\beta_1+(1+\alpha_2)(-\alpha_1)(-\alpha_2)+\alpha_4(1-\alpha_1)^2+(1-\alpha_1)(-\alpha_2)^2\right)- \\ -\left(\alpha_2\alpha_4\beta_1+\alpha_4(-\alpha_1)^2+\alpha_2(1-\alpha_1)(-\alpha_2)-\alpha_1(-\alpha_2)^2\right)=0 \\ \left(\alpha_1\alpha_4\beta_1-\alpha_1\alpha_4\beta_1+\alpha_1(1-\alpha_1)(-\alpha_2)+\beta_1(-\alpha_2)^2\right)- \\ -\left((1+\alpha_2)^2\beta_1+(1+\alpha_2)(-\alpha_1)(1-\alpha_1)+(1+\alpha_2)(1-\alpha_1)^2+(1-\alpha_1)^2(-\alpha_2)\right)=0 \end{cases}$$

Simplifying, we get an inconsistent system of equations. Thus, there is no endo-commutative algebra among A_1(c).

Similarly, if $A = A_2(c) = \begin{pmatrix} \alpha_1 & 0 & 0 & \alpha_4 \\ 1 & \beta_2 & 1-\alpha_1 & 0 \end{pmatrix}$, where $c = (\alpha_1, \alpha_4, \beta_2) \in \mathbb{F}^3$ and $\alpha_4 \neq 0$, then system (2.6) implies $\alpha_4 = 0$, which contradicts with $\alpha_4 \neq 0$. Hence, A_2(c) also does not contain an endo-commutative algebra.

Let consider $A = A_3(c) = \begin{pmatrix} \alpha_1 & 0 & 0 & \alpha_4 \\ 0 & \beta_2 & 1-\alpha_1 & 0 \end{pmatrix}$, where $c = (\alpha_1, \alpha_4, \beta_2) \in \mathbb{F}^3$. Then the system of equations (2.6) for A_3(c) looks like

$$\begin{cases} \alpha_1{}^2\alpha_4-\alpha_4\beta_2{}^2=0 \\ \alpha_1{}^2\alpha_4-(1-\alpha_1)^2\alpha_4=0 \\ (1-\alpha_1)^2\alpha_4-\alpha_4\beta_2{}^2=0 \end{cases}$$

which has the following solutions: (α_1, 0, β_2), $\left(\frac{1}{2},\alpha_4,-\frac{1}{2}\right)$ and $\left(\frac{1}{2},\alpha_4,\frac{1}{2}\right)$. These solutions produce the endo-commutative algebras:

$$A_3(\alpha_1,0,\beta_2)=\begin{pmatrix}\alpha_1 & 0 & 0 & 0\\ 0 & \beta_2 & 1-\alpha_1 & 0\end{pmatrix},\ A_3\left(\frac{1}{2},\alpha_4,\frac{1}{2}\right)=\begin{pmatrix}\frac{1}{2} & 0 & 0 & \alpha_4\\ 0 & \frac{1}{2} & \frac{1}{2} & 0\end{pmatrix}$$

and
$$A_3\left(\frac{1}{2},\alpha_4,-\frac{1}{2}\right)=\begin{pmatrix}\frac{1}{2} & 0 & 0 & \alpha_4\\ 0 & -\frac{1}{2} & \frac{1}{2} & 0\end{pmatrix},$$

where $\alpha_1,\alpha_4,\beta_2\in\mathbb{F}$ and $\alpha_4\neq 0$.

Letting $A=A_4(c)=\begin{pmatrix}0 & 1 & 1 & 0\\ \beta_1 & \beta_2 & 1 & -1\end{pmatrix}$, where $c=(\beta_1,\beta_2)\in\mathbb{F}^2$, we get an inconsistent system of equations.

If $A=A_5(c)=\begin{pmatrix}\alpha_1 & 0 & 0 & 0\\ 1 & 2\alpha_1-1 & 1-\alpha_1 & 0\end{pmatrix}$, where $c=\alpha_1\in\mathbb{F}$, then all the equations of system (2.6) become identities; therefore, all the algebras in this class are endo-commutative.

