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THE LINDELÖF NUMBER IS *fu*-INVARIANT

Two Tychonoff spaces X and Y are said to be l-equivalent (u-equivalent) if $C_p(X)$ and $C_p(Y)$ are linearly (uniformly) homeomorphic. N.V.Velichko proved that the Lindelöf property is preserved by the relation of l-equivalence. A. Bouziad strengthened this result and proved that the Lindelöf number is preserved by the relation of l-equivalence. In this paper the concept of the support different variants of which can be founded in the papers of S.P. Gul′ko and O.G. Okunev is introduced. Using this concept we introduce an equivalence relation on the class of topological spaces. Two Tychonoff spaces X and Y are said to be fu-equivalent if there exists an uniform homeomorphism h: $C_p(Y) \to C_p(X)$ such that $\text{supp}^h x$ and $\text{supp}^{h^{-1}} x$ are finite sets for all $x \in X$ and $y \in Y$. This is an intermediate relation between relations of u- and l-equivalence. In this paper it has been proved that the Lindelöf number is preserved by the relation of fu-equivalence.

Keywords: *u-equivalence; Lindelöf number; Function spaces; Set-valued mappings.*

Introduction

All spaces below are assumed to be Tychonoff. $\mathbf{R}^X$ is a space of all real-valued functions on X, $C_p(X)$ is a space of all real-valued continuous functions on X equipped with the topology of pointwise convergence.

$C_p(X\,|F) = \{f \in C_p(X): f(x) = 0$ for all $x \in F\}$, where F is a subset of X. The restriction of a function f to a subset A is denoted by $f|_A$. The cardinality of a set A is denoted by $|A|$. $\aleph_0$ is the countable cardinal, $l(X)$ is the Lindelöf number of X. Fin A is a family of all finite subsets of a set A. For a set-valued mapping p: $X \to Y$ and sets $A \subset X$ and $B \subset Y$, we define the image of A as a set $p(A) = \bigcup\{p(x) : x \in A\}$, and preimage of B as a set $p^{-1}(B) = \{x \in X : p(x) \cap B \neq \varnothing\}$. A set-valued mapping p: $X \to Y$ is called lower semicontinuous if a preimage of every open subset of Y is open in X. It is called surjective if for any $y \in Y$ there exist $x \in X$ such that $y \in p(x)$.

1. Concept of the support

Definition 1.1. Let X be a topological space. A linear subspace $A \subset \mathbf{R}^X$ is called suitable if for any point $x \in X$, its open neighborhood O_x, and two functions $f', f'' \in A$ there exist a function $f \in A$ such that $f(x) = f''(x)$ and $f(x') = f'(x')$ for all $x' \in X \setminus O_x$. A point $x \in X$ is said to be a zero-point of a family $A \subset \mathbf{R}^X$ if $f(x) = 0$ for all $f \in A$. Denote by ker A the set of all zero-points of a family A.

The examples of suitable subspaces are $C_p(X)$, $C_p(X\,|F)$, where F is a closed subset of a space X, and then ker $C_p(X\,|F) = F$. Following definitions are analogous to the definitions introduced by O.G.Okunev in [4] for t-equivalent spaces.

Definition 1.2. Let X, Y be topological spaces, A, B – suitable subspaces of spaces $\mathbf{R}^X$ and $\mathbf{R}^Y$, respectively, and let h:$B \to A$ be an uniform homeomorphism such that the image of the zero-function $0_Y \in B$ under h is the zero-function $0_X \in A$. Fix a point $x \in X$

and $\varepsilon > 0$. We call a point $y \in Y$ ε-essential for x (under h) if for any open neighborhood O_y of y there exist functions g', $g'' \in B$, coinciding on the set $Y \setminus O_y$ and satisfying the following inequality:

$$|h(g')(x) - h(g'')(x)| > \varepsilon. \tag{1}$$

Furthermore, we say that a point y is ε-inessential for x if it is not ε-essential for x, and call the set of all points that are ε-essential for x the ε-support of x (under h) and denote it by $\operatorname{supp}_\varepsilon^h x$. The union of ε-supports of x (under h) over all positive ε is called the support of x (under h) and is denoted by $\operatorname{supp}^h x$. If h is fixed, then we write $\operatorname{supp}_\varepsilon x$ ($\operatorname{supp} x$, respectively).

Remark 1.3. If $x \in \ker A$, then $\operatorname{supp} x = \varnothing$.

It is clear that if $\varepsilon < \delta$, then $\operatorname{supp}_\delta x \subset \operatorname{supp}_\varepsilon x$, therefore $\operatorname{supp} x = \bigcup_{n \in \mathbf{N}} \operatorname{supp}_{1/n} x$. It is not difficult to verify that $\operatorname{supp}_\varepsilon x$ is a closed set. Then we have the following two properties of the support:

(i) $\operatorname{supp}_\varepsilon x$ is a nonempty finite subset of Y for any $\varepsilon > 0$ (if $x \notin \ker A$);

(ii) $\operatorname{supp}: X \to Y$ is a countable-valued, surjective, lower semicontinuous mapping.

To prove these properties, we note some results of S.P.Gul'ko [3]. Let X, Y be topological spaces, A, B – suitable subspaces of spaces $\mathbf{R}^X$ and $\mathbf{R}^Y$, respectively, and let $h: B \to A$ be an uniform homeomorphism such that the image of the zero function $0_Y \in B$ under h is the zero function $0_X \in A$. Let $x \in X$, $\delta > 0$, and let $K \subset Y$ be a finite subset. We introduce into consideration the quantity

$$a(x,K,\delta) = \sup |h(g')(x) - h(g'')(x)|, \tag{2}$$

where the supremum is taken over all g', $g'' \in B$ such that $|g'(y) - g''(y)| < \delta$ for all $y \in K$. This definition was introduced by S.P.Gul'ko in [3]. We also define

$$a(x,K,0) = \sup |h(g')(x) - h(g'')(x)|, \tag{3}$$

where the supremum is taken over all g', $g'' \in B$ coinciding on K (if K is empty, then the supremum is taken over all $g', g'' \in B$). It is obvious that if $0 \le \delta_1 \le \delta_2$, then $a(x,K,\delta_1) \le a(x,K,\delta_2)$, and if $K_1 \subset K_2 \subset Y$, then $a(x,K_2,\delta) \le a(x, K_1,\delta)$ for all $\delta \ge 0$. In [3], it was proved that for all $x \in X \setminus \ker A$ there exist a nonempty finite subset $K(x) \subset Y$ such that

(1) $a(x,K(x),\delta) < \infty$ for any $\delta > 0$,

(2) $a(x,K',\delta) = \infty$ for any proper subset K' of $K(x)$ and any $\delta > 0$.

For all $x \in \ker A$ we put $K(x) = \varnothing$. We now prove that the set $K(x)$ has a stronger property which we get substituting $\delta > 0$ for $\delta \ge 0$ in (2). To prove this, we need the following

Lemma 1.4. If $a(x,K,0) < \infty$, then $a(x,K,\delta) < \infty$ for all $\delta > 0$.

Proof. Fix $x \in X$ and finite $K \subset Y$ such that $a(x,K,0) < \infty$. We prove that the function $\delta \mapsto a(x,K,\delta)$ is continuous at the point 0. Let $\delta > 0$. Since h is an uniform homeomorphism, there exist a finite set $K' \subset Y$ and $\delta > 0$ such that for all g', $g'' \in B$ we have the implication

$$(|g'(y) - g''(y)| < \delta \text{ for all } y \in K') \Rightarrow |h(g')(x) - h(g'')(x)| < \varepsilon.$$

Let g', $g'' \in B$ and $|g'(y) - g''(y)| < \delta$ for all $y \in K$. Since B is a suitable subspace, there is a function $g \in B$ such that

$$g(y) = \begin{cases} g'(y), & y \in K; \\ g''(y), & y \in K' \setminus K. \end{cases} \tag{4}$$

Then $|g(y) - g''(y)| < \delta$ for all $y \in K'$, hence, $|h(g)(x) - h(g'')(x)| < \varepsilon$. Now by the triangle inequality we obtain

$$|h(g')(x) - h(g'')(x)| \le |h(g')(x) - h(g)(x)| + |h(g)(x) - h(g'')(x)| < a(x,K,0)+\varepsilon.$$

Passing to the supremum over all $g', g'' \in B$ such that $|g'(y) - g''(y)| < \delta$ for all $y \in K$ we have inequality $a(x,K,\delta) \le a(x,K,0)+\varepsilon$, which implies that the function $\delta \mapsto a(x,K,\delta)$ is continuous at the point 0. ■

For any $x \in X$ put $a(x) = a(x,K(x),0)$. Now we can rewrite the properties of the set-valued mapping $x \mapsto K(x)$ in new notations.

(K1) If $g', g'' \in B$ and $g'|_{K(x)} = g''|_{K(x)}$, then $|h(g')(x) - h(g'')(x)| \le a(x)$.

(K2) For any proper subset K' of $K(x)$ and any real b there exist functions $g', g'' \in B$ such that $g'|_{K'} = g''|_{K'}$ and $|h(g')(x) - h(g'')(x)| > b$.

Besides, this mapping surjectively maps the space $X \setminus \ker A$ onto $Y \setminus \ker B$ (see [3]), i.e., for any $y \in Y \setminus \ker B$ there exist $x \in X \setminus \ker A$ such that $y \in K(x)$. Now we shall prove the properties (i) and (ii) of the support $\operatorname{supp}_\varepsilon x$.

Lemma 1.5. $K(x) \subset \operatorname{supp}_\varepsilon x$ for any $\varepsilon > 0$.

Proof. Let $x \notin \ker A$. Fix a point $y_0 \in K(x)$ and $\varepsilon > 0$. We shall show that y_0 is ε-essential for x. Put $a = \max(\varepsilon, a(x))$, $K' = K(x) \setminus \{y_0\}$. By the property (K2), there exist functions $g', g'' \in B$ such that $g'|_{K'} = g''|_{K'}$ and $|h(g')(x) - h(g'')(x)| > 2a$. Then there is a neighborhood U of y_0 that does not meet K'. Choose a function $g \in B$ such that $g|_{Y \setminus U} = g'|_{Y \setminus U}$ and $g(y_0) = g''(y_0)$. Then we have $g|_{K(x)} = g''|_{K(x)}$ and $|h(g)(x) - h(g'')(x)| \le a(x) \le a$. By the triangle inequality we obtain that $|h(g)(x) - h(g')(x)| > a \ge \varepsilon$. Besides, g coincides with g' on the set $Y \setminus U$. By definition this means that y_0 is ε-essential for x. ■

So, lemma 1.5 implies that the set $\operatorname{supp}_\varepsilon x$ is nonempty for any $\varepsilon > 0$ and any $x \notin \ker A$, and it also implies that the set-valued mapping $x \mapsto \operatorname{supp} x$ from $X \setminus \ker A$ onto $Y \setminus \ker B$ is surjective.

Lemma 1.6. The set $\operatorname{supp}_\varepsilon x$ is finite for any $\varepsilon > 0$.