For $A=A_6(c)=\begin{pmatrix}\alpha_1 & 0 & 0 & \alpha_4\\ 1 & 1-\alpha_1 & -\alpha_1 & 0\end{pmatrix}$, where $c=(\alpha_1,\alpha_4)\in\mathbb{F}^2$ and $\alpha_4\neq 0$, the system of equations (2.6) gives $\alpha_4 = 0$ which is a contradiction.

If $A=A_7(c)=\begin{pmatrix}\alpha_1 & 0 & 0 & \alpha_4\\ 0 & 1-\alpha_1 & -\alpha_1 & 0\end{pmatrix}$, where $c=(\alpha_1,\alpha_4)\in\mathbb{F}^2$ then the system of equations (2.6) is equivalent to $\alpha_1^2\alpha_4-\alpha_4(1-\alpha_1)^2=0$ and we have the solutions (α_1, 0) and $\left(\frac{1}{2},\alpha_4\right)$, where $\alpha_4\neq 0$, i.e.,

$$A_7(\alpha_1,0)=\begin{pmatrix}\alpha_1 & 0 & 0 & 0\\ 0 & 1-\alpha_1 & -\alpha_1 & 0\end{pmatrix},\ \alpha_1\in\mathbb{F}$$

and
$$A_7\left(\frac{1}{2},\alpha_4\right)=\begin{pmatrix}\frac{1}{2} & 0 & 0 & \alpha_4\\ 0 & \frac{1}{2} & -\frac{1}{2} & 0\end{pmatrix},\ \alpha_4\neq 0\in\mathbb{F}$$

are endo-commutative.

Letting
$$A=A_8(c)=\begin{pmatrix}0 & 1 & 1 & 0\\ \beta_1 & 1 & 0 & -1\end{pmatrix},$$
where $c=\beta_1\in\mathbb{F}$, we obtain an inconsistent system of equations.

Verifying the system of equations (2.6) for A_9, A_{10}(c), A_{11}(c), A_{12}(c), and A_{13}, we get identities, i.e., all these algebras turn out to be endo-commutative algebras.

3.2. Classification over $\mathbb{F}$ *with* $\mathrm{Char}(\mathbb{F}) = 2, 3$.

The description of two-dimensional endo-commutative algebras over $\mathbb{F}$ for the cases $\mathrm{Char}(\mathbb{F}) = 2$ and $\mathrm{Char}(\mathbb{F}) = 3$ can be obtained by using Theorems 2 and 3, respectively. The proof is similar to that of Theorem 4. Therefore, we give here the results without the proof.

Theorem 5. Any nontrivial two-dimensional endo-commutative algebra over a field $\mathbb{F}$ with $\mathrm{Char}(\mathbb{F}) = 2$ is isomorphic to only one of the following algebras listed by their matrices of structure constants:

- $A_{2,2}(\alpha_1, 0, 1) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 1 & 1 & 1+\alpha_1 & 0 \end{pmatrix}$, where $\alpha_1 \in \mathbb{F}$;
- $A_{3,2}(\alpha_1, 0, \beta_2) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \beta_2 & 1+\alpha_1 & 0 \end{pmatrix}$, where $\alpha_1, \beta_2 \in \mathbb{F}$;
- $A_{4,2}(\alpha_1, \beta_1, 0) = \begin{pmatrix} \alpha_1 & 1 & 1 & 0 \\ \beta_1 & 0 & 1+\alpha_1 & 1 \end{pmatrix} \simeq \begin{pmatrix} \alpha_1 & 1 & 1 & 0 \\ \beta_1 + a + a^2 & 0 & 1+\alpha_1 & 1 \end{pmatrix}$, where $a, \alpha_1, \beta_1 \in \mathbb{F}$;
- $A_{5,2}(1, 0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}$;
- $A_{6,2}(\alpha_1, 0) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & 1+\alpha_1 & \alpha_1 & 0 \end{pmatrix}$, where $\alpha_1 \in \mathbb{F}$;
- $A_{7,2}(0, \beta_1) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_1 & 1 & 0 & 1 \end{pmatrix} \simeq \begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_1 + a + a^2 & 1 & 0 & 1 \end{pmatrix}$, where $\beta_1, a \in \mathbb{F}$;
- $A_{8,2}(\beta_1) = \begin{pmatrix} 0 & 1 & 1 & 1 \\ \beta_1 & 0 & 0 & 1 \end{pmatrix} \simeq \begin{pmatrix} 0 & 1 & 1 & 1 \\ \beta_1'(a) & 0 & 0 & 1 \end{pmatrix}$, where the polynomial $(\beta_1 t^3 + t + 1)(\beta_1 t^2 + \beta_1 t + 1)$ has no roots in $\mathbb{F}$, $a \in \mathbb{F}$ and $\beta_1'(t) = \dfrac{\left(\beta_1^2 t^3 + \beta_1 t + \beta_1\right)^2}{\left(\beta_1 t^2 + \beta_1 t + 1\right)^3}$;
- $A_{9,2}(\beta_1) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ \beta_1 & 0 & 0 & 0 \end{pmatrix} \simeq \begin{pmatrix} 0 & 0 & 0 & 1 \\ a^3\beta_1^{\pm 1} & 0 & 0 & 0 \end{pmatrix}$, where $\beta_1, a \in \mathbb{F}$, $a \neq 0$ and the polynomial $\beta_1 + t^3$ has no roots in $\mathbb{F}$;
- $A_{10,2}(\beta_1) = \begin{pmatrix} 1 & 1 & 1 & 0 \\ \beta_1 & 1 & 1 & 1 \end{pmatrix} \simeq \begin{pmatrix} 1 & 1 & 1 & 0 \\ \beta_1 + a + a^2 & 1 & 1 & 1 \end{pmatrix}$, where $\beta_1, a \in \mathbb{F}$;
- $A_{11,2}(\beta_1) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_1 & 0 & 0 & 1 \end{pmatrix} \simeq \begin{pmatrix} 0 & 1 & 1 & 0 \\ b^2\left(\beta_1 + a^2\right) & 0 & 0 & 1 \end{pmatrix}$, where $a, b \in \mathbb{F}$ and $b \neq 0$;
- $A_{12,2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

If $\mathbb{F}=\mathbb{Z}_2$, we obtain the list of representatives of endo-commutative algebras over $\mathbb{Z}_2$ as a particular case of the theorem, as follows.

Corollary 3.1. Any nontrivial two-dimensional endo-commutative algebra over a field $\mathbb{F}=\mathbb{Z}_2$ is isomorphic to only one of the following algebras listed by their matrices of structure constants:

- $A_{2,2}(0,0,1)=\begin{pmatrix}0&0&0&0\\1&1&1&0\end{pmatrix}$;
- $A_{2,2}(1,0,1)=\begin{pmatrix}1&0&0&0\\1&1&0&0\end{pmatrix}$;
- $A_{3,2}(0,0,0)=\begin{pmatrix}0&0&0&0\\0&0&1&0\end{pmatrix}$;
- $A_{3,2}(0,0,1)=\begin{pmatrix}0&0&0&0\\0&1&1&0\end{pmatrix}$;
- $A_{3,2}(1,0,0)=\begin{pmatrix}1&0&0&0\\0&0&0&0\end{pmatrix}$;
- $A_{3,2}(1,0,1)=\begin{pmatrix}1&0&0&0\\0&1&0&0\end{pmatrix}$;
- $A_{4,2}(0,0,0)=\begin{pmatrix}0&1&1&0\\0&0&1&1\end{pmatrix}$;
- $A_{4,2}(0,1,0)=\begin{pmatrix}0&1&1&0\\1&0&1&1\end{pmatrix}$;
- $A_{4,2}(1,0,0)=\begin{pmatrix}1&1&1&0\\0&0&0&1\end{pmatrix}$;
- $A_{4,2}(1,1,0)=\begin{pmatrix}1&1&1&0\\1&0&0&1\end{pmatrix}$;
- $A_{5,2}(1,0)=\begin{pmatrix}1&0&0&0\\1&0&1&0\end{pmatrix}$;
- $A_{6,2}(0,0)=\begin{pmatrix}0&0&0&0\\0&1&0&0\end{pmatrix}$;
- $A_{6,2}(1,0)=\begin{pmatrix}1&0&0&0\\0&0&1&0\end{pmatrix}$;
- $A_{7,2}(0,0)=\begin{pmatrix}0&1&1&0\\0&1&0&1\end{pmatrix}$;
- $A_{7,2}(0,1)=\begin{pmatrix}0&1&1&0\\1&1&0&1\end{pmatrix}$;
- $A_{8,2}(1)=\begin{pmatrix}0&1&1&1\\1&0&0&1\end{pmatrix}$;
- $A_{10,2}(0)=\begin{pmatrix}1&1&1&0\\0&1&1&1\end{pmatrix}$;
- $A_{10,2}(1)=\begin{pmatrix}1&1&1&0\\1&1&1&1\end{pmatrix}$;
- $A_{11,2}(0)=\begin{pmatrix}0&1&1&0\\0&0&0&1\end{pmatrix}$;
- $A_{12,2}=\begin{pmatrix}0&0&0&0\\1&0&0&0\end{pmatrix}$.

Theorem 6. Any nontrivial two-dimensional endo-commutative algebra over a field $\mathbb{F}$ with $\mathrm{Char}(\mathbb{F})=3$ is isomorphic to only one of the following algebras listed by their matrices of structure constants:

- $A_{3,3}(\alpha_1,0,\beta_2)=\begin{pmatrix}\alpha_1&0&0&0\\0&\beta_2&1-\alpha_1&0\end{pmatrix}$, where $\alpha_1,\beta_2\in\mathbb{F}$;
- $A_{3,3}(2,\alpha_4,1)=\begin{pmatrix}2&0&0&\alpha_4\\0&1&2&0\end{pmatrix}\simeq\begin{pmatrix}2&0&0&a^2\alpha_4\\0&1&2&0\end{pmatrix}$, where $a,\alpha_4\in\mathbb{F}$ and $a,\alpha_4\neq 0$;
- $A_{3,3}(2,\alpha_4,2)=\begin{pmatrix}2&0&0&\alpha_4\\0&2&2&0\end{pmatrix}\simeq\begin{pmatrix}2&0&0&a^2\alpha_4\\0&2&2&0\end{pmatrix}$, where $a,\alpha_4\in\mathbb{F}$ and $a,\alpha_4\neq 0$;
- $A_{5,3}(\alpha_1)=\begin{pmatrix}\alpha_1&0&0&0\\1&2\alpha_1-1&1-\alpha_1&0\end{pmatrix}$, where $\alpha_1\in\mathbb{F}$;

- $A_{7,3}(\alpha_1, 0) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & 1-\alpha_1 & -\alpha_1 & 0 \end{pmatrix}$, where $\alpha_1 \in \mathbb{F}$;
- $A_{7,3}(2, \alpha_4) = \begin{pmatrix} 2 & 0 & 0 & \alpha_4 \\ 0 & 2 & 1 & 0 \end{pmatrix} \simeq \begin{pmatrix} 2 & 0 & 0 & a^2\alpha_4 \\ 0 & 2 & 1 & 0 \end{pmatrix}$, where $a, \alpha_4 \in \mathbb{F}$ and $a, \alpha_4 \neq 0$;
- $A_{9,3}(\beta_1) = \begin{pmatrix} 0 & 1 & 1 & 1 \\ \beta_1 & 0 & 0 & -1 \end{pmatrix} \simeq \begin{pmatrix} 0 & 1 & 1 & 1 \\ \beta_1'(a) & 0 & 0 & -1 \end{pmatrix}$, where the polynomial