Proof. Let $x \notin \ker A$ and $\varepsilon > 0$. Since h is an uniform homeomorphism, there exist a finite set $K \subset Y$ and $\delta > 0$ such that for all $g', g'' \in B$ we have the implication ($|g'(y) - g''(y)| < \delta$ for all $y \in K) \Rightarrow |h(g')(x) - h(g'')(x)| < \varepsilon$. Let us show that $\operatorname{supp}_\varepsilon x \subset K$. Fix y_0 in $Y \setminus K$. Then there is a neighborhood U of y_0 that does not meet K. Choose functions $g', g'' \in B$ coinciding on the set $Y \setminus U$. Then they coincide on K, hence, $|h(g')(x) - h(g'')(x)| < \varepsilon$. By the definition this means that y_0 is ε-inessential for x, i.e., $y_0 \notin \operatorname{supp}_\varepsilon x$. Thus, $\operatorname{supp}_\varepsilon x \subset K$. ■

Property (i) of the support is proved. For the proof of property (ii) we introduce into consideration the set $K_\varepsilon(x) \subset Y$ for any $x \in X \setminus \ker A$ and any $\varepsilon > 0$, satisfying the following properties:

(KE0) $K_\varepsilon(x)$ is finite and nonempty;

(KE1) $a(x, K_\varepsilon(x), 0) \le \varepsilon$;

(KE2) $a(x,K',0) > \varepsilon$ for any proper subset K' of $K_\varepsilon(x)$.

Such a set we could obtain from the set K from the previous proof, decreasing this set while it satisfies point (KE2). There can be several sets, satisfying properties (KE1) and (KE2), then we denote by $K_\varepsilon(x)$ anyone of them. The following lemma is an analogous to result obtained by O.G.Okunev [4] for t-equivalence.

Lemma 1.7. Let $A \subset C_p(X)$, $x_0 \in X \setminus \ker A$, $\varepsilon > 0$, G is open subset of Y such that $\operatorname{supp}_\varepsilon x_0 \cap G \ne \emptyset$. Then there is an open neighborhood U of x_0 such that $K_\varepsilon(x) \cap G \ne \emptyset$ for all x from U.

Proof. We may assume that $\mathrm{supp}_\varepsilon x_0 \cap G = \{y_0\}$, where y_0 is any ε-essential point for x_0. By definition, for the neighborhood G of y_0 there exist functions $g', g''\in B$ coinciding on $Y\setminus G$ such that $|h(g')(x_0) - h(g'')(x_0)| > \varepsilon$. Put $U = \{x\in X\colon |h(g')(x) - h(g'')(x)| > \varepsilon\}$; then U is an open neighborhood of x_0. Let us check that $K_\varepsilon(x)\cap G\neq\varnothing$ for all x from U. Assume the converse. Let $x\in U$ be a point such that $K_\varepsilon(x)\cap G = \varnothing$. Then g' coincides with g'' on $K_\varepsilon(x)$. Therefore $|h(g')(x) - h(g'')(x)|\leq\varepsilon$, A contradiction with $x\in U$. ■

From the result obtained by O.G.Okunev for *t*-equivalence it follows that the set-valued mapping $\mathrm{supp}_\varepsilon$ has a weaker property then the lower semicontinuity. In our case this property of $\mathrm{supp}_\varepsilon$ implies the lower semicontinuity of the mapping supp. To prove this fact we shall need

Lemma 1.8. Let $x\in X\setminus\ker A$, $\varepsilon > 0$. There exists $\delta > 0$, such that $K_\varepsilon(x)\subset\mathrm{supp}_\delta x$.

Proof. Fix a point $y_0\in K_\varepsilon(x)$. Put $K' = K_\varepsilon(x)\setminus\{y_0\}$. By definition of $K_\varepsilon(x)$, there exist functions $g', g''\in B$ coinciding on K' such that $|h(g')(x) - h(g'')(x)| > \varepsilon$. There exists $\delta_0 > 0$ such that

$$|h(g')(x) - h(g'')(x)| > \varepsilon+\delta_0. \qquad (5)$$

Let us show that $y_0 \in \mathrm{supp}_{\delta_0} x$. Choose a neighborhood U of y_0 that does not meet K', and a function $g\in B$ coinciding with g' on $Y\setminus U$ such that $g(y_0) = g''(y_0)$. Then g coincides with g' on $K_\varepsilon(x)$, hence, $|h(g'')(x) - h(g)(x)|\leq\varepsilon$. From this and inequality (5) it follows that $|h(g')(x) - h(g)(x)| > \delta_0$. But g' coincide with g on $Y\setminus U$, consequently, y_0 is δ_0-essential for x. Let's enumerate all the points of the set $K_\varepsilon(x) = \{y_1,\ldots,y_n\}$, and for any y_i chose δ_i so that y_i is δ_i-essential for x. Put $\delta = \min\{\delta_i\colon i\leq n\}$; then $K_\varepsilon(x)\subset\mathrm{supp}_\delta x$. ■

Theorem 1.9. If $A\subset C_p(X)$, then the set-valued mapping $\mathrm{supp}\colon X\to Y$ is lower semicontinuous.

Proof. Put $\varphi(x) = \mathrm{supp}\, x$. We need to show that for any nonempty open set $G\subset Y$ it's preimage $\varphi^{-1}(G) = \{x\in X\colon \varphi(x)\cap G\neq\varnothing\}$ is open in X. Let $G\subset Y$ is any nonempty open set such that $\varphi^{-1}(G)\neq\varnothing$, and let $x\in\varphi^{-1}(G)$. Then there exists $\varepsilon > 0$ such that $\mathrm{supp}_\varepsilon x\cap G\neq\varnothing$. By Lemma 1.7 there exists an open neighborhood U of x such that $K_\varepsilon(z)\cap G\neq\varnothing$ for all z from U. By Lemma 1.8, for all $z\in X$ and $\varepsilon > 0$ we can find $\delta_0 > 0$ (depending on z and ε) such that $K_\varepsilon(z)\subset\mathrm{supp}_{\delta_0} z\subset\mathrm{supp}\, z$, i.e., $\mathrm{supp}\, z\cap G\neq\varnothing$, hence, $\varphi^{-1}(G)$ is open, and the mapping supp is lower semicontinuous. ■

Besides, the set supp x has the following property.

Theorem 1.10. Let $x\in X$.

(a) If two functions $g', g''\in B$ coincide on the set supp x, then $h(g')(x) = h(g'')(x)$.