$(1 - t^3)(\beta_1 t^2 + \beta_1 t + 1)(\beta_1^2 t^3 + \beta_1 - 2)$ has no roots in $\mathbb{F}$, $a \in \mathbb{F}$ and $\beta_1'(t) = \frac{\left(\beta_1^2 t^3 + \beta_1 - 2\right)^2}{\left(\beta_1 t^2 + \beta_1 t + 1\right)^3}$;

- $A_{10,3}(\beta_1) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ \beta_1 & 0 & 0 & 0 \end{pmatrix} \simeq \begin{pmatrix} 0 & 0 & 0 & 1 \\ a^3\beta_1^{\pm 1} & 0 & 0 & 0 \end{pmatrix}$, where polynomial $\beta_1 - t^3$ has no

roots, $a, \beta_1 \in \mathbb{F}$ and $a, \beta_1 \neq 0$;

- $A_{11,3}(\beta_1) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_1 & 0 & 0 & -1 \end{pmatrix} \simeq \begin{pmatrix} 0 & 1 & 1 & 0 \\ a^2\beta_1 & 0 & 0 & -1 \end{pmatrix}$, where $a, \beta_1 \in \mathbb{F}$ and $a \neq 0$;
- $A_{12,3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 \end{pmatrix}$;
- $A_{13,3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

4. Two-dimensional curled algebras

Let A be a two-dimensional algebra and

$$A = \begin{pmatrix} \alpha_1' & \alpha_2' & \alpha_3' & \alpha_4' \\ \beta_1' & \beta_2' & \beta_3' & \beta_4' \end{pmatrix}$$

be its MSC with respect to a basis e = (e_1, e_2). Then (2.3) in terms of the elements of A and the coordinate vector of x is written as follows:

$$\begin{cases} \alpha_1' x_1^2 + \left(\alpha_2' + \alpha_3'\right) x_1 x_2 + \alpha_4' x_2^2 - \lambda(x_1, x_2) x_1 = 0, \\ \beta_1' x_1^2 + \left(\beta_2' + \beta_3'\right) x_1 x_2 + \beta_4' x_2^2 - \lambda(x_1, x_2) x_2 = 0. \end{cases} \quad (4.1)$$

We consider two options:

(1) There exists a nonzero x such that $\lambda(x) \neq 0$. Since (4.1) is an identity, it must hold for any $x = (x_1, x_2) \in \mathbb{F}^2$. Particularly, it must hold for $x = (0, x_2)$ and $x = (x_1, 0)$ as well. Therefore, $x_1 = 0, x_2 \neq 0$ ($x_1 \neq 0, x_2 = 0$) implies $\alpha_4' = 0$, $\lambda(0, x_2) = \beta_4' x_2$ (respectively, $\beta_1' = 0$, $\lambda(x_1, 0) = \alpha_1' x_1$) and (4.1) becomes

$$\begin{cases} \alpha_1' x_1^2 + \left(\alpha_2' + \alpha_3'\right) x_1 x_2 - \lambda(x_1, x_2) x_1 = 0, \\ \left(\beta_2' + \beta_3'\right) x_1 x_2 + \beta_4' x_2^2 - \lambda(x_1, x_2) x_2 = 0. \end{cases}$$

Thus, if $x_1 \neq 0$, $x_2 \neq 0$, then system (4.1) can be rewritten as follows:

$$\begin{cases} \alpha_1' x_1 + \left(\alpha_2' + \alpha_3'\right) x_2 - \lambda\left(x_1, x_2\right) = 0, \\ \left(\beta_2' + \beta_3'\right) x_1 + \beta_4' x_2 - \lambda\left(x_1, x_2\right) = 0. \end{cases}$$