(b) If F is a closed subspace of Y such that $h(g')(x) = h(g'')(x)$ for any two functions $g', g''\in B$ coinciding on F, then $\mathrm{supp}\, x\subset F$.

Proof. (a) Let $\varepsilon > 0$. Fix $K_\varepsilon(x)$. Let functions $g', g''\in B$ coincide on the set supp x. By Lemma 1.8, $K_\varepsilon(x)\subset\mathrm{supp}\, x$, therefore, $|h(g')(x) - h(g'')(x)|\leq\varepsilon$. Since ε is arbitrary, we obtain $h(g')(x) = h(g'')(x)$.

(b) Assume the converse. Let $y_0\in(\mathrm{supp}\, x)\setminus F\neq\varnothing$. There exists $\varepsilon > 0$ such that $y_0\in\mathrm{supp}_\varepsilon x$. Let U is an open neighborhood of y_0 that does not meet F. Then there are $g', g''\in B$ coinciding on $Y\setminus U$ such that $|h(g')(x) - h(g'')(x)| > \varepsilon$. But in this case g' coincides with g'' on F, whence $h(g')(x) = h(g'')(x)$. This contradiction proves the theorem. ■

The concept of the support can be generalized.

Definition 1.11. If $h\colon B\to A$ is an arbitrary uniform homeomorphism we shall define a mapping $h^*\colon B\to A$ by the formula $h^*(g) = h(g) - h(0_Y)$ for all $g\in B$. Then h^* is also an uniform homeomorphism and $h^*(0_Y) = 0_X$. Put

$$\operatorname{supp}_\varepsilon^h x = \operatorname{supp}_\varepsilon^{h^*} x, \quad \operatorname{supp}^h x = \operatorname{supp}^{h^*} x,$$

$$\operatorname{supp}_\varepsilon^{h^{-1}} y = \operatorname{supp}_\varepsilon^{(h^*)^{-1}} y, \quad \operatorname{supp}^{h^{-1}} y = \operatorname{supp}^{(h^*)^{-1}} y.$$

2. Main result

Definition 2.1. Two Tychonoff spaces X and Y are said to be *fu*-equivalent if there exists an uniform homeomorphism $h:C_p(Y)\to C_p(X)$ such that $\operatorname{supp}^h x$ and $\operatorname{supp}^{h^{-1}} y$ are finite sets for all $x\in X$ and $y\in Y$.

The main result of the paper is following.

Theorem 2.2. If X and Y are *fu*-equivalent then $l(X) = l(Y)$.

For the proof we need some notions.

Definition 2.3. Let $\varphi:X\to Y$ be a finite-valued, surjective, lower semicontinuous mapping of X to Y. For φ and any $U\subset Y$ we put $\varphi^*(U) = \{x\in X: \varphi(x)\subset U\}$.

We denote by $\mathcal{T}$ the family of all open subsets of Y.

Definition 2.4. A set-valued mapping $G:\mathrm{T}\to X$ is said to be φ-extractor (simply extractor) if the following conditions hold:

S(1) $\varphi^*(U)\subset G(U)$ for any $U\in\mathcal{T}$;

S(2) For any increasing consequence $(U_n)_{n\in\mathbf{N}}$, $U_n\in\mathcal{T}$ such that

$$X = \bigcup_{k\in N}\bigcap_{n\ge k} G(U_n) \tag{6}$$

the following equality holds:

$$Y = \bigcup_{n\in N} U_n. \tag{7}$$

The complement of $G(U)$ to X we denote by $F(U)$ and the set-valued mapping $F:\mathrm{T}\to X$ we call φ-co-extractor (simply co-extractor).

The concept of extractor was introduced by A.Bouziad in [1].

Let $\mathcal{U}$ be an open cover of Y closed with respect to finite unions. Fix any infinite cardinal τ. Let us introduce some notations. Put $[\mathrm{U}]_\tau = \{\bigcup \mathrm{U}' : \mathrm{U}' \subset \mathrm{U}, |\mathrm{U}'| \le \tau\}$.

We say that the set $A\subset X$ has a type F_τ in X or A is F_τ-subset of X, where τ is a cardinal, if A can be represented as a union of a family $\mathcal{F}$ of closed subsets so that $|\mathrm{F}|\le\tau$. By $\mathcal{F}_\tau$ we denote the family of all subsets of X that have a type F_τ. Denote by $\mathcal{L}_\tau$ the family of all subsets A of X such that $l(A)\le\tau$.

Let $l(X)\le\tau$. Then $\mathcal{F}_\tau\subset\mathcal{L}_\tau$. Define the mapping

$$U:\mathrm{Fin}\,\mathrm{F}_\tau\to[\mathrm{U}]_\tau, \quad U = U(\mathrm{F}), \quad \mathrm{F}\in\mathrm{Fin}\,\mathrm{F}_\tau, \tag{8}$$

which will be called φ-constructor from $\mathcal{U}$ (simply constructor). For this aim we define a number $\rho(x) = |\varphi(x)|$ for any $x\in X$. For any $F\subset X$ we put $\rho(F) = \min\{\rho(x): x\in F\}$. The number $\rho(F)$ is said to be level of F. Further, for any $U\in\mathcal{T}$ and $k\in\mathbf{N}$ put $U^{[k]} = \{x\in X: |\varphi(x)\cap U|\ge k\}$. Since φ is lower semicontinuous, we see that $U^{[k]}$ is open in X for any $k\in\mathbf{N}$ (it can be empty for $k\ge 2$).

Suppose $\mathrm{F}=\{F_1,\ldots,F_n\}\subset \mathrm{F}_\tau$. Put $F_A=\bigcap_{i\in A} F_i$ for any $A\subset\{1, \ldots, n\}$, $A\neq\varnothing$ and put $\overline{\mathrm{F}}=\{F_A\neq\varnothing:\varnothing\neq A\subset\{1,\ldots,n\}\}$, i.e., $\overline{\mathrm{F}}$ is the family of all nonempty intersections of elements from F. Let $F\in\overline{\mathrm{F}}$ and $k=\rho(F)$. It is not difficult to verify that the family $\mathrm{U}^{[k]}=\{U^{[k]}:U\in\mathrm{U}\}$ is an open cover of F which Lindelöf number does not exceed τ, hence it contains a subcover $\{U^{[k]}:U\in\mathrm{U}_F\}$ of F, where $\mathrm{U}_F\subset\mathrm{U}$ such that $|\mathrm{U}_F|\leq\tau$. Put $U(\mathrm{F})=\bigcup_{F\in\overline{\mathrm{F}}}(\bigcup\mathrm{U}_F)$. Obviously, $U(\mathrm{F})\in[\mathrm{U}]_\tau$, and if $\mathrm{F}_1\subset\mathrm{F}_2$, then $U(\mathrm{F}_1)\subset U(\mathrm{F}_2)$. The constructor (8) is defined. Note one property of the constructor.

(*) For any $\mathrm{F}\in\mathrm{Fin}\,\mathrm{F}_\tau$, nonempty $F\in\overline{\mathrm{F}}$, and $x\in F$ the following inequality holds:

$$|\varphi(x)\cap U(\mathrm{F})|\geq\rho(F). \tag{9}$$

Definition 2.5. An open cover $\mathcal{U}$ of Y is said to be τ-trivial if it has subcover which cardinality does not exceed τ. Otherwise this cover is said to be τ-nontrivial.

Proof of theorem 2.2. Let $h{:}C_p(Y)\to C_p(X)$ be an uniform homeomorphism such that $\mathrm{supp}^h\,x$ and $\mathrm{supp}^{h^{-1}}\,y$ are finite sets for all $x\in X$ and $y\in Y$, and let $l(X)\leq\tau$. Every uniform homeomorphism between C_p-spaces can be extended to some uniform homeomorphism h^* between the spaces of all functions (see [2]). The following lemma states that the mapping supp^h will not changes if we substitute the uniform homeomorphism $h{:}C_p(Y)\to C_p(X)$ for its extension $h^*{:}\mathbf{R}^Y\to\mathbf{R}^X$.

Lemma 2.6. Let $h{:}C_p(Y)\to C_p(X)$ be an uniform homeomorphism and $h^*{:}\mathbf{R}^Y\to\mathbf{R}^X$ – its uniformly continuous extension. Then $\mathrm{supp}^h\,x=\mathrm{supp}^{h^*}\,x$ for any $x\in X$.

Proof. Let y be ε-essential point for x under h for some $\varepsilon>0$. It follows from definition, that y is ε-essential for x under h^*. Therefore, $\mathrm{supp}_\varepsilon^h\,x\subset\mathrm{supp}_\varepsilon^{h^*}\,x$. Then, we shall prove that $\mathrm{supp}_\varepsilon^{h^*}\,x\subset\mathrm{supp}_\delta^h\,x$ if $0<\delta<\varepsilon$. Let y be ε-essential point for x under h^* for some $\varepsilon>0$, and let $0<\delta<\varepsilon$. Put $\varepsilon_0=(\varepsilon-\delta)/2$. Let O_y be an open neighborhood of y. There exist functions g_0', $g_0''\in\mathbf{R}^Y$, coinciding on the set $Y\setminus O_y$ and satisfying the following inequality:

$$|h^*(g_0')(x)-h^*(g_0'')(x)|>\varepsilon.$$

Since h^* is an uniform homeomorphism, there exist a finite set $K\subset Y$ and $\Delta>0$ such that for all g', $g''\in\mathbf{R}^Y$ we have the implication

$$(|g'(y)-g''(y)|<\Delta \text{ for all } y\in K)\Rightarrow|h^*(g')(x)-h^*(g'')(x)|<\varepsilon_0. \tag{10}$$

Put $F=K\cap O_y$. There is a function $g_1\in C_p(Y)$ such that

$$g_1|_K=g_0'|_K, \tag{11}$$

and a function $g_2\in C_p(Y)$ such that $g_2|_{Y\setminus O_y}=g_1|_{Y\setminus O_y}$, $g_2|_F=g_0''|_F$. Then

$$g_2|_K=g_0''|_K. \tag{12}$$

By (11), (12), and (10) we have

$$|h^*(g_0')(x)-h(g_1)(x)|<\varepsilon_0,\ |h^*(g_0'')(x)-h^*(g_2)(x)|<\varepsilon_0,$$

hence,

$$|h(g_1)(x)-h(g_2)(x)|\geq|h^*(g_0')(x)-h^*(g_0'')(x)|-|h^*(g_0')(x)-h(g_1)(x)|-$$
$$-|h(g_2)(x)-h^*(g_0'')(x)|>\varepsilon-2\varepsilon_0=\delta.$$

Inclusion $\mathrm{supp}_\varepsilon^{h^*}\,x\subset\mathrm{supp}_\delta^h\,x$ is proved. This completes the proof. ■

Further, we can assume without loss of generality that h is an uniform homeomorphism from $\mathbf{R}^Y$ to $\mathbf{R}^X$ satisfying following conditions:

1. $h(C_p(Y)) = C_p(X)$ and $h^{-1}(C_p(X)) = C_p(Y)$;
2. h takes zero-function $0_Y \in C_p(Y)$ to zero-function $0_X \in C_p(X)$;
3. $\mathrm{supp}^h x$ and $\mathrm{supp}^{h^{-1}} y$ are finite sets for all $x \in X$ and $y \in Y$.

Suppose that $l(Y) > \tau$ to obtain a contradiction. In our terminology it means that there exists τ-nontrivial open cover $\mathcal{U}$ of Y. We can assume without loss of generality that $\mathcal{U}$ is closed with respect to finite unions and $\mathcal{U} \subset \mathcal{B}$, where $\mathcal{B}$ is a base of Y which consists of all functionally open subsets of Y. Family $\mathcal{B}$ is closed with respect to finite unions. Let $\varphi = \mathrm{supp}^h : X \to Y$. Note an important property of φ.