This yields $(\alpha_1' - \beta_2' - \beta_3')x_1 + (\alpha_2' + \alpha_3' - \beta_4')x_2$ and, if $\mathrm{Card}\left(\mathbb{F}\right) > 2$ (that is, $\mathbb{F} \neq \mathbb{Z}_2$), then the latter is nothing else than $\alpha_1' = \beta_2' + \beta_3'$, $\beta_4' = \alpha_2' + \alpha_3'$. So, if $\mathbb{F} \neq \mathbb{Z}_2$, then we have $\alpha_1' x_1 + \beta_4' x_2 = \lambda(x_1, x_2)$, at least one of α_1', β_4' is not zero, and

$$A = \begin{pmatrix} \alpha_1' & \alpha_2' & \beta_4' - \alpha_2' & 0 \\ 0 & \beta_2' & \alpha_1' - \beta_2' & \beta_4' \end{pmatrix} \tag{4.2}$$

If $\mathbb{F} = \mathbb{Z}_2$, then $\alpha_1' x_1 + (a - \beta_2' - \beta_3')x_2 = \lambda(x_1, x_2)$, at least one of α_1', $a - \beta_2' - \beta_3'$ is not zero, and one has

$$A = \begin{pmatrix} \alpha_1' & \alpha_2' & a - \alpha_1' - \alpha_2' & 0 \\ 0 & \beta_2' & \beta_3' & a - \beta_2' - \beta_3' \end{pmatrix}, \tag{4.3}$$

where $a = \lambda(1,1)$.

(2) Let $\lambda(x)$ be identically zero on $\mathbb{F}^2 \setminus \left(0,0\right)$. In this case, system (4.1) is equivalent to $\alpha_1' = \alpha_4' = \beta_1' = \beta_4' = \alpha_2' + \alpha_3' = \beta_2' + \beta_3' = 0$ and, therefore, we have

$$A = \begin{pmatrix} 0 & \alpha_2' & -\alpha_2' & 0 \\ 0 & \beta_2' & -\beta_2' & 0 \end{pmatrix}. \tag{4.4}$$

These types of algebras were said to be zeropotent (see [1, 20, 23, 24]). The forms (4.2), (4.3) and (4.4) for MSC are the conditions for A to be curled. Now having the conditions (4.2), (4.3) and (4.4) we can give the description of two-dimensional curled algebras by using Theorems 1, 2, and 3. For example, in the case of $\mathbb{F}$ with $\mathrm{Char}\left(\mathbb{F}\right) \neq 2,3$, a two-dimensional algebra given by

$$A = A_1\left(c\right) = \begin{pmatrix} \alpha_1 & \alpha_2 & 1+\alpha_2 & \alpha_4 \\ \beta_1 & -\alpha_1 & 1-\alpha_1 & -\alpha_2 \end{pmatrix},$$

where $c = \left(\alpha_1, \alpha_2, \alpha_4, \beta_1\right) \in \mathbb{F}^4$, is a curled algebra if and only if A has the following form:

$$A = \begin{pmatrix} \alpha_1' & \alpha_2' & \beta_4' - \alpha_2' & 0 \\ 0 & \beta_2' & \alpha_1' - \beta_2' & \beta_4' \end{pmatrix},$$

that is,

$$\begin{cases} \alpha_1' = \alpha_1 & \alpha_2' = \alpha_2 \\ \beta_4' - \alpha_2' = 1 + \alpha_2 & 0 = \alpha_4 \\ 0 = \beta_1 & \beta_2' = -\alpha_1 \\ \alpha_1' - \beta_2' = 1 - \alpha_1 & \beta_4' = -\alpha_2 \end{cases}$$

which takes place if and only if $\alpha_1 = \frac{1}{3}$, $\alpha_2 = -\frac{1}{3}$, $\alpha_4 = 0$, $\beta_1 = 0$. Therefore, among the algebras $A_1(c)$, only $A_1\left(\frac{1}{3}, -\frac{1}{3}, 0, 0\right)$ is curled.

Going through $A_2(c) - A_{13}$ of Theorem 1 in this manner, one comes to the following result.