(Φ) If $g', g'' \in \mathbf{R}^Y$ and $g'|_{\varphi(x)} = g''|_{\varphi(x)}$, then $h(g')(x) = h(g'')(x)$.

For any $A \subset Y$ define the function $e_A \in \mathbf{R}^Y$ by the formula

$$e_A(y) = \begin{cases} 0, & y \in A; \\ 1, & y \notin A. \end{cases} \tag{13}$$

For every open set $V \in \mathcal{T}$ put

$$G(V) = \{x \in X : h(e_V)(x) = 0\},\ F(V) = \{x \in X : h(e_V)(x) \neq 0\}.$$

Lemma 2.7. G is φ-extractor.

Proof. Check that condition S(1) is fulfilled. Let $V \in \mathcal{T}$ and $x \in \varphi^*(V)$. Since $\varphi(x) \in V$, we have $e_V|_{\varphi(x)} \equiv 0$, hence, by property ($\Phi$), we get $h(e_V)(x) = 0$, i.e., $x \in G(V)$.

It remains to check S(2). Let $(U_n)_{n\in\mathbf{N}}$, $U_n \in \mathcal{T}$, be any increasing consequence, satisfying condition (6). Assume that $Y \neq \bigcup_{n\in N} U_n$. Put $U = \bigcup_{n\in N} U_n$. Let $y \in Y \setminus U$. Choose a finite subset $K = \{x_1, \dots, x_p\} \subset X$ and $\delta > 0$ so that for any function $f \in \mathbf{R}^X$ the following implication holds:

$$(\,|f(x_i)| \leq \delta \text{ for all } i \in \{1, \dots, p\}) \Rightarrow |h^{-1}(f)(y)| < 1.$$

Such a choice is possible because of the continuity of the mapping h^{-1} and the condition $h^{-1}(0_X) = 0_Y$. By condition (6), we can choose a number N such that $x_i \in G(U_n)$ for all $n > N$ and $i \in \{1, \dots, p\}$, i.e., $h(e_{U_n})(x_i) = 0$. Passing to the limit as $n \to \infty$, we obtain $h(e_U)(x_i) = 0$ for all $i \in \{1, \dots, p\}$. Then from (13) we have $|e_U(y)| < 1$, hence, $y \in U$. This contradiction concludes the proof. ■

Now denote by C the family of all functionally closed subsets of Y. Any functionally open subset V admits a decomposition $V = \bigcup_{n\in N} F_n$ where $F_n \in C$ and $F_n \subset F_{n+1}$ for all $n \in \mathbf{N}$. If there exists a decomposition satisfying the following condition:

$$\varphi^*(V) \setminus \varphi^*(F_n) \neq \varnothing \text{ for all } n \in \mathbf{N},$$

then we say that the subset V is adequate. This notion was introduced by A.Bouziad in [1].

Lemma 2.8. Let τ be an infinite cardinal, $\mathcal{U} \subset \mathcal{B}$ – an open, τ-nontrivial cover of Y, closed with respect to finite unions, and $\{U_t\}_{t\in T} \subset \mathcal{U}$ – a subfamily such that $|T| \leq \tau$. Then there is a family $\{V_s\}_{s\in S} \subset [\mathrm{U}]_{\aleph_0}$, closed with respect to finite unions, such that

1. $|S| \leq \tau$,
2. each set V_s is adequate,

3. $\bigcup_{t\in T} U_t \subset \bigcup_{s\in S} V_s$.

Proof. Put $U = \bigcup_{t\in T} U_t$. Since the cover $\mathcal{U}$ is τ-nontrivial, we have $Y\setminus U\neq\varnothing$. Choose $x_1\in X$ such that $\varphi(x_1)\not\subset U$ and a set $V_1\in\mathcal{U}$ such that $\varphi(x_1)\subset V_1$. Suppose $x_1, \dots , x_n$ and $V_1, \dots ,V_n$ are already chosen. The set $Y\setminus (U\cup V_1\cup \dots \cup V_n)$ is nonempty, hence there is an element $x_{n+1}\in X$ such that $\varphi(x_{n+1})\not\subset U\cup V_1\cup \dots \cup V_n$. There exist $V_{n+1}\in\mathcal{U}$ such that $\varphi(x_{n+1})\subset V_{n+1}$. We get two consequences $(x_n)_{n\in\mathbf{N}}$, $x_n\in X$ and $(V_n)_{n\in\mathbf{N}}$, $V_n\in\mathcal{U}$ for all $n\in\mathbf{N}$. Put $V = \bigcup_{n\in N} V_n$. Let $\{W_s\}_{s\in S}$ be the family of all finite unions of sets from the family $\{U_t\}_{t\in T}$. For each $s\in S$ put $V_s = W_s\cup V$. It is clear that the family $\{V_s\}_{s\in S} \subset [\mathrm{U}]_{\aleph_0}$ is closed with respect to finite unions, $|S|\leq\tau$, and $\bigcup_{t\in T} U_t \subset \bigcup_{s\in S} V_s$. Let us check that each V_s is adequate. Let $s\in S$. Fix a decomposition $(F_n^s)_{n\in N}$ of W_s and a decomposition $(F_n^k)_{n\in N}$ of V_k, $k\in\mathbf{N}$. The consequence $(G_n^s)_{n\in N}$, where $G_n^s = F_n^s \cup F_n^1 \cup \dots \cup F_n^n$, is a required decomposition of V_s. Besides, we have $(x_n)_{n\in\mathbf{N}}\subset\varphi^*(V_s)$ and $x_{n+1} \notin \varphi^*(G_n^s)$ for all $n\in\mathbf{N}$. ■

Lemma 2.9. Let S be infinite set and $\{V_s\}_{s\in S}$ be a family of adequate functionally open subsets of Y, closed with respect to finite unions. Then $F(\bigcup_{s\in S} V_s)$ is F_τ-subset of X, where $\tau = |S|$.

Proof. Put $V = \bigcup_{s\in S} V_s$. Let $(F_n^s)_{n\in N}$ be a decomposition of V_s such that $F_n^s \in \mathrm{C}$ and $F_n^s \subset F_{n+1}^s$ for all $n\in\mathbf{N}$. For any natural n and $s\in S$ choose a function $g_n^s \in C(Y)$ such that

$$g_n^s\Big|_{F_n^s} \equiv 0, \quad g_n^s\Big|_{Y\setminus V_s} \equiv 1.$$

For every $x\in\varphi^*(V_s)$, and every natural k put

$$U_k^s(x) = \left\{x' \in X : \left|h\left(g_{k+n(x,s)}^s\right)(x') - h\left(g_{k+n(x,s)}^s\right)(x)\right| < 1/k\right\},$$

where $n(x, s)$ is the least number n such that $\varphi(x) \subset F_n^s$. Then $U_k^s(x)$ is an open neighborhood of $x\in X$. Put

$$A_s = \bigcap_{k\in N} \bigcup_{x\in\varphi^*(V_s)} U_k^s(x),\ B_s = \{x\in X\colon \varphi(x)\cap(V\setminus V_s) \neq \varnothing\},\ A = \bigcap_{s\in S} (A_s \cup B_s).$$

We now prove that $G(V) = A$. Since the sets A_s and B_s are G_δ-subsets of X, it will be enough for the proof of the lemma. First we shall show that

$$F(V) \subset X\setminus A. \tag{14}$$

Let $x'\in F(V)$. Then there exists $\varepsilon > 0$ such that $|h(e_V)(x')| > \varepsilon$. Since $\varphi(x')$ is a finite set and the family $\{V_s\}_{s\in S}$ closed with respect to finite unions, there exists $s\in S$ such that $\varphi(x')\cap V\subset V_s$, i.e., $x'\notin B_s$. Note that, since, $e_V\big|_{\varphi(x')}=e_{V_s}\big|_{\varphi(x')}$ by (Φ) we have $h(e_{V_s})(x') = h(e_V)(x')$. Since $\left|h(e_{V_s})(x')\right| = |h(e_V)(x')| > \varepsilon$ and $\lim_{n\to\infty} g_n^s = e_{V_s}$, there exists natural N such that

$$\left|f(g_n^s)(x')\right| > \varepsilon \text{ for all } n > N.$$

Choose a number k such that $k \geq \max\{N, 1/\varepsilon\}$. We shall check that

$$x' \notin \bigcup_{x\in\varphi^*(V_s)} U_k^s(x).$$

It will imply that $x' \notin A_s$, and inclusion (14) will be proved. Let $x \in \varphi^*(V_s)$. Note that, since $g_{k+n(x,s)}^s \big|_{\varphi(x)} \equiv 0$, we have $h(g_{k+n(x,s)}^s)(x) = 0$. Then

$$\left|h\left(g_{k+n(x,s)}^s\right)(x') - h\left(g_{k+n(x,s)}^s\right)(x)\right| = \left|h\left(g_{k+n(x,s)}^s\right)(x')\right| > \varepsilon \geq 1/k,$$

i.e., $x' \notin U_k^s(x)$. Inclusion (14) is proved.

Let us prove the inverse inclusion $X \setminus A \subset F(V)$. Since the sets V_s are adequate, we can assume that there decompositions satisfy the following property: $\varphi^*(V_s) \setminus \varphi^*(F_n^s) \neq \varnothing$ for all $n \in \mathbf{N}$. Let $x' \notin A$. Choose $s \in S$ such that $x' \notin A_s \cup B_s$. Then we have $\varphi(x') \cap V \subset V_s$. Fix natural k such that $x' \notin \bigcup_{x\in\varphi^*(V_s)} U_k^s(x)$, natural m such that $\varphi(x') \cap V \subset F_m^s$, and $x_0 \in \varphi^*(V_s)$ such that $\varphi(x_0) \not\subset F_m^s$. Then we have $n(x_0, s) > m$ and

$$\varphi(x') \cap V = \varphi(x') \cap V_s \subset F_m^s \subset F_{k+n(x_0,s)}^s.$$

Put $i = k+n(x_0, s)$. Since $x' \notin U_k^s(x_0)$, we have $\left|h(g_i^s)(x') - h(g_i^s)(x_0)\right| \geq 1/k$. Besides, $h(g_i^s)(x_0) = 0$. From this we obtain that $\left|h(g_i^s)(x')\right| \geq 1/k$. But since $e_V \big|_{\varphi(x')} = e_{V_s} \big|_{\varphi(x')}$ and $e_{V_s} \big|_{\varphi(x')} = g_i^s \big|_{\varphi(x')}$, we have $h(g_i^s)(x') = h(e_V)(x')$, and, finally, $|h(e_V)(x')| \geq 1/k$, i.e., $x' \in F(V)$. The statement of the lemma is proved. ■

By 2.9 and 2.8 we have

Theorem 2.10. Let τ be an infinite cardinal and $\mathcal{U} \subset \mathcal{B}$ – an open, τ-nontrivial cover of Y, closed with respect to finite unions, $\{U_t\}_{t\in T} \subset \mathcal{U}$ – a subfamily such that $|T| \leq \tau$. Then there is $V \in [\mathcal{U}]_\tau$ such that $\bigcup_{t\in T} U_t \subset V$ and $F(V)$ is F_τ-subset of X.