Theorem 7. Any nontrivial two-dimensional curled algebra over a field $\mathbb{F}$ with $\mathrm{Char}(\mathbb{F}) \neq 2,3$ is isomorphic to only one of the following algebras listed by their matrices of structure constants:

- $A_1\left(\frac{1}{3}, -\frac{1}{3}, 0, 0\right) = \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \end{pmatrix}$;
- $A_3(\alpha_1, 0, 2\alpha_1 - 1) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & 2\alpha_1 - 1 & 1 - \alpha_1 & 0 \end{pmatrix}$, where $\alpha_1 \in \mathbb{F}$;
- $A_7\left(\frac{1}{3}, 0\right) = \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{2}{3} & -\frac{1}{3} & 0 \end{pmatrix}$.

Among the listed algebras, only $A_3(0, 0, 1)$ is zeropotent.

Similarly, in the cases of $\mathrm{Char}(\mathbb{F}) = 2,3$, the following results hold true.

Theorem 8. Any nontrivial two-dimensional curled algebra over a field $\mathbb{F} \neq \mathbb{Z}_2$ with $\mathrm{Char}(\mathbb{F}) = 2$ is isomorphic to only one of the following algebras listed by their matrices of structure constants:

- $A_{1,2}(1,1,0,0) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$;
- $A_{3,2}(\alpha_1, 0, 1) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & 1 & 1 + \alpha_1 & 0 \end{pmatrix}$, where $\alpha_1 \in \mathbb{F}$;
- $A_{6,2}(1,0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$.

And among them only $A_{3,2}(0, 0, 1)$ is zeropotent.

Theorem 9. Any nontrivial two-dimensional curled algebra over a field $\mathbb{F} = \mathbb{Z}_2$ is isomorphic to only one of the following algebras listed by their matrices of structure constants:

- $A_{1,2}(1,1,0,0) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$;
- $A_{3,2}(\alpha_1, 0, 1) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & 1 & 1 + \alpha_1 & 0 \end{pmatrix}$, where $\alpha_1 \in \mathbb{Z}_2$;

- $A_{4,2}(\alpha_1,0,0)=\begin{pmatrix}\alpha_1 & 1 & 1 & 0\\ 0 & 0 & 1+\alpha_1 & 1\end{pmatrix}$, where $\alpha_1\in\mathbb{Z}_2$;
- $A_{6,2}(1,0)=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\end{pmatrix}$;
- $A_{7,2}(0,0)=\begin{pmatrix}0 & 1 & 1 & 0\\ 0 & 1 & 0 & 1\end{pmatrix}$;
- $A_{10,2}(0)=\begin{pmatrix}1 & 1 & 1 & 0\\ 0 & 1 & 1 & 1\end{pmatrix}$.

Among these algebras, only $A_{3,2}(0, 0, 1)$ is zeropotent.

Theorem 10. Any nontrivial two-dimensional curled algebra over a field $\mathbb{F}$ with $\mathrm{Char}(\mathbb{F})=3$ is isomorphic to only one of the following algebras listed by their matrices of structure constants:

- $A_{3,3}(\alpha_1,0,2+2\alpha_1)=\begin{pmatrix}\alpha_1 & 0 & 0 & 0\\ 0 & 2+2\alpha & 1+2\alpha_1 & 0\end{pmatrix}$, where $\alpha_1\in\mathbb{F}$;
- $A_{4,3}(0,2)=\begin{pmatrix}0 & 1 & 1 & 0\\ 0 & 2 & 1 & 2\end{pmatrix}$;
- $A_{11,3}(0)=\begin{pmatrix}0 & 1 & 1 & 0\\ 0 & 0 & 0 & 2\end{pmatrix}$.

Among the listed algebras, only $A_{3,3}(0, 0, 2)$ is zeropotent.

5. Two-dimensional endo-commutative curled algebras

In this section, we make use of the results of the last two sections to get a classification of two-dimensional endo-commutative curled algebras up to isomorphism.