Now we have all the facts necessary to prove the result, formulated in the beginning. We shall construct by induction increasing consequence $(V_n)_{n\in\mathbf{N}}$, $V_n \in [\mathcal{U}]_\tau$ such that $Y = \bigcup_{n\in N} V_n$. Simultaneously we shall construct the consequence $(\mathcal{F}_n)_{n\in\mathbf{N}}$, $\mathcal{F}_n \in \mathrm{Fin}\, \mathcal{F}_\tau$, such that $\mathcal{F}_{n'} \subset \mathcal{F}_{n''}$ for $n' < n''$. For this aim we shall use the constructor U and the co-extractor F defined above.

Put $\mathcal{F}_0 = X$. Pick a set $V_1 \in [\mathcal{U}]_\tau$ such that $U(\mathcal{F}_0) \subset V_1$ and $F(V_1) \in \mathcal{F}_\tau$ (this is possible by Theorem 2.10), and put $\mathcal{F}_1 = \{X, F(V_1)\}$. Choose the set $V_2 \in [\mathcal{U}]_\tau$ so that $V_1 \cup U(\mathcal{F}_1) \subset V_2$ and $F(V_2) \in \mathcal{F}_\tau$. Suppose we have already defined the sets $V_i \in [\mathcal{U}]_\tau$ and $\mathcal{F}_i \in \mathrm{Fin}\, \mathcal{F}_\tau$ for all numbers i such that $1 \leq i \leq k$, satisfying the following conditions:

1. $F(V_i) \in \mathcal{F}_\tau$, $1 \leq i \leq k$;
2. $V_i \cup U(\mathcal{F}_i) \subset V_{i+1}$, $1 \leq i \leq k-1$, where $\mathcal{F}_i = \{X, F(V_1), \dots, F(V_i)\}$, $1 \leq i \leq k$.

Choose the set $V_{k+1}\in[\mathcal{U}]_\tau$ so that the following conditions hold:

$$V_k\cup U(\mathcal{F}_k)\subset V_{k+1} \text{ and } F(V_{k+1})\in\mathcal{F}_\tau. \tag{15}$$

Put $\mathcal{F}_{k+1} = \{X, F(V_1), \dots , F(V_{k+1})\}$. The consequences $(V_n)_{n\in\mathbf{N}}$, $V_n\in[\mathcal{U}]_\tau$ and $(\mathcal{F}_n)_{n\in\mathbf{N}}$, $\mathcal{F}_n\in\text{Fin }\mathcal{F}_\tau$ are already defined. Prove by induction over n the following statement:

(ST) For any natural n and any subset $\{j_1, \dots ,j_k\}\subset\{1, \dots , n\}$ such that $F(V_{j_1})\cap\ldots\cap F(V_{j_k})\neq\varnothing$, the following inequality holds:

$$\rho(F(V_{j_1})\cap\ldots\cap F(V_{j_k}))\geq k+1. \tag{16}$$

For $k = 1$ it is enough to show that $\rho(F(V_n))\geq2$. For any $x\in X$ by inequality (9) we have $|\varphi(x)\cap V_n| \geq |\varphi(x)\cap V_1| \geq |\varphi(x)\cap U(\mathcal{F}_0)| \geq \rho(X) \geq 1$. Therefore, if $\rho(x) = 1$ for some $x\in X$, then $\varphi(x)\subset V_n$, consequently by S(1) we have $x\notin F(V_n)$, and thus, $\rho(F(V_n)) \geq 2$.

Suppose statement (ST) is true for all numbers n such that $1\leq n\leq N$. Let us prove that it is true for $n = N+1$. It suffices to show that for any subset $\{j_1, \dots ,j_k\}\subset\{1, \dots , N\}$ such that $F = F(V_{j_1})\cap\ldots\cap F(V_{j_k})\cap F(V_{N+1})\neq\varnothing$, we have $\rho(F) \geq k+2$. Put $F' = F(V_{j_1})\cap\ldots\cap F(V_{j_k})$, then $F = F'\cap F(V_{N+1})$. By the inductive assumption, we have $\rho(F') \geq k+1$. Assume that $\rho(F) = k+1$ to obtain a contradiction. Let $x\in F$ such that $|\varphi(x)| = k+1$. Since $F'\in\overline{\mathrm{F}_N}$, we see that by condition (15) and inequality (9), it follows that $|\varphi(x)\cap V_{N+1}| \geq |\varphi(x)\cap U(\mathcal{F}_N)| \geq \rho(F') \geq k+1$. Hence, $\varphi(x)\subset V_{N+1}$, and by condition S(2), it follows that $x\notin F(V_N+1)$, therefore, $x\notin F$. This contradiction concludes the proof of statement (ST). In particular, inequality (16) involves that for any $x\in X$ there is a number n_0 such that $x\notin F(V_n)$ for all $n > n_0$, i.e., $x\in G(V_n)$. In other words, equality (6) holds. So, by condition S(2), we have $Y = \bigcup_{n\in N} V_n$. Since $V_n\in[\mathcal{U}]_\tau$ for all $n\in\mathbf{N}$, we see that the cover $\mathcal{U}$ of Y is τ-trivial, a contradiction. So, $l(Y)\leq\tau$. Consequently, $l(Y)\leq l(X)$. Analogously, $l(X)\leq l(Y)$. Theorem is proved. ■

REFERENCES

1. *Bouziad A.* Le degre de Lindelöf est l-invariant // Proceedings of the American Mathematical Society. 2000. V. 129. No. 3. P. 913 – 919.
2. *Engelking R.* General Topology (PWN, Warszawa, 1977).
3. *Gul'ko S.P.* On uniform homeomorphisms of spaces of continuous functions // Proceedings of the Steklov Institute of Mathematics. 1993. V. 3. P. 87 – 93.
4. *Okunev O.* Homeomorphisms of function spaces and hereditary cardinal invariants // Topol. and its Appl. 1997. V. 80. P. 177 – 188.
5. *Velichko N.V.* The Lindelöf property is *l*-invariant // Topol. and its Appl. 1998. V. 89. P. 277 – 283.

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