Theorem 11. Any nontrivial two-dimensional endo-commutative curled algebra over a field $\mathbb{F}$ with $\mathrm{Char}(\mathbb{F})\neq 2,3$ is isomorphic to only one of the following algebras listed by their matrices of structure constants:

- $A_3(\alpha_1,0,2\alpha_1-1)=\begin{pmatrix}\alpha_1 & 0 & 0 & 0\\ 0 & 2\alpha_1-1 & 1-\alpha_1 & 0\end{pmatrix}$, where $\alpha_1\in\mathbb{F}$;
- $A_7\left(\frac{1}{3},0\right)=\begin{pmatrix}\frac{1}{3} & 0 & 0 & 0\\ 0 & \frac{2}{3} & -\frac{1}{3} & 0\end{pmatrix}$.

Among them, only $A_3(0, 0, -1)$ is zeropotent.

Theorem 12. Any non-trivial two-dimensional endo-commutative curled algebra over a field $\mathbb{F}\neq\mathbb{Z}_2$ with $Char(\mathbb{F})=2$ is isomorphic to only one of the following, listed by their matrices of structure constants, such algebras:

- $A_{3,2}(\alpha_1,0,1)=\begin{pmatrix}\alpha_1 & 0 & 0 & 0\\ 0 & 1 & 1+\alpha_1 & 0\end{pmatrix}$, where $\alpha_1\in\mathbb{F}$;

- $A_{6,2}(1,0)=\begin{pmatrix}1&0&0&0\\0&0&1&0\end{pmatrix}$.

Among the algebras listed above, only $A_{3,2}(0, 0, 1)$ is a zeropotent algebra.

Theorem 13. Any nontrivial two-dimensional endo-commutative curled algebra over a field $\mathbb{F}=\mathbb{Z}_2$ is isomorphic to only one of the following algebras listed by their matrices of structure constants:

- $A_{3,2}(0,0,1)=\begin{pmatrix}0&0&0&0\\0&1&1&0\end{pmatrix}$;
- $A_{3,2}(1,0,1)=\begin{pmatrix}1&0&0&0\\0&1&0&0\end{pmatrix}$;
- $A_{4,2}(0,1,0)=\begin{pmatrix}0&1&1&0\\1&0&1&1\end{pmatrix}$;
- $A_{4,2}(1,1,0)=\begin{pmatrix}1&1&1&0\\1&0&0&1\end{pmatrix}$;
- $A_{6,2}(1,0)=\begin{pmatrix}1&0&0&0\\0&0&1&0\end{pmatrix}$;
- $A_{7,2}(0,0)=\begin{pmatrix}0&1&1&0\\0&1&0&1\end{pmatrix}$;
- $A_{10,2}(0)=\begin{pmatrix}1&1&1&0\\0&1&1&1\end{pmatrix}$.

Here, only $A_{3,2}(0, 0, 1)$ is zeropotent.

Note that the endo-commutative curled algebras given in [19] can be found in Theorem 13 as $A_{3,2}(0,0,1)\cong C_4$, $A_{3,2}(1,0,1)\cong C_3$, $A_{4,2}(0,1,0)\cong C_{12}$, $A_{4,2}(1,1,0)\cong C_7$, $A_{6,2}(1,0)\cong C_2$, $A_{7,2}(0,0)\cong C_{13}$, and $A_{10,2}(0)\cong C_1$.

Theorem 14. Any nontrivial two-dimensional endo-commutative curled algebra over a field $\mathbb{F}$ with $\mathrm{Char}(\mathbb{F})=3$ is isomorphic to only one of the following algebras listed by their matrices of structure constants:

- $A_{3,3}(\alpha_1,0,2\alpha_1-1)=\begin{pmatrix}\alpha_1&0&0&0\\0&2\alpha_1-1&1-\alpha_1&0\end{pmatrix}$, where $\alpha_1\in\mathbb{F}$;
- $A_{11,3}(0)=\begin{pmatrix}0&1&1&0\\0&0&0&2\end{pmatrix}$.

Among the algebras listed above, only $A_{3,3}(0, 0, –1)$ is zeropotent.

